Innovative Applications of O.R.
Incorporating quantity discounts and their inventory impacts into the centralized purchasing decision

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\textbf{ABSTRACT}

Multi-site organizations must balance conflicting forces to determine the appropriate degree of purchasing centralization for their respective supplies. The ability to garner quantity discounts represents one of the primary reasons that organizations centralize procurement. This paper provides methodologies to calculate optimal order quantities and compute total purchasing and inventory costs when products have quantity discount pricing. Procedures for both all-units and incremental quantity discount schedules are provided for four different strategic purchasing configurations (scenarios): complete decentralization, centralized pricing with decentralized purchasing, centralized purchasing with local distribution, and centralized purchasing and warehousing. For ordering decisions under local distribution, procedures to determine optimal order quantities and costs are presented in a precise form that could be easily implemented into spreadsheets by practicing managers. For the more complicated multi-echelon scenarios, we introduce a single-cycle policy with a tailored aggregation refinement step that performs very well under experimentation when compared to a conservative bound.

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1. Introduction and background

The typical manufacturer spends more than 60\% of sales on purchased materials and services (Krajewski et al., 2007); thus, firms should exert significant effort to optimize their respective internal purchasing organizations. A key decision that any company with multiple facilities must make is the degree of centralization of the purchasing function.

Centralized purchasing has a number of advantages and disadvantages (see, e.g., Munson, 2007; Vagstad, 2000; Corey, 1978). In fact, most companies seem to practice some combination of centralized and decentralized purchasing (Benton, 2007). One attempt to encapsulate the best aspects of both centralization and decentralization is called “centralized pricing with decentralized purchasing”. In that structure, a centralized purchasing group selects suppliers and negotiates contracts for the entire company. Each location orders from those contracts, but each chooses its own order quantities, orders exactly when it wants to, and has the products shipped directly. Even small firms can pursue such a strategy by joining industry buying groups such as pharmacy cooperatives (Schotanus, 2007). “P-Cards” have increased in popularity and have become an effective method for implementing centralized pricing with decentralized purchasing (Roy, 2003). Other descriptions of hybrid models can be found in Schotanus et al. (2008), Reese and Pohlman (2005) and Monczka et al. (2002).

Thus, we observe that firms first must choose whether to price centrally or not, then whether to actually purchase centrally or not. And since many firms manage central warehouses while others do not, the third centralization decision is then whether or not to actually store the goods at a central location or to have suppliers ship directly to field sites. Of these $2^3 = 8$ possible combinations of decisions, we choose in this paper to examine the four logical ones. That is, if a firm prices locally, then it will presumably purchase locally and have local shipment (Scenario 1). If the firm prices centrally, then it could still purchase locally, presumably then also using local delivery (Scenario 2). On the other hand, if the firm prices and purchases centrally, then it could choose either to warehouse the goods centrally (Scenario 3) or have local delivery (Scenario 4). Fig. 1 displays our classification.

A surprisingly small number of theoretical research papers have directly addressed the centralized purchasing issue from a cost optimization perspective. Munson (2007) provides a model that allocates different products to one of three different centralized purchasing schemes based on a comparative analysis of seven cost categories. Aspects of the centralized purchasing issue do arise in related research,
2.1. Centralization scenarios

The modeling framework

The centralization scenarios.

2. The modeling framework

2.1. Centralization scenarios

Fig. 1 provides a flow diagram of the four centralization scenarios that we analyze.

Scenario 1 – Complete decentralization: Each site purchases completely on its own and stores its own units.

Scenario 2 – Centralized pricing with decentralized purchasing: The company negotiates company-wide contracts with vendors. All sites purchase for themselves using the negotiated prices, and they store their own units. While purchase prices should generally be at least as low as those obtained using Scenario 1, the CPO’s vendors may be located far away from certain specific sites, resulting in large transportation costs compared to using local vendors.

Scenario 3 – Centralized purchasing and warehousing: The items are purchased from one location and delivered to a central warehouse where they may stay for some time until requested for distribution to the individual sites. Thus, the field sites receive bulk deliveries from the warehouse periodically. In this case, demand at the field sites is constant, but the demand at the warehouse is lumpy. Compared to Scenario 4, this represents a multiechelon inventory system allowing for potentially cheaper holding costs at a warehouse and potentially different order frequencies at the field sites. Scenario 3 is designed to capture certain benefits from both Scenarios 2 and 4.

Scenario 4 – Centralized purchasing with local distribution: The company purchases the items from a CPO, but all items are shipped directly to the local sites (no central warehousing). Purchases are made on a periodic basis, so this represents a type of joint replenishment problem. Compared to Scenario 2, this scenario may provide purchase price benefits due to larger orders, but at the cost of inventory positions at each site that do not necessarily minimize the site-specific inventory costs.
2.2. Assumptions and notation

(1) The company always purchases FOB vendor (or at least knows exactly how much of the purchase cost is for the transportation expense). Ideally, transportation cost should be considered in our models, because it represents a crucial component in any centralized purchasing decision with significant geographic dispersion. Furthermore, separating transportation cost from purchase price allows for more accurate holding cost calculations in the multi-echelon case (Scenario 3).

(2) No transshipments are allowed between the field sites. (Transshipments represent a special inventory management problem (see, e.g., Schwarz, 1981), and they are most applicable when the field sites have close geographic proximity.)

(3) As with the bulk of the quantity discount literature, typical EOQ assumptions apply, e.g., infinite planning horizon; infinite production rate; deterministic, constant, and uniform demand; no backorders; unlimited capacity; deterministic lead times; etc.

(4) Basic notation includes: \( i \) = site number, \( i = 0, 1, \ldots, I \) (site 0 is the CPO and warehouse, if applicable); \( D_i \) = annual demand at site \( i \); \( D_0 = \sum_{i=1}^{I} D_i \); \( S_i \) = setup cost per order at site \( i \); \( H_i \) = annual holding cost expressed as a percentage of value of the product at site \( i \); \( C_p \) = total purchasing cost of \( q_{ij} \) units at site \( i \) (see Section 2.4); and \( Q_i \) = amount per order at site \( i \). For notational convenience, unless otherwise indicated, \( \sum_{i} \) means \( \sum_{i=1}^{I} \) (i.e. the summation over the field sites not including the warehouse).

**Fig. 2.** Flow diagram of the centralization scenarios.
2.3. Transportation costs

For the purposes of this study, transportation costs are assumed to be linear. (If transportation costs are imbedded in the purchase price schedules, then the transportation cost terms can be ignored in our formulas. Alternatively, the formulas could be modified to incorporate non-linear transportation costs, when applicable.) The term “site k’s vendor” refers to the vendor who offers the price schedule to site k. Since transportation costs are part of the cost of the product and they would differ depending on the centralization scenario chosen, they are included in the inventory holding cost calculations in our models. We define the following:

\[ T_i^{(k)} = \text{per-unit transportation cost directly from site k's vendor to site } i, \quad (i, k) = 0, 1, \ldots, I; \]

\[ T_i^W = \text{per-unit transportation cost directly from the central warehouse to site } i, \quad i = 1, 2, \ldots, I. \]

Under this scheme, transportation costs for the four scenarios are: Scenario 1 = \[ \sum D_i T_i^{(k)} \], Scenarios 2 and 4 = \[ \sum D_i T_i^W \], and Scenario 3 = \[ \sum D_i (T_i^{(k)} + T_i^W) \].

