Linearized numerical schemes for the Boussinesq equation

A. G. Bratsos∗, Ch. Tsitouras∗∗, and D. G. Natsis∗∗∗

1 Department of Mathematics, Technological Educational Institution (T.E.I.) of Athens, GR 122 10 Egaleo, Athens, Greece.
2 Department of Applied Sciences, Technological Educational Institution (T.E.I.) of Chalkis, GR 34 400 Psahna, Greece.

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Two different linearized schemes are applied to a parametric finite-difference scheme concerning the numerical solution of the Boussinesq equation. At the first linearized scheme the nonlinear term of the equation is substituted by an appropriate value, while at the second scheme we use Taylor’s expansion. Both schemes are analyzed for local truncation error, stability and convergence. The results of the experiments are examined for their accuracy for the single and the double-soliton waves to known from the bibliography numerical schemes.

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1 Introduction

In the past few years interest has increased in the solution of partial differential equations governing nonlinear waves in dispersive media. As a result several texts and numerous research papers have been devoted to the subject. In parallel with the mathematical treatment a considerable literature has grown dealing with the numerical solution of such problems. Among them a great interest has been developed for equations, which possess special solutions in the form of pulses, which retain their shapes and velocities after interaction amongst themselves. Such solutions are called solitons. Solitons are of great importance in many physical areas, as for example, in dislocation theory of crystals, plasma and fluid dynamics, magnetohydrodynamics, laser and fiber optics etc., as well as in the study of the water waves. A part of the latter one is going to be examined at the present paper.

Three different methods have been developed independently concerning the analytical solutions of soliton type equations. Ablowitz and Segur [1] implemented the inverse scattering transform method to handle the nonlinear equations of physical significance, where soliton solutions were developed, Hirota [11]-[13] constructed the \( N \)-soliton solutions of the evolution equation by reducing it to the bilinear form, Nimmo and Freeman [22]-[23] introduced an alternative formulation of the \( N \)-soliton solutions in terms of some function of the Wronskian determinant of \( N \) functions.

The archetypal equation introduced by Korteweg & de Vries (KdV) [15], which describes long gravity waves moving over stationary water, is written as

\[
\frac{\partial u}{\partial t} + 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \quad t > 0, \quad x \in \mathbb{R},
\]

Once the general method of solution of the KdV equation was obtained (see Gardner [10]), many other equations and mathematical approaches followed. In the particular field of water waves, two families of evolution equations occur: one is the KdV family of equations and the other is based on the nonlinear Schrödinger (NLS) equation

\[
\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + q |u|^2 = 0; \quad L_0 < x < L_1, \quad t > t_0.
\]

* Corresponding author: e-mail: bratsos@teiath.gr, Phone: 00 30210 5385 308, Fax: 00 30201 9630 842
** Second author footnote: e-mail: tsitoura@teihal.gr
*** Third author footnote: e-mail: natsisd@otenet.gr

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where \( \imath = \sqrt{-1} \), \( q \geq 0 \) is a real parameter and \( u = u(x, t) = v(x, t) + \imath w(x, t) \) a sufficiently differentiable complex-valued function.

The Boussinesq (BS) nonlinear equation, which belongs to the KdV family of equations, describes motions of long waves in shallow water under gravity and in a one-dimensional nonlinear lattice is given by

\[
\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + q \frac{\partial u}{\partial x} + \frac{\partial^2 (u^2)}{\partial x^2}; \quad L_0 < x < L_1, \quad t > t_0.
\]  

(1.2)

where \( u = u(x, t) \) and \( |q| = 1 \) is a real parameter. Taking \( q = -1 \) gives the Good Boussinesq or well-posed equation (GB), while taking \( q = 1 \) gives the Bad Boussinesq or ill-posed equation (BB).

Numerical methods for solving the BS equation are reported by Ambrosi [2], Bratsos [3], Daripa and Hua [7], Feng [8], Furihata [9], Ismail and Bratsos [14], Wazwaz [26] etc., while, recently, an interest has been developed (see for example Bratsos [4]-[6]) for numerical solutions of the BS equation, in which the nonlinear term is properly linearized. The numerical schemes which are going to be examined in this paper belong to the last category of methods. More precisely in section 2, after the definition of alternative boundary conditions, a parametric finite-difference scheme is applied to BS equation. Then in order to avoid solving the resulting nonlinear system, two different types of linearization were performed and the behavior of the resulting linear system at the numerical solution of BS equation was examined in section 3.