2.4. Discount schedules

In purchasing, the two most common quantity discount forms are “all-units” (where the lower price applies to all units purchased, not just those above the price break) and “incremental” (where only those units within a price break interval receive that interval’s discount). While the all-units variety seems to occur more frequently, the incremental variety appears often enough that it deserves careful attention as well (Munson and Rosenblatt, 1998).

We define \( q_0 = \text{the } j^{th} \text{ cost breakpoint in the price schedule at site } i; \quad j = 1, \ldots, j[i]; \quad q_{0} = 0; \quad \text{and } q_{(j[i]-1)} = \infty. \)

For the all-units discount schedule, \( P_i = \text{the per-unit price for all units purchased at site } i \text{ when } q_0 < q_i \leq q_{(j[i]-1)}; \) \( q_{(j[i]-1)} = 1 \), and \( C_i = P_i q_i. \) For the incremental discount schedule, \( P_i = \text{the per-unit price for quantities } q_i \text{ purchased at site } i \text{ such that } q_0 < q_i \leq q_{(j[i]-1)}; \) \( j = 1, \ldots, j[i]; \quad q_{(j[i]-1)} = 1 \); \( P_i = P_{i0} q_i + P_{i1} (q_i - q_0) + \ldots + P_{i(j[i]-1)} (q_{(j[i]-1)} - q_{(j[i]-1)}). \) (We follow the long-held notational convention (Hadley and Whitin, 1963; Chopra and Meindl, 2007) defining \( q_0 \) as the first quantity to receive the lower price \( P_i \) in the all-units case, while defining \( q_0 + 1 \) as the first quantity to receive the lower price \( P_i \) in the incremental case. Each convention simplifies the notation in the resulting formulas for its respective discount type.)

3. Optimal purchasing and inventory policies for each scenario

We analyze the four centralization scenarios first for all-units discounts (Section 3.1) and then for incremental discounts (Section 3.2).

3.1. All-units quantity discounts

3.1.1. Scenario 1 – Complete decentralization, all-units discounts

Procedures to determine the optimal lot size in an EOQ environment given an all-units quantity discount schedule can be found in most introductory operations management textbooks (e.g., Heizer and Render, 2008). It is easy to show that the procedure can be written as a single function by using Min and Max operators, which may allow for streamlined implementation into spreadsheets. We will refer to this technique as the ADOQ (all-units discount order quantity) method. For each site \( i \) and discount interval \( j \), define \( Q_i = \left[ 2SD_i / \left[ H_i (P_i^0 + T_i^{(k)}) \right] \right]^{1/2} \) and \( Z_i = D_i P_i^0 / \left[ H_i (P_i^0 + T_i^{(k)}) / 2 \right] \max(q_0, Q_i) + \left[ (D_i S_i) / \max(q_0, Q_i) \right]. \) Then the optimal lot size at site \( i \) is \( Q_i^* = \max(q_0, Q_i), \) where \( j = \text{Argmin}_{j_0 \ldots j_{(i-1)}} Z_j. \) And the purchasing and inventory costs for site \( i \) are: \( \Omega_i^* = Z_i. \)

3.1.2. Scenario 2 – Centralized pricing with decentralized purchasing, all-units discounts

In this case, each site purchases from the CPO’s price schedule, so, using the ADOQ method, \( Q_i = \left[ 2SD_i / \left[ H_i (P_i^0 + T_i^{(k)}) \right] \right]^{1/2} \) and \( Z_i = D_i P_i^0 + \left[ H_i (P_i^0 + T_i^{(k)}) / 2 \right] \max(q_0, Q_i) + \left[ (D_i S_i) / \max(q_0, Q_i) \right]. \) The optimal lot size at site \( i \) is \( Q_i^* = \max(q_0, Q_i), \) where \( j = \text{Argmin}_{j_0 \ldots j_{(i-1)}} Z_j. \) And the purchasing and inventory costs for site \( i \) are: \( \Omega_i^* = Z_i. \)

3.1.3. Scenario 3 – Centralized purchasing and warehousing, all-units discounts

In Scenario 3, units are first delivered in lots to the warehouse where, after possibly remaining as inventory for some time, they are subsequently delivered to the field sites. The units go through a materials handling process twice, so the field sites have a setup cost \( S_i^W \) that relates to their orders from the warehouse. (Typically, \( S_i^W < S_i \) that is applicable to orders directly from vendors.)

As first introduced by Clark and Scarf (1960), we shall define the holding cost parameters as “echelon holding costs”. This is a common practice in the multiechelon literature that simplifies the inventory calculations. Specifically, define: \( H_0 = \text{annual cost to hold inventory anywhere in the system, expressed as a percentage of value}; \) and \( H_i = \text{additional annual cost to hold inventory at site } i, \) expressed as a percentage of value. In other words, for a price \( P \), the holding cost per unit at the warehouse equals \( H_0 P, \) while the holding cost per unit at field site \( i \) equals \( H_i P = (H_0 + H_i) P. \) (We invoke the common assumption that it is not cheaper to hold inventory at any field site than at the warehouse, i.e. \( H_i > 0 \).)

The traditional one-warehouse, multi-retailer inventory problem is very difficult to solve optimally, in general. Certain heuristics in the form of tailored aggregation (Chopra and Meindl, 2007), q-optimal integer ratio or optimal power-of-two policies (Roundy, 1985) have been proposed in the literature. The incorporation of quantity discounts into purchase price makes the Roundy approaches rather unwieldy for implementation. A stricter assumption that is much more tractable would utilize a single-cycle policy with common delivery frequency among the field sites (as in Blumenfeld et al., 1985). For this policy (Step 1), lot sizes are computed such that the inventory position at each of the field sites equals zero simultaneously, i.e. the warehouse makes a delivery of size \( Q_i \) to each field site \( i \) at the same time. To ensure a
common delivery frequency, the lot sizes must be set such that $Q_i/D_i = Q_j/D_j$, for field sites $i$ and $k$, respectively. The warehouse orders an integer multiple of $n$ of $\sum Q_i$. The solution technique will have some similarities to the EOQ-based method, although additional steps may be necessary due to the integrality of $n$.

A single-cycle policy as just described can be thought of to be “approximate” across echelons, but it retains the disadvantage of assuming that every field site uses the same order frequency – a clearly inferior policy when the parameters at the sites differ significantly. Therefore, we also propose a refinement heuristic (Step 2) that modifies the single-cycle policy by recalculating order frequencies for each field site after determining a good (actually optimal for the single-cycle case) overall order size (and cycle length) for the central warehouse. For each field site, given a total lot size allocated from the central warehouse during the warehouse’s cycle length, the site can order the full amount once, half the amount twice, one-third the amount three times, etc. Since each site can order the full allocated amount divided by $n$, this policy will perform at least as well as the single-cycle policy, and it should often perform better.