The initial displacement associated with Eq. (1.2) will be assumed to be of the form,

\[
u(x, t_0) = f(x); \quad L_0 \leq x \leq L_1,
\]  

(1.3)

with initial velocity,

\[
\frac{\partial u(x, t_0)}{\partial t} = g(x); \quad L_0 \leq x \leq L_1.
\]  

(1.4)

Following Manoranjan et al [21] for the single-soliton the function

\[
u(x, t) = q_1 \left\{ A \tanh^2 \left[ \sqrt{\frac{A}{6}} (x - ct + x_1^0) \right] + \left( b - q_1 \frac{1}{2} \right) \right\},
\]  

(1.5)

where \( A \) is the amplitude of the pulse, \( b \) an arbitrary parameter, \( x_1^0 \) the initial position of the pulse and \( c = \pm 2q_1 (b + A/3)^{1/2} \) the velocity, when \( q_1 = 1 \), satisfies the BB equation, while, when \( q_1 = -1 \) the relevant GB. Then an \( N \)-soliton solution can be obtained as

\[
u(x, t) = q_1 \left\{ \sum_{i=1}^{N} A_i \tanh^2 \left[ \sqrt{\frac{A_i}{6}} (x - c_i t + x_i^0) \right] + \left( b_i - q_1 \frac{1}{2} \right) \right\}.
\]  

(1.6)

where \( c_i = \pm 2q_1 (b_i + A_i/3)^{1/2} \); \( i = 1, 2, ..., N \) is the velocity for the \( i \)-soliton, \( A_i \) is the corresponding amplitude, \( b_i \) an arbitrary parameter and \( x_i^0 \) the initial position.

2 The numerical method

2.1 Grid and solution vector

To obtain numerical solutions the time and the space partial derivatives are replaced by central difference replacements. To this effect the region \( R = [t > t_0] \times [L_0 < x < L_1] \) with its boundary \( \partial \Omega \), consisting of the lines \( x = L_0, x = L_1 \) and \( t = t_0 \), is covered with a rectangular mesh, \( G \), of points with co-ordinates \( (x, t) = (m, n) \) with \( m = 0, 1, ..., N+1 \) and \( n = 0, 1, ..., L_1 - L_0 \). The theoretical solution of Eq. (1.2) at the typical mesh point \( (x_m, t_n) \) is \( u(x_m, t_n) \) which may be denoted, when convenient, by \( u^n \). The solution of an approximating difference scheme at the same point will be denoted by \( U^n \). For the purpose of analyzing stability, the numerical value of \( U^n \) actually obtained (subject, for instance, to computer round-off errors) will be denoted by \( \tilde{U}^n \).

Let the solution vector be

\[
U^n = U(t_n) = [U_1^n, U_2^n, ..., U_N^n]^T.
\]  

(2.1)

\( T \) denoting transpose. Obviously there are \( N \) values to be determined at each time step.
2.2 The boundary conditions

Instead of the boundary conditions used by Bratsos [3, 4] the following boundary conditions introduced by Daripa and Hua [7] are going to be used for the numerical solution of Eq. (1.2)

\[ u(L_0, t) = \frac{1}{2} [u(L_0 + h, t) + u(L_0 + 2h, t)] \quad \text{and} \quad u(L_1, t) = \frac{1}{2} [u(L_1 - h, t) + u(L_1 - 2h, t)], \]

otherwise using the notation (2.1)

\[ U^n_0 = \frac{1}{2} (U^n_1 + U^n_2) \quad \text{and} \quad U^n_{N+1} = \frac{1}{2} (U^n_{N-1} + U^n_N), \quad (2.2) \]

and,

\[ u(L_0 - h, t) = \frac{1}{2} [-3u(L_0, t) + 6u(L_0 + h, t) - u(L_0 + 2h, t) - 6u'(L_0, t) h], \]

\[ u(L_1 + h, t) = \frac{1}{2} [-3u(L_1, t) + 6u(L_1 - h, t) - u(L_1 - 2h, t) + 6u'(L_1, t) h], \]

which, when using (2.1), are written as

\[ U^n_{-1} = \frac{1}{4} \left[ 9U^n_1 - 5U^n_2 - 12h (u^n_0)' \right] \quad \text{and} \quad U^n_{N+2} = \frac{1}{4} \left[ 9U^n_N - 5U^n_{N-1} + 12h (u^n_{N+1})' \right]. \quad (2.3) \]

where \((u^n_0)'\) and \((u^n_{N+1})'\) are substituted by the relevant theoretical solution given either by Eq. (1.5) for the single or Eq. (1.6) for the double-soliton wave at time level \(t = n\ell \); \(n = 0, 1, \ldots \)