3.1.3.1. Step 1: Development of the single-cycle policy. The quantity discount applies to the amount $n\sum Q_i$ purchased by the CPO. For

$$Q_0 < n\sum Q_i < q_{0(j-1)}$$

the purchasing and base inventory costs for the system are $D_0P_0^a + (D_0/n\sum Q_i)S_0 + (n\sum Q_i/2)H_0\left(P_0^a + T_0^{\text{by}}\right)$. The incremental inventory costs for site $i$ are $(D_i/Q_i)S_i^w + (Q_i/2)H_i\left(P_i^a + T_i^{\text{by}}\right) + H_iT_i^w$. For notational convenience, we define $V_i = P_i^a + T_i^{\text{by}}$ (i.e. the value of the item at the warehouse when purchased in discount interval $j$). Then, combining the base system and incremental site costs, the problem of minimizing the total system purchasing and inventory costs when $q_0 < n\sum Q_i < q_{0(j-1)}$ can be written as:

$$\begin{align*}
\text{Min}_{nQ_i} & \quad D_0P_0^a + \frac{D_0}{n\sum Q_i}S_0 + \frac{n\sum Q_i}{2}H_0V_0 + \sum_i \left[\frac{D_i}{Q_i}S_i^w + \frac{Q_i}{2}H_iV_i + H_iT_i^w\right] \\
\text{Subject to} & \quad Q_i/D_i = Q_j/D_j, \quad \text{for } i = 2, \ldots, I.
\end{align*}$$

The function $\Phi_i(n, Q_i)$ has $I+1$ decision variables. By substituting each constraint into the objective function we are left with the following unconstrained total system purchasing and inventory cost function in two variables, $n$ and $Q_i$:

$$\Phi_i(n, Q_i) = D_0P_0^a + D_i \left(S_0 + n\sum S_i^w\right)/n + nD_iH_iV_i + \sum_i D_i\left(H_iV_i + H_iT_i^w\right)$$

Setting $\partial \Phi_i(n, Q_i)/\partial Q_i = 0$ yields the cost minimizing $Q_i$ for a given $n$:

$$Q_i^*(n) = \left\{2(D_i)^2\left(S_0 + n\sum S_i^w\right)^{1/2}/\left[nD_iH_iV_i + \sum_i D_i\left(H_iV_i + H_iT_i^w\right)\right]\right\}^{1/2}$$

Using the common frequency constraints, the lot sizes for the other field sites are $Q_i^*(n) = D_iQ_i^*(n)/D_i$. And the total cost evaluated at $Q_i = Q_i^*(n)$ equals:

$$\Phi_i^*(n) = D_0P_0^a + \left\{2\left(S_0 + n\sum S_i^w\right)\left[nD_iH_iV_i + \sum_i D_i\left(H_iV_i + H_iT_i^w\right)\right]/n\right\}^{1/2}$$

As a function of $n$, $\Phi_i^*(n)$ is minimized by minimizing the following function with respect to $n$:

$$\Phi_i(n) = S_0 + n\sum D_i\left(H_iV_i + H_iT_i^w\right) + nD_iH_iV_i + \sum_i D_i\left(H_iV_i + H_iT_i^w\right)$$

The optimal $n = n_i'$ that minimizes $\Phi_i(n)$ must satisfy $a(n_i') < a(n_i'+1)$ and $a(n_i') < a(n_i'-1)$, or $n_i'-1 < n_i$ such that $\Phi_i(n_i', 1) < \Phi_i(n_i', 2)$. Using the quadratic formula and defining $|x|$ as the greatest integer $\leq x$, a closed-form solution that satisfies these inequalities can be shown to be equal to:

$$n_i' = \frac{1}{2} \left[1 + \sqrt{1 + 4S_i\sum D_i\left(H_iV_i + H_iT_i^w\right)}/(D_iH_iV_i + \sum D_iH_iT_i^w)\right]$$

Of course, $n_i'$ and $Q_i^*(n_i')$ are only valid if $q_0 < n_i\sum Q_i(n_i') < q_{0(j-1)}$.

For a given $n$ and $Q_i$, $\Phi_i(n, Q_i)$ is decreasing in $j$ (since $P_i^a$ is decreasing in $j$); therefore, for a given $n$, $\Phi_i^*(n)$ must also be decreasing in $j$. Consequently, as with the ADOQ algorithm, the only potential combination of $n_i'$ and $Q_i^*(n_i')$ that minimizes total costs over all intervals is the first feasible one ($q_0 < n_i\sum Q_i(n_i') < q_{0(j-1)}$) starting with $j = J/0$ and working backwards.

However, as with the ADOQ algorithm, once the first feasible combination of $n_i'$ and $Q_i^*(n_i')$ is found, we still need to check the best order quantity for all intervals $> j$ due to the nature of the all-units discount schedule. There is only one decision variable ($Q_i$) with the ADOQ method, so given the convexity of the cost function, the price breakpoint is the only point that needs to be checked for intervals coming after the one containing the first feasible EOQ. However, there are (effectively) two decision variables in the multiechelon case, one of which is an integer. Therefore, it is not a priori obvious which combinations of $n_i'$ and $Q_i^*(n_i')$ would minimize $\Phi_i(n, Q_i)$. However, because $Q_i^*(n_i')$ is not feasible. As Proposition 1 below shows, the costs for two different order quantity combinations will have to be checked for each quantity interval $\geq j$. First we need the following two remarks.

**Remark 1.** The quantity $n\sum Q_i(n)$ is increasing in $n$.

**Proof.** See Appendix A.

**Remark 2.** The function $\Phi_i^*(n)$ is nondecreasing in integer $n$ starting with $n_i'$.
Proof. The function \( a(n) \) is strictly convex in \( n \), with \( n^*_i \) equal to the integer minimizer (or one of two integer minimizers) of \( \Phi_i(n) \). The result holds since the square root function is a monotonic transformation. 

**Proposition 1.** Let \( k \) be the first interval (working backwards from interval \( j[0] \)) containing the first feasible combination of \( n^*_i \) and \( Q_i^*(n^*_i) \). For every interval \( j > k \), the feasible order quantity combinations that may provide a lower cost than \( n^*_i \) and \( Q_i^*(n^*_i) \), the only ones that need to be considered in comparison to \( n^*_i \) and \( Q_i^*(n^*_i) \) are: (A) the best combination such that \( n^*_i \sum Q_i = q_0 \), and (B) the first combination (moving forwards from \( n^*_i + 1 \) of \( n \) and \( Q_i^*(n) \)) such that \( n^*_i \sum Q_i(n) > q_0 \).

**Proof.** See Appendix A.

Before stating the solution algorithm, we must determine how to compute the order quantities referred to in Parts A and B of Proposition 1.

To calculate the combination from Part A, if \( n^*_i \sum Q_i = q_0 \), then we can substitute \( Q_i = q_0 D_i/(nD_o) \) into the cost function and then minimize it with respect to \( n \) : 
\[
\Phi_i(n, Q_i) = D_i P_i^o + (D_o/q_0) \left( S_0 + n^*_i \sum S_i^w \right) + \frac{q_0}{2nD_o} \left[ nD_o H_c V_q + \sum D_i (H_i V_q + H_i T_i^w) \right] 
\]
which is minimized by minimizing the function \( b(n) = nD_o \sum S_i^w / q_0 + q_0 \sum D_i (H_i V_q + H_i T_i^w) / (2nD_o) \) with respect to \( n \). The optimal \( n = n^*_i \), which minimizes \( b(n) \) must satisfy \( b(n^*_i) < b(n^*_i - 1) \), or
\[
n^*_i = \left\lfloor 1/2 \left( 1 + \sqrt{1 + 2q_0 \sum D_i (H_i V_q + H_i T_i^w) (D_o \sum S_i^w)^{-1}} \right) \right\rfloor.
\]

And the optimal \( Q_i = Q_i^* = D_i q_0 / (D_o n^*_i) \).