2.3 The nonlinear numerical scheme

Consider the approximation for the time derivative

\[ \frac{\partial^2 u(x, t)}{\partial t^2} = \lambda \left( \frac{\partial^2 u(x, t)}{\partial x^2} \right)^n + (1 - 2\lambda) \left( \frac{\partial^2 u(x, t)}{\partial x^2} \right)^n \]

\[ = \ell^{-2} \left[ \lambda \left( U^{n+1}_{m-1} - 2U^n_m + U^{n-1}_{m+1} \right) + (1 - 2\lambda) \left( U^{n+1}_m - 2U^n_{m+1} + U^{n-1}_m \right) \right. \]

\[ + \lambda \left( U^{n+1}_m - 2U^n_{m+1} + U^{n-1}_m \right) \right]; \quad \lambda \in [0, 1], \quad (2.4) \]

after using familiar formulae and analogous approximations for the space derivatives of the form

\[ \frac{\partial^2 u(x, t)}{\partial x^2} = \mu_1 \left( \frac{\partial^2 u(x, t)}{\partial x^2} \right)^{n+1}_m + (1 - 2\mu_1) \left( \frac{\partial^2 u(x, t)}{\partial x^2} \right)^n_m + \mu_1 \left( \frac{\partial^2 u(x, t)}{\partial x^2} \right)^{n-1}_m, \quad (2.5) \]

\[ \frac{\partial^2 u^2(x, t)}{\partial x^2} = \mu_2 \left( \frac{\partial^2 u^2(x, t)}{\partial x^2} \right)^{n+1}_m + (1 - 2\mu_2) \left( \frac{\partial^2 u^2(x, t)}{\partial x^2} \right)^n_m + \mu_2 \left( \frac{\partial^2 u^2(x, t)}{\partial x^2} \right)^{n-1}_m, \quad (2.6) \]

\[ \frac{\partial^4 u(x, t)}{\partial x^4} = \mu_3 \left( \frac{\partial^4 u(x, t)}{\partial x^4} \right)^{n+1}_m + (1 - 2\mu_3) \left( \frac{\partial^4 u(x, t)}{\partial x^4} \right)^n_m + \mu_3 \left( \frac{\partial^4 u(x, t)}{\partial x^4} \right)^{n-1}_m, \quad (2.7) \]

where \(\mu_i \in [0, 1]; \; i = 1, 2, 3.\)

Then Eqs. (2.4)-(2.7), when applied to Eq. (1.2) at each interior point of the grid \(G\), lead to the following three-time level finite-difference scheme

\[ -\mu_3 p^2 q \left( U^{n+1}_{m+2} + U^{n+1}_{m-2} \right) + (\lambda - \mu_1 r^2 + 4\mu_3 p^2 q) \left( U^{n+1}_{m+1} + U^{n+1}_{m-1} \right) \]

\[ + (1 - 2\lambda + 2\mu_1 r^2 - 6\mu_3 p^2 q) U^{n+1}_m - \mu_2 r^2 \left( (U^{n+1}_{m+1})^2 - 2 (U^{n+1}_m)^2 + (U^{n+1}_{m-1})^2 \right) \]

\[ = (1 - 2\mu_3) p^2 q \left( U^{n+1}_{m+2} + U^{n+1}_{m-2} \right) + 2\lambda + (1 - 2\mu_1) r^2 - 4 (1 - 2\mu_3) p^2 q \left( U^{n+1}_{m+1} + U^{n+1}_{m-1} \right) \]
For the stability analysis of the scheme which tends to zero as \( n \to \infty \), \( \lambda = \mu_1 r^2 + 4 \mu_3 p^2 q \), \( U_{m-1}^{n+1} - 2 U_m^{n+1} + U_{m+1}^{n+1} \) is written as

\[
+2 \left( \frac{1}{2} \left( \lambda - \mu_1 r^2 + 4 \mu_3 p^2 q \right) U_{m-1}^{n+1} \right) + \left( 1 - 2 \mu_1 r^2 + 6 \mu_3 p^2 q \right) U_{m-1}^{n+1} + \mu_2 r^2 \left( \left( U_{m-1}^{n+1} \right)^2 - 2 (U_m^{n+1})^2 + (U_{m+1}^{n+1})^2 \right),
\]

(2.8)

for \( m = 1, 2, \ldots, N \), which forms the following nonlinear system

\[
F(U^{n+1}) = 0.
\]

(2.9)

### 2.3.1 Local truncation error

The principal part of the local truncation error of the numerical scheme arising from Eq. (2.8) is

\[
L(x, t) = \ell^2 \left[ \frac{1}{12} \frac{\partial^4}{\partial x^4} \left( \frac{\partial^4}{\partial x^4} \right) \right]
\]

\[+ h^2 \left[ - \frac{1}{12} \frac{\partial^4}{\partial x^4} \left( u + u'' \right) + \mu \frac{\partial^4}{\partial x^4} \left( \frac{\partial^4}{\partial x^4} \right) \right] + O(\ell^4 + h^4). \tag{2.10}
\]

which tends to zero as \( h, \ell \to 0 \) simultaneously, so the scheme is consistent with Eq. (1.2).

### 2.3.2 Stability analysis

For the stability analysis of the scheme (2.8) the Fourier method with a linearizing technique applied to the nonlinear terms of Eq. (2.8) was used. Let

\[
U_B = \max_{m=1,2,\ldots,N} U_m^0,
\]

(2.11)

be a constant typical value of \( U_{m-1}^{n+1} \) and \( U_{m+1}^{n+1} \). Then Eq. (2.8) is written as

\[
\begin{align*}
U_{m-1}^{n+1} &+ \left( \lambda - \mu_1 r^2 + 4 \mu_3 p^2 q \right) \left( U_{m-1}^{n+1} - 2 U_m^{n+1} + U_{m+1}^{n+1} \right) - 2 \mu_2 r^2 U_B \left( U_{m-1}^{n+1} - 2 U_m^{n+1} + U_{m+1}^{n+1} \right) \\
&- \mu_2 r^2 U_B \left( U_{m-1}^{n+1} - 2 U_m^{n+1} + U_{m+1}^{n+1} \right) \\
&= 2 U_{m-1}^{n+1} + \left[ \lambda + 1 - 2 \mu_1 r^2 - 4 \left( 1 - 2 \mu_3 p^2 q \right) \right] \left( U_{m-1}^{n+1} - 2 U_m^{n+1} + U_{m+1}^{n+1} \right) \\
&+ \left( 1 - 2 \mu_3 p^2 q \right) \left( U_{m-2}^{n+1} + U_{m+2}^{n+1} \right) + \left( 1 - 2 \mu_2 r^2 \right) U_B \left( U_{m-1}^{n+1} - 2 U_m^{n+1} + U_{m+1}^{n+1} \right) \\
&\left( U_{m-1}^{n+1} - 2 U_m^{n+1} + U_{m+1}^{n+1} \right) + \mu_2 r^2 U_B \left( U_{m-1}^{n+1} - 2 U_m^{n+1} + U_{m+1}^{n+1} \right).
\end{align*}
\]

(2.12)

If \( Z_m = U_m - \tilde{U}_m \), in which \( Z_m = e^{\alpha n} e^{i m \beta h} \), where \( \alpha \) is complex and \( \beta \) is real, it leads, after canceling both sides of Eq. (2.12) by \( e^{\alpha n} e^{i m \beta h} \) and using Euler’s formula \( e^{ix} = \cos x + i \sin x \), to the following stability equation

\[
\begin{align*}
&\left[ 1 - 4 \left( \lambda - \mu_1 r^2 - \mu_2 r^2 U_B + 4 \mu_3 p^2 q \right) \sin^2 \frac{\beta h}{2} + 4 \mu_3 p^2 q \sin^2 \beta h \right] \xi^2 \\
&- 2 \left( 1 - 2 \left( \lambda + 1 - 2 \mu_1 r^2 + (1 - 2 \mu_2 r^2) U_B - 4 \left( 1 - 2 \mu_3 p^2 q \right) \sin^2 \frac{\beta h}{2} \right) \xi + 1 + 4 \mu_3 p^2 q \sin^2 \beta h \right) \xi + 4 \left( \lambda - \mu_1 r^2 - \mu_2 r^2 U_B + 4 \mu_3 p^2 q \right) \sin^2 \frac{\beta h}{2} h = 0.
\end{align*}
\]