To calculate the combination for Part B of Proposition 1, we must find the smallest \( n \) such that \( n^*_i \sum Q_i(n) > q_0 \), or (after squaring both sides) 
\[
2D_i \left( \sum S_i^w \right)^2 + D_o \left( 2D_o S_0 - H_c V_q q_0 \right) n - D_o \sum D_i (H_i V_q + H_i T_i^w) \right] q_0 > 0. \text{ Define } \lfloor x \rfloor \text{ as the smallest integer } \geq x. \text{ Then the smallest integer } n \text{ that satisfies the inequality (where the subscript } I \text{ stands for “interior”): }
\]
\[
n_I^* = \max \left\{ j, \left( \frac{2D_o S_0 - H_c V_q q_0}{2} + 8 \left( \sum S_i^w \right) \left( \sum D_i (H_i V_q + H_i T_i^w) \right) q_0^2 \right) / (4D_o \sum S_i^w) \right\}.
\]

And the optimal \( Q_i = Q_i^*(n_I^*) \).

Algorithm 1 finds the optimal combination of \( n^* \) and \( Q_i^* \) that minimizes the total system purchasing and inventory cost for the all-units case when using a single-cycle policy.

**Algorithm 1. Single-Cycle Solution Method for Scenario 3, All-Units Discounts**

Step 1. Set \( j = 0 \).

Step 2. Compute \( n^*_i \) [from (3)] and \( Q_i^*(n^*_i) \) [from (2)].

Step 3. If \( n^*_i Q_i^*(n_i^*) > q_0 \), set \( n = n^*_i, Q_i = Q_i^*(n^*_i) \). Otherwise, set \( \Phi = \Phi_i(n^*_i, Q_i^*(n^*_i)) \) [from (1)], \( j = j + 1 \), and go to Step 2.

Step 4. Set \( j = j - 1 \) and go to Step 2.

Step 5. If \( j = 0 \) then go to Step 7.

Step 6. For \( j = 1 \) to \( j[0] \), compute \( n^*_i \) [from (4)], and set \( Q_i = D_i q_0 / (D_o n^*_i) \).

If \( \Phi_i(n^*_i, Q_i^*_i) < \Phi \), set \( \Phi_i = \Phi_i(n^*_i, Q_i^*_i) \), \( n = n^*_i \), \( Q_i = Q_i^*_i \), and \( j = j + 1 \).

Step 7. For \( i = 2 \) to \( I \), set \( Q_i = D_i Q_i^*_i / D_o, \text{ Stop} \).

3.1.3.2. Step 2: Tailored aggregation refinement of the single-cycle policy. Algorithm 1 provides lot sizes for the central warehouse and each site, where each site is assumed to have the same order frequency. Step 2 allows sites to modify their order frequencies into integer multiples \( m_i \), evenly spread forward the warehouse’s cycle length (the single-cycle solution sets all \( m_i = n \)). In other words, from Algorithm 1, the central warehouse will order \( Q_c = n^*_c Q_c \) units every \( Q_c / D_o \) years. The current solution has site \( i \) ordering \( Q_i^* \) units every \( \sum Q_i / D_o \). Given that site \( i \) must order a total of \( n^* Q_i^* \) units during the warehouse’s cycle of length \( Q_c / D_o \), our refinement procedure searches for the best multiplier \( m_i \) producing a lot size of \( n^* Q_i^* / m_i \) that minimizes the field site’s (incremental) setup and holding costs: 
\[
\gamma(m_i) = \left( m_i D_i / n^* Q_i^* \right) S_i^w + \left( n^* Q_i^* / 2m_i \right) \left[ H_i V_q + H_i T_i^w \right].
\]
The optimal \( m_i = m_i^* \) that minimizes \( \gamma(m_i) \) must satisfy \( \gamma(m_i) \leq \gamma(m_i + 1) \) and \( \gamma(m_i) < \gamma(m_i - 1) \), or
\[
m_i^* = \left\lfloor 1/2 \left( 1 + \sqrt{1 + 2 \left( n^* Q_i^* / 2 \right) \left[ H_i V_q + H_i T_i^w \right] / \left( D_o S_i^w \right)^{-1}} \right) \right\rfloor.
\]
Each new optimal \( Q_i^* \) equals \( n^* Q_i^* / m_i^* \), for \( i = 1, 2, \ldots, I \). Finally, after completing Algorithm 1 and the refinement heuristic in Step 2, the total purchasing and inventory costs for the system are:
\[
\Omega' = D_o P_i^o + D_o S_0 / Q_o + Q_o H_c V_q / 2 + \sum D_i S_i^w / Q_i + Q_i \left( H_i V_q + H_i T_i^w \right) / 2.
\]
3.4. Scenario 4 – Centralized purchasing with local distribution, all-units discounts

We assume a single-cycle policy for this case so that the CPO has a stationary policy and receives the same quantity discount (if any) every order. There should be more opportunities for quantity discounts, but worse individual inventory optimization when compared to Scenario 2. In this case, the CPO has purchasing and setup costs, while the field sites have holding costs. The problem of minimizing the total system purchasing and inventory costs when $q_0 < \sum Q_i < q_{(j+1)}$ is:

$$\min_{Q_i} \ D_0 P_{q_0} + \sum_{i=1}^{n} D_i (S_i + C_i - P_i Q_i) + \sum_{i=1}^{n} Q_i (H_i (P_i + T_i))$$

subject to $Q_i/D_i = q_i/D_i$ for $i = 2, 3, \ldots, I$.

The objective function has $I$ decision variables. By substituting each constraint into the objective function we are left with the following unconstrained total system purchasing and inventory cost function in one variable $Q_i : \Phi_i (Q_i) = D_0 P_{q_0} + D_i S_i + \frac{Q_i}{2} H_i (P_i + T_i)$.

For interval $j$, $\Phi_j (Q_j)$ is minimized by setting $Q_j = 2 (S_j + C_j - P_j Q_j) D_j H_j (P_j + T_j)^{1/2}$. In addition, to set $Q_i$ equal to the endpoint $q_i$, $Q_i = D_i q_i/D_i$. With these formulas, for each discount interval $j$, define $Z_j = D_0 P_{q_0} + \frac{(\sum D_i H_i (P_i + T_i)^{1/2})}{(2 D_i)} \text{Max}(D_i q_i/D_i, Q_i) + [(D_0 S_0)/\text{Max}(D_0 q_0/D_0, Q_0)])].$ Then the optimal lot size at site $i$ is $Q_i^* = \text{Min}(D_i q_i/D_i, Q_i)$. And for $i = 2$ to $I$, $Q_i^* = D_i q_i/D_i$. The total purchasing and inventory costs for this policy are: $\Omega = \sum Z_i$.

3.2. Incremental quantity discounts

3.2.1. Scenario 1 – Complete decentralization, incremental discounts

The method for determining the optimal order quantity when faced with an incremental quantity discount schedule found in most textbooks (e.g., Chopra and Meindl, 2007) can be rewritten more compactly and streamlined to eliminate steps. We refer to the following technique as the IDIQ (incremental discount order quantity) method, which modifies the streamlined method provided in Hu and Munson (2002) to fit the model in this paper.

For each site $i$ and discount interval $j$, define $Z_i = D_0 P_{q_0} + \frac{(\sum D_i H_i (P_i + T_i)^{1/2})}{(2 D_i)} \text{Max}(D_i q_i/D_i, Q_i) + [(D_0 S_0)/\text{Max}(D_0 q_0/D_0, Q_0)])].$ Then the optimal lot size at site $i$ is $Q_i^* = \text{Min}(D_i q_i/D_i, Q_i)$. And the purchasing and inventory costs for site $i$ are: $\Omega^* = Z_i$.

3.2.2. Scenario 2 – Centralized pricing with decentralized purchasing, incremental discounts

In this case, each site purchases from the CPO's price schedule, so using the IDIQ method, $Z_j = D_0 P_{q_0} + \frac{(\sum D_i H_i (P_i + T_i)^{1/2})}{(2 D_i)} \text{Max}(D_i q_i/D_i, Q_i) + [(D_0 S_0)/\text{Max}(D_0 q_0/D_0, Q_0)])].$ Then the optimal lot size at site $i$ is $Q_i^* = \text{Min}(D_i q_i/D_i, Q_i)$. And the purchasing and inventory costs for site $i$ are: $\Omega^* = Z_i$.

3.2.3. Scenario 3 – Centralized purchasing and warehousing, incremental discounts

As with the all-units case, we will first assume that all of the field sites have the same delivery frequency $Q_i/D_i$, and the warehouse orders an integer multiple $n$ of $\sum Q_i$. In Step 2, we will then apply the same type of refinement heuristic to allow the field sites to incur different delivery frequencies. Even with a single-cycle policy, the solution technique for the incremental case will be more involved than the IDIQ method. Specifically, for each discount interval, it will be necessary to search over a range of values of $n$.

3.2.3.1. Step 1: Development of the single-cycle policy. The quantity discount applies to the amount $n \sum Q_i$ purchased by the CPO. The total purchasing cost of $n \sum Q_i$, units, for $q_0 < n \sum Q_i < q_{(j+1)}$, equals $C_0 + P_0 (n \sum Q_i - q_0)$. The average purchasing cost per unit equals $C_0/(n \sum Q_i)$. The average purchasing cost per unit equals $C_0 + P_0 (n \sum Q_i - q_0)$. The average purchasing cost per unit equals $C_0/(n \sum Q_i) + P_0 (n \sum Q_i - q_0)/(n \sum Q_i)$.

Therefore, the purchasing and base inventory costs for the system are $D_0 P_{q_0} + \frac{(\sum D_i H_i (P_i + T_i)^{1/2})}{(2 D_i)} \sum D_i S_i + \frac{(\sum D_i H_i (P_i + T_i)^{1/2})}{2} \sum D_i (C_i - P_i q_i)/(n \sum Q_i) + (\sum D_i H_i (P_i + T_i)^{1/2})/2$. The incremental inventory costs for site $i$ are $(D_i q_i S_i)^{1/2} + (C_i - P_i q_i)/(n \sum Q_i) + (\sum D_i H_i (P_i + T_i)^{1/2})/2$. For notational convenience, we define $V_i = P_i (n \sum Q_i)^{1/2}$. Then, combining the base system and incremental site costs, the problem of minimizing the total system purchasing and inventory costs when $q_0 < n \sum Q_i < q_{(j+1)}$, can be written as:

$$\min_{Q_i} \ D_0 P_{q_0} + \sum_{i=1}^{n} D_i (S_i + C_i - P_i Q_i) + \sum_{i=1}^{n} Q_i (H_i (P_i + T_i)) + \sum_{i=1}^{n} \left[ D_i S_i Q_i + \frac{Q_i}{2} H_i (V_i + \frac{C_i - P_i q_i}{n \sum Q_i}) + H_i T_i^i \right]$$

subject to $Q_i/D_i = q_i/D_i$ for $i = 2, 3, \ldots, I$.

The objective function has $I + 1$ decision variables. By substituting each constraint into the objective function we are left with the following unconstrained total system purchasing and inventory cost function in two variables, $n$ and $Q_1 : \Phi_1 (n, Q_1) = D_0 P_{q_0} + \sum_{i=1}^{n} D_i (S_i + C_i - P_i q_i)/(n \sum Q_i) + (n \sum Q_i)/(n \sum Q_i) + \sum_{i=1}^{n} D_i (H_i V_i + H_i T_i^i)/2 + \frac{Q_i}{2} (n \sum Q_i)/(n \sum Q_i) + \sum_{i=1}^{n} D_i (H_i V_i + H_i T_i^i)/2$. Setting $\partial \Phi_1 (n, Q_1)/\partial Q_1 = 0$ yields the cost minimizing $Q_1$ for a given $n$:

$$Q_1^*(n) = \left\{ \frac{2 D_0 P_{q_0} + \sum_{i=1}^{n} D_i (S_i + C_i - P_i q_i)/(n \sum Q_i) + (n \sum Q_i)/(n \sum Q_i) + \sum_{i=1}^{n} D_i (H_i V_i + H_i T_i^i)/2 + \frac{Q_i}{2} (n \sum Q_i)/(n \sum Q_i) + \sum_{i=1}^{n} D_i (H_i V_i + H_i T_i^i)/2}{2 n \sum Q_i} \right\}$$
Proof. Same as the proof of Remark 1 with $q_j$ equaling:

$$
\Phi_j(n) = D_0 P_{q_0} + \left[2 \left( (S_0 + C_0 - P'_{q_0} q_0) + n \sum_i S_i \right) \right] \left[ n D_0 H_0 V_{q_0} + \sum_i D_i (H_i V_{q_0} + H_i T_{ij}^{Q_i}) \right] / n^{1/2} \\
+ (C_0 - P'_{q_0} q_0) / (n D_0).
$$

(7)

Unfortunately, $\Phi_j(n)$ is not necessarily convex in $n$ (unless $H_i'(n) = 0 \forall i$). The cost function may have up to three positive critical points. Thus, unlike the all-units case, we cannot derive a formula for the cost minimizing $n$ for each function $\Phi_j(n)$, $j = 0, 1, \ldots, j[0]$. Instead, we propose a set of bounds to use in a search procedure over $n$ for each discount interval.

The total cost function $\Phi(n, Q_4)$ is composed of $\Phi_j(n)$ for $q_0 < \sum_i Q_i \leq q_{i+1}$, $j = 0, 1, \ldots, j[0]$. The following remark is used in the proof of Theorem 1.

Remark 3. For a given $n$, $\Phi(n, Q_4)$ is continuous in positive $Q_1$.


It is known from the development of the IDQ algorithm that, unlike the all-units case, for the basic ordering problem with incremental discount the buyer should never order a quantity equal to one of the price breakpoints in the quantity discount schedule (Hadley and Whitin, 1963). Theorem 1 below shows that this property also holds for the single-cycle multiechelon ordering problem examined here.

Theorem 1. The warehouse should never order a quantity equal to one of the price breakpoints in the cost schedule, i.e. the optimal $n \sum_i Q_i \neq q_j$, for $j = 1, 2, \ldots, j[0]$.

Proof. Similar in structure to the basic ordering problem proof found in Hadley and Whitin (1963). For details, see Hu (2004).