(2.13)
with $\xi = e^{\alpha t}$ the amplification factor. Eq. (2.13) is of the form

$$A\xi^2 - 2B\xi + A = 0,$$  \hspace{1cm} (2.14)

with $A, B \in \mathbb{R}$. Then following the von Neumann necessary criterion for stability Eq. (2.14) will have roots $\xi_1$, $\xi_2$ with modulus less than or equal to unity, if,

$$\left| B \right| \leq \left| A \right|.$$  \hspace{1cm} (2.15)

In (2.15) for Eq. (2.13) gives

$$-\dot{A} \leq \dot{A} - 2 \left[ r^2 (1 + U_B) - 4p^2 q \right] \sin^2 \frac{\beta h}{2} - 2p^2 q \sin^2 \beta h \leq \dot{A},$$  \hspace{1cm} (2.16)

with,

$$\dot{A} = 1 - 4 \left( \lambda - \mu_1 r^2 - \mu_2 r^2 U_B + 4\mu_3 p^2 q \right) \sin^2 \frac{\beta h}{2} + 4\mu_3 p^2 q \sin^2 \beta h.$$  \hspace{1cm} (2.17)

The left-hand side of In. (2.16) leads to

$$\left[ r^2 (1 + U_B) - 4r^2 (\mu_1 + \mu_2 U_B) - 4p^2 q (1 - 4\mu_3) \right] \sin^2 \frac{\beta h}{2}$$

$$+ p^2 q (1 - 4\mu_3) \sin^2 \beta h \leq 1 - 4\lambda \sin^2 \frac{\beta h}{2}.$$  \hspace{1cm} (2.18)

If $\lambda = 0$, then In. (2.18) leads to $r^2 (1 + |U_B|) + 5p^2 \leq 1$, which gives the following restriction for the time step

$$\ell \leq \frac{h}{5} \left( 1 + |U_B| + \frac{5}{h^2} \right)^{-1/2}.$$  \hspace{1cm} (2.19)

otherwise, if $\lambda \neq 0$, In. (2.18) is satisfied when,

$$\left[ r^2 (1 + U_B) - 4r^2 (\mu_1 + \mu_2 U_B) - 4p^2 q (1 - 4\mu_3) \right] \sin^2 \frac{\beta h}{2}$$

$$+ p^2 q (1 - 4\mu_3) \sin^2 \beta h \leq 1 - 4\lambda,$$  \hspace{1cm} (2.20)

which for $\lambda \in (0, 0.25)$ it leads to the following restriction for the time step

$$\ell \leq \frac{h}{5} (1 - 4\lambda) \left( 1 + |U_B| + \frac{5}{h^2} \right)^{-1/2}.$$  \hspace{1cm} (2.21)

The right-hand side of In. (2.16) gives $[1 + U_B] r^2 - 4p^2 q \sin^2 \frac{\beta h}{2} + p^2 q \sin^2 \beta h \geq 0$, which assuming that $\sin^2 (\beta h/2) \neq 0$, otherwise holds as equality, it leads to

$$h^2 (1 + U_B) \geq 4q \sin^2 \frac{\beta h}{2},$$  \hspace{1cm} (2.22)

If $1 + U_B > 0$, then In. (2.22) is always satisfied for $q = -1$, while for $q = 1$ it gives the following restriction for the space step

$$h \geq 2 (1 + U_B)^{-1/2},$$  \hspace{1cm} (2.23)

If $1 + U_B = 0$, In. (2.22) is always satisfied for $q = -1$, while for $q = 1$ it is impossible. Finally, if $1 + U_B < 0$, In. (2.22) is impossible for $q = 1$, while for $q = -1$ it leads to the following restriction for the space step