Theorem 1 is important because it allows us to limit the search over values of $n$ to those $n$ such that (for interval $j$) $q_j < \sum_i Q_i(n) \leq q_{j+1}$. This result follows because $\Phi_j(n, Q_4)$ is convex in $Q_1$. First, if $\sum_i Q_i(n) > q_{j+1}$, then the best feasible value of $Q_1$ for that $n$ is $D_j(q_{j+1})/(n D_0)$, which cannot be optimal by Theorem 1. Second, if $\sum_i Q_i(n) < q_j$, then the best feasible value of $Q_1$ for that $n$ is $D_j(q_j + 1)/(n D_0)$, which has a larger cost than the cost at $D_j q_j/(n D_0)$, which cannot be optimal by Theorem 1.

Next we develop bounds on $n$ for each quantity interval. First we need Remark 4.

Remark 4. The quantity $n \sum_i Q_i(n)$ is increasing in $n$.

Proof. Same as the proof of Remark 1 with $P'_{q_0}$ replaced by $P'_{q_1}$ and $S_0$ replaced by $\left( S_0 + C_0 - P'_{q_0} q_0 \right)$. □

From Remark 4, the lower bound on $n$ for interval $j$, $j = 0, 1, \ldots, j[0]$, should be the first $n$ such that $\sum_i Q_i(n) > q_j$. The derivation is very similar to the all-units case (5), resulting in a lower bound for interval $j$ of:

$$
n_j^l = \max \left\{ 1, \frac{-b + b^2 - 4ac}{2a} \right\}, \quad \text{where}
\begin{align*}
a &= 2 D_0 \sum_i S_i, \\
b &= 2 D_0 \left( S_0 + C_0 - P'_{q_0} q_0 \right) - H_0 V_{q_0}^2 q_0^2, \\
c &= - \sum_i D_i \left( H_i V_{q_0} + H_i T_{ij}^{Q_i} \right) q_0^2 / D_0
\end{align*}
$$

(8)

For the upper bound on $n$ for interval $j$, $j = 0, 1, \ldots, j[0]$, we need the largest $n$ such that $\sum_i Q_i(n) \leq q_{j+1}$, or:

$$
n_j^c = \max \left\{ 1, \frac{-b + b^2 - 4ac}{2a} \right\}, \quad \text{where}
\begin{align*}
a &= 2 D_0 \sum_i S_i, \\
b &= 2 D_0 \left( S_0 + C_0 - P'_{q_0} q_0 \right) - H_0 V_{q_0}^2 q_{j+1}^2, \\
c &= - \sum_i D_i \left( H_i V_{q_0} + H_i T_{ij}^{Q_i} \right) q_{j+1}^2 / D_0
\end{align*}
$$

(9)

Since $q_{j+1}(j[0]+1) = \infty$, we need to compute the upper bound for the last interval differently. Lemma 1 describes this computation.

Lemma 1. An upper bound $n_{j[0]}^b$ is the smallest integer $n$ that satisfies $An^2 > 2D[An + B + C]^{1/2} + B$; where $A = 2V_{q_{j[0]}} D_j H_j \sum_i S_i$, $B = 2 \left( S_0 + C_0 - P'_{q_{j[0]}} q_{j[0]} \right) \sum_i D_i \left( H_i V_{q_{j[0]}} + H_i T_{ij}^{Q_i} \right)$, $C = 2 \left( V_{q_{j[0]}} D_j H_j \left( S_0 + C_0 - P'_{q_{j[0]}} q_{j[0]} \right) + \sum_i S_i \right)$ and $D = \left( C_{q_{j[0]}} - P'_{q_{j[0]}} q_{j[0]} \right) \sum_i (D_i H_i)' / (2D_0)$.

Proof. See Appendix A.

Algorithm 2 finds the optimal combination of $n$ and $Q_i$ that minimizes the total system purchasing and inventory cost for the incremental discount case when using a single-cycle policy.


Step 1. Set $\Phi^* = \infty$.

Step 2. For $j = 0$ to $j[0]$

Step 3. Calculate $n_j^l$ from (8) and $n_j^c$ from (9) for $j = 0, \ldots, j[0] - 1$; and from Lemma 1 for $j = j[0]$.

Step 4. If $n_j^c > n_j^l$, goto Step 2.

Step 5. For $n = n_j^l$ to $n_j^c$

Step 6. For $i = 0$ to $n_j^c$

Step 7. For $h = 0$ to $j[0]$

Step 8. For $j = 0$ to $j[0]$
Step 6. Calculate $\Phi = \Phi'(n)$ [from (7)].
Step 7. If $\Phi < \Phi'$, set $\Phi = \Phi'$, $n = n$, $Q_i = Q_i'(n)$ [from (6)], and $j' = j$.
Step 8. For $i = 2$ to $I$, set $Q_i = D(Q_i/D_i)$. Stop.

3.2.3.2. Step 2: Tailored aggregation refinement of the single-cycle policy. As in the all-units case, we introduce a second step that allows sites to modify their order frequencies into integer multiples $m_i$ evenly spread throughout the warehouse’s cycle length (the single-cycle solution sets $m = n$). Given that site $i$ must order a total of $n' Q_i^n$ units during the warehouse’s cycle of length $Q_i/D_i$, our refinement procedure searches for the best multiplier $m_i$ producing a lot size of $n' Q_i^n/m_i$ that minimizes the field site’s (incremental) setup and holding costs:

$$
\gamma(m_i) = (m_i D_i/n' Q_i^n)^{2/3} + (n' Q_i^n/2m_i) \left\{ H_i \left( V_{i0'} + (C_{0i} - P_{0i} q_{0i})/Q_o \right) + H_i T_i^{W} \right\}.
$$

The optimal $m_i = m_i'$ that minimizes $\gamma(m_i)$ must satisfy $\gamma(m_i') < \gamma(m_i' + 1)$ and $\gamma(m_i') < \gamma(m_i' - 1)$, or

$$
m_i' = \left[ 1/2 \left( 1 + \sqrt{1 + 4 \left\{ \frac{H_i \left( V_{i0'} + (C_{0i} - P_{0i} q_{0i})/Q_o \right) + H_i T_i^{W}}{D_i S_i'} \right\} } \right) \right].
$$

And the new optimal $Q_i'$ equals $n' Q_i^n/m_i'$, for $i = 1, 2, \ldots, I$ Finally, after completing Algorithm 2 and the refinement heuristic in Step 2, the total purchasing and inventory costs for the system are:

$$
\Omega' = D_i p_{0i} + D_i \left( S_i + C_{0i} - P_{0i} q_{0i} \right)/Q_o + Q_i H_i V_{i0'}/2 + \left\{ C_{0i} - P_{0i} q_{0i} \right\} H_i/2 + \sum_i \left\{ D_i S_i'/Q_i + Q_i \left\{ H_i \left( V_{i0'} + (C_{0i} - P_{0i} q_{0i})/Q_o \right) + H_i T_i^{W} \right\} / 2 \right\}.
$$

3.2.4. Scenario 4 – Centralized purchasing with local distribution, incremental discounts

As with the all-units case, we assume a single-cycle policy for this scenario. The problem of minimizing the total system purchasing and inventory costs when $q_{0i} < \sum_j Q_j < q_{0i+1}$, is:

$$
\min_{Q_{0i}, Z_i} D_i p_{0i} + D_i \sum_{Q_i} \left( S_i + C_{0i} - P_{0i} q_{0i} \right)/Q_o + \sum_i \left( \frac{Q_i}{2} H_i \left( P_{0i} + C_{0i} - P_{0i} q_{0i} + T_i^{W} \right) \right)
$$

Subject to $Q_i/D_i = Q_i$, for $i = 2, 3, \ldots, I$.

The objective function has $I$ decision variables. By substituting each constraint into the objective function we are left with the following unconstrained total system purchasing and inventory cost function in one variable $Q_i$: $\Phi(Q_i) = D_i p_{0i} + D_i \left( S_i + C_{0i} - P_{0i} q_{0i} \right)/Q_o + Q_i \left\{ D_i H_i \left( P_{0i} + T_i^{W} \right) \right\} / (2D_i) + \sum_i \left\{ D_i H_i \left( C_{0i} - P_{0i} q_{0i} \right) / (2D_i) \right\} + \left\{ 2 \left( S_i + C_{0i} - P_{0i} q_{0i} \right) \sum_i \left\{ D_i H_i \left( P_{0i} + T_i^{W} \right) \right\} \right\}^{1/2}$. Then the optimal lot size at site 1 is $Q_1' = \left\{ 2D_i \left( S_i + C_{0i} - P_{0i} q_{0i} \right) / \left\{ D_i H_i \left( P_{0i} + T_i^{W} \right) \right\} \right\}^{1/2}$, where $j' = \operatorname{Argmin}_{j=0,1,\ldots,I} Z_j$. And for $i = 2$ to $I$, $Q_i' = D_i Q_{i-1}' / D_i$. The total purchasing and inventory costs for this policy are: $\Omega = Z_i$.

4. Numerical analysis

4.1. Example

In this section we present a “base case” numerical example to observe how purchasing quantities and costs may vary depending upon the centralization scenario chosen. The parameter values are kept the same for both the all-units and incremental discount cases. The example includes 5 field sites with annual demands of 500; 5000; 5000; 10,000; and 10,000 units, respectively. All discount schedules have price breakpoints at 4000; 8000; 20,000; and 50,000 units. The associated prices for the field sites are $40.00, 37.50, 35.00, 32.50, and 30.00, while the associated prices for the CPO are $36.00, $34.00, $32.00, $30.00, and $28.00. The other parameters are: $S_i = 60$ for $i = 1, \ldots, 5$, $S_i = 80$ for $i = 1, \ldots, 5$, $S_i = 80$ for $i = 1, \ldots, 5$, $H_i = 25\%$ for $i = 1, \ldots, 5$, $H_i = 20\%$, $T_i^{W} = 2$ for $i = 0, \ldots, 5$, $T_i^{W} = 3$ for $i \neq j$; and $T_i^{W} = 81 for $i = 1, \ldots, 5$.

Table 1 presents the results of the base case example. For these parameter values, Scenario 3 was best for the all-units schedule, while Scenario 4 was best for the incremental schedule. Scenario 2’s order sizes were larger than those of Scenario 1 (and the purchasing and inventory costs were lower) due to lower holding costs stemming from buying from the cheaper price schedule. The lower purchasing and inventory costs were somewhat offset by higher transportation costs from using the CPO’s vendors. Furthermore, the “all-units” form of pricing had a significant impact on all four scenarios. Sites 4 and 5 ordered much larger quantities than their respective EOQs would suggest, while the CPO ordered 20,000 units under Scenarios 3 and 4 to take advantage of the significant price break at that level. For the all-units case, the ability to store many of the 20,000 units at the cheaper warehouse instead of at the individual sites produced lower inventory costs for Scenario 3 than Scenario 4. Under the incremental case, price breaks are less valuable, and in this example, the order quantities were accordingly smaller than their all-units counterparts (except in the cases where actual EOQs at the highest price were being ordered). The CPO still ordered over 10,000 units under Scenario 4, but under Scenario 3 the order size was less than 1000 units. In that case, the purchase price savings under the incremental schedule did not justify the extra setup costs that central warehousing would impose.
4.2. Test of the single-cycle policy and refinement procedure for Scenario 3

For both the all-units and incremental discounts, we have introduced a tailored aggregation refinement step in the lot sizing for Scenario 3 (centralized purchasing and warehousing). We ran a computational study to examine the effectiveness of the refinement procedure. We also computed a lower bound on costs to test the effectiveness of both the single-cycle policy and the refinement procedure compared to this bound.

The lower bound was computed by “decoupling” all of the field sites and their relationship with the central warehouse. The purchasing costs and warehouse holding and setup costs were set equal to a solution that optimizes costs for the warehouse with no regard to inventory effects at the field sites. For the field site costs, we took the conservative approach of computing the optimal holding and setup costs at each of the field sites, assuming that they could order their individual EOQs using the cheapest price from the CPO’s schedule, without introducing any inventory coordination constraints with either the other sites or the warehouse.

We used the base case example as a starting point and then made order-of-magnitude changes to certain parameters and ran a full-factorial experiment. The large differences in parameter values were introduced to attempt to identify any main effects where the policies were performing particularly poorly, and if so, to then investigate them further.

Three purchase price schedules for the CPO were used: (1) the base case, (2) 10 times the base case, and (3) 0.1 times the base case. For each of these schedules, dollar discounts were also doubled to provide another three price schedules. Field site holding costs varied randomly by site, and had two different levels: (1) UNIFORM[20%, 30%] and (2) UNIFORM[20%, 60%]. Demand also varied randomly by field site via three different levels: (1) INTEGER UNIFORM[50, 150], (2) INTEGER UNIFORM[50, 1500], and (3) INTEGER UNIFORM[50, 15000]. Finally, each of these 36 experimental settings was implemented for a system with 5 field sites and for another with 50 field sites, resulting in 72 total experiments. Since the refinement procedure is the same in spirit for both all-units and incremental discounts, we only analyzed the all-units case.

When comparing via percentages, the denominator clearly impacts the size of the percent change. To try to avoid too much dilution in the denominator, we ignored costs unrelated to the lot sizing decision (transportation and “base purchasing cost”). As the lot sizing decision may affect quantity discounts, we included in our costs the “incremental purchasing cost,” which equals total annual purchasing cost minus total annual purchasing cost if the best possible purchasing price had been used. We added holding and setup costs at the warehouse and all field sites to the incremental purchasing cost to derive our total relevant costs for this analysis.

Table 2 shows the summary results of the 72 experiments. Jumping from 5 to 50 field sites did not seem to significantly affect the results. The single-cycle policy actually performed quite well on its own, with only a 2.32% cost penalty over the lower bound. The penalty exceeded 5% in fewer than 10% of the trials. The refinement procedure did help, averaging a 0.78% savings over the single-cycle policy.

Table 2
Numerical test of the Scenario 3 single-cycle policy and refinement procedure.

<table>
<thead>
<tr>
<th>Summary measure</th>
<th>Savings(^a) using refinement procedure</th>
<th>Penalty(^b) over lower bound</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Single-cycle policy</td>
</tr>
<tr>
<td>5 field sites (36 samples)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Average</td>
<td>0.30%</td>
<td>2.11%</td>
</tr>
<tr>
<td>Maximum</td>
<td>2.18%</td>
<td>6.54%</td>
</tr>
<tr>
<td>Minimum</td>
<td>0.00%</td>
<td>0.04%</td>
</tr>
<tr>
<td>50 field sites (36 samples)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Average</td>
<td>1.26%</td>
<td>2.53%</td>
</tr>
<tr>
<td>Maximum</td>
<td>3.90%</td>
<td>10.51%</td>
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<tr>
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<tr>
<td>Maximum</td>
<td>3.90%</td>
<td>10.51%</td>
</tr>
<tr>
<td>Minimum</td>
<td>0.00%</td>
<td>0.04%</td>
</tr>
</tbody>
</table>

\(^a\) Cost basis includes the holding and setup costs at the warehouse and all field sites, plus the “incremental purchasing cost”, defined as total purchase cost minus total purchase cost if the lowest per-unit price had been used.

\(^b\) CPO’s lot size = 20,000 units (\(\alpha^* = 1\)); single-cycle policy cost $230 more and had order sizes of 29.81, 298.06, and 596.13.

\(^c\) CPO’s lot size = 968.74 units (\(\alpha^* = 1\)); single-cycle policy was the same.
Comparing to the lower bound, the refinement procedure averaged a 1.52% penalty, exceeding 5% in only 5 out of the 72 trials. The maximum penalty vs. the lower bound with the refinement procedure for this experiment was only 7.23%.

These results are encouraging overall. When quantity discounts exist, the single-cycle policy seems to perform quite well when compared with a conservative lower bound. The results in this experiment are in the guaranteed performance range of the more complicated "Q-optimal" and "optimal power-of-two" policies proposed for multiechelon systems with no quantity discounts. Furthermore, our refinement procedure appended to a single-cycle policy improves performance further, and it has the attractive feature of being much easier to implement than an "optimal power-of-two" policy would be in the quantity discount environment. Thus, not only is this solution approach potentially faster than more complicated ones might be, but it should also be easier to incorporate into more comprehensive centralized purchasing cost models that might examine additional issues such as safety stock, pipeline inventory, supplier management, etc.

5. Conclusion

In this paper we have provided methods to compute lot sizes under four different purchasing frameworks when offered prices from either all-units or incremental quantity discount schedules. The four frameworks represent some of the most common configurations of centralized versus decentralized pricing, purchasing, warehousing, and delivery. The methods provided here can aid managers in lot-sizing decisions for a given purchasing configuration, or they can be combined with other cost considerations to assist in making informed decisions about the proper degree of purchasing centralization itself. Future researchers, in fact, may wish to incorporate our lot-sizing models into more comprehensive centralized purchasing cost models.

The numerical example in Section 4.1 illustrates the sometimes very different purchasing plans that may result from different centralized purchasing configurations, with the more centralized plans leading to larger purchase quantities while a central warehouse may allow field sites to hold little inventory yet still attain a discounted purchase price. Proper choice of purchasing configuration may significantly reduce costs—in our all-units example, the Scenario 3 cost was 15% lower than the Scenario 1 cost. The example further illustrates that under some conditions, even with the same price breakpoints and prices, an incremental discount schedule may produce a reduced incentive to order larger quantities than the same all-units schedule would. For instance, the CPO's order size under the all-units schedule was more than 20 times larger than its order size under the incremental schedule in our example.

Finally, the numerical experiments in Section 4.2 suggest that our assumption of a single-cycle policy, particularly when combined with our "refinement step," seems to perform quite well when compared with a conservative lower bound under the existence of quantity discounts. The single-cycle policy averaged only a 2.32% cost penalty over the conservative lower bound, while the introduction of the refinement step lowered that average penalty to 1.52%. The simplicity and robustness of our policies should make them attractive to purchasing managers, especially since other policies such as "power-of-two" are developed under no quantity discounts and are in much more complex step lowered that average penalty to 1.52%. The simplicity and robustness of our policies should make them attractive to purchasing managers, especially since other policies such as "power-of-two" are developed under no quantity discounts and are in much more complex

Appendix A

Proof of Remark 1. Define \( c(n) = \left( nS_0 + n^2 \sum_{j=0}^{C_j} q_j \right) / \left( nD_0H_0V_0 + \sum_i D_i \left( H_iV_0 + H_i^W \right) \right) \). Then \( n\sum_{j=0}^{C_j} q_j(n) = D_0 \sqrt{c(n)} \), so clearly \( n\sum_{j=0}^{C_j} q_j \) is increasing in \( n \) if \( c(n) \) is increasing in \( n \). Taking the derivative \( c'(n): \)

\[
c'(n) = \frac{(S_0 + 2n\sum_{j=0}^{C_j} q_j) \sum_i D_i \left( H_iV_0 + H_i^W \right) + n^2 D_0H_0V_0 \sum_{j=0}^{C_j} q_j \left( nD_0H_0V_0 + \sum_i D_i \left( H_iV_0 + H_i^W \right) \right)^2 > 0, \]

which proves the remark. \( \square \)

Proof of Proposition 1. First note that for a given \( n \), \( \Phi_j(n, Q_j) \) is convex in \( Q_j \). Let \( q_j \) denote the value of \( n \) that belongs to the cheapest feasible combination in interval \( j \). If \( q_j \sum_{j=0}^{C_j} q_j(q_j) < q_j \), then, due to the convexity of \( \Phi_j(n, Q_j) \), the value of \( Q_j \) that makes the order combination feasible in the cheapest manner is \( Q_j = q_jD_1/nD_0 \), establishing Part A of the proposition.

If \( q_j < q_j \sum_{j=0}^{C_j} q_j(q_j) < q_j(q_j+1) \), then the optimal order quantity for interval \( j \) lies in the interior of the interval. In this case, the order quantity does not lie at the endpoint \( q_j \), so we need to ensure that we search for that point in case it exists. From Remark 1, we know that the order quantity \( n\sum_{j=0}^{C_j} q_j \) approaches and eventually passes \( q_j \) as \( n \) increases starting with \( n_j \). Let \( q_j \) denote the value of \( n \) that first causes \( n\sum_{j=0}^{C_j} q_j \) to equal or surpass \( q_j \). As noted above, until that point is reached, the best order quantity given \( n \) will be at an endpoint. From Remark 2, the cost cannot improve for values of \( n \) larger than \( q_j \). Therefore, combinations including values of \( n \) larger than \( q_j \) do not need to be considered. Finally, if \( q_j \sum_{j=0}^{C_j} q_j(q_j) > q_j(q_j+1) \), then the combination of \( q_j \) and \( Q_j(q_j) \) will produce a cheaper cost in a higher interval than \( j \) (since \( \Phi_j(n) \) is decreasing in \( j \)), implying that the optimal order quantity does not lie in \( j \) and it will do no harm to compare \( \Phi_j(n_j, Q_j(q_j)) \) to the costs from other intervals. \( \square \)

Proof of Lemma 1. With \( A, B, C, \) and \( D \) as defined in the Lemma, \( \Phi_{y_{0j}}(n) \) (Eq. (7)) can be written as:

\[
\Phi_{y_{0j}}(n) = D_0P_{y_{0j}}^j + \sqrt{An + B/n + C} + (C_{0j} - P_{y_{0j}}q_{y_{0j}})H_0/2 + D/n.
\]

Now the bound \( n_{y_{0j}} \) represents a value of \( n \) such that \( \Phi_{y_{0j}}(n) \) is always increasing for \( n > n_{y_{0j}} \). Thus, we need to find the range of \( n \) such that \( d\Phi_{y_{0j}}(n)/dn > 0 \), or

\[
\frac{A - B/n^2}{2\sqrt{An + B/n + C}} > D/n^2, \quad \text{or} \quad A > 2D\sqrt{An + B/n + C} + B/n^2.
\]

The proof follows because the right-hand side of the inequality is strictly decreasing in \( n \). \( \square \)
References