$$h \leq 2 |1 + U_B|^{-1/2}.$$  \hspace{1cm} (2.24)
2.3.3 Convergence analysis

Let \( U^m_n = e^{i\psi} e^{im\theta} \) where \( \psi \) is complex and \( \theta \) is real. It can be easily proved that 
\[
(U^m_{n-1})^2 - 2(U^m_n)^2 + (U^m_{n+1})^2 = -4e^{2i\psi} e^{2im\theta} \sin^2 \theta.
\]
Then substituting in Eq. (2.8) after canceling both sides by \( e^{i\psi} e^{im\theta} \) it leads to 
\[
\left[ -\mu_3 p^2 q e^{2i\theta} + \left( \lambda - \mu_3 r^2 + 4\mu_3 p^2 q \right) (e^{i\theta} + e^{-i\theta}) \right] + 1 - 2\lambda + 2\mu_1 r^2 - 6\mu_3 p^2 q \right] (e^{i\psi} + e^{-i\psi}) + 4\mu_2 r^2 e^{i\psi} e^{im\theta} (e^{2i\psi} + e^{-2i\psi}) \sin^2 \theta 
\]
\[
= (1 - 2\mu_3) p^2 q (e^{2i\theta} + e^{-2i\theta}) + [2\lambda + (1 - 2\mu_1) r^2 - 4(1 - 2\mu_3) p^2 q] (e^{i\theta} + e^{-i\theta}) 
\]
\[
+ 2 \left[ 1 - 2\lambda - (1 - 2\mu_1) r^2 + 3(1 - 2\mu_3) p^2 q \right] - 4(1 - 2\mu_2) r^2 e^{i\psi} e^{im\theta} \sin^2 \theta, 
\]
or using Euler’s identity 
\[
2 \left[ 1 + 4\mu_3 p^2 q \sin^2 \theta - 4 \left( \lambda - \mu_3 r^2 + 4\mu_3 p^2 q \right) \sin^2 \frac{\theta}{2} \right] \sin \psi 
\]
\[
= 2 - 4(1 - 2\mu_3) p^2 q \sin^2 \theta - 4 \left[ 2\lambda + (1 - 2\mu_1) r^2 - 4(1 - 2\mu_3) p^2 q \right] \sin^2 \frac{\theta}{2} 
\]
\[
- 4(1 - 2\mu_2) r^2 e^{i\psi} e^{im\theta} \sin^2 \theta - 4\mu_2 r^2 e^{i\psi} e^{im\theta} (e^{2i\psi} + e^{-2i\psi}) \sin^2 \theta, 
\]
(2.25)

Since \( \sin \psi = 1 - 2 \sin^2 \frac{\psi}{2} \), Eq. (2.25) gives the following convergence equation 
\[
\left[ 1 + 4\mu_3 p^2 q \sin^2 \theta - 4 \left( \lambda - \mu_3 r^2 + 4\mu_3 p^2 q \right) \sin^2 \frac{\theta}{2} \right] \sin \frac{\psi}{2} 
\]
\[
= p^2 q \sin^2 \theta + (r^2 - 4p^2 q) \sin^2 \frac{\theta}{2} 
\]
\[
+ (1 - 2\mu_2) r^2 e^{i\psi} e^{im\theta} \sin^2 \theta + \mu_2 r^2 e^{i\psi} e^{im\theta} (e^{2i\psi} + e^{-2i\psi}) \sin^2 \theta. 
\]
(2.26)

So, whenever \( \left| \sin \frac{\psi}{2} \right| \leq 1 \), that is, whenever \( \psi \) is real, so that \( U^m_n \) is bounded, Eq. (2.26) gives 
\[
5p^2 + 6r^2 \leq 1
\]
which leads to the following restriction for the time step 
\[
\ell \leq \left[ \frac{1}{h^2} \left( 6 + \frac{5}{h^2} \right) \right]^{-1/2}.
\]
(2.27)

At the experiments the most restrictive of Ins. (2.19), (2.21) and (2.27) will be used.

2.4 The linearized methods

To overcome solving the nonlinear system (2.9) the following two types of linearization to the nonlinear term of the BS equation were used.
2.4.1 Method I

This method can be derived from linearization (2.11) by considering the approximation

\[(U_{m+1}^n)^2 \approx U_{m+1}^n U_L.\]  \hspace{1cm} (2.28)

where \(U_L = U_D\) and \(U_D\) is defined by Eq. (2.11). Then the finite-difference scheme defined by Eq. (2.8) is written as

\[-\mu_3 p^2 q U_3^{n+1} + \frac{1}{2} \left( 3 \left( \lambda - \mu_1 r^2 - \mu_2 r^2 U_L \right) + \frac{29}{4} \mu_3 p^2 q \right) U_2^{n+1} + \left[ 1 - \frac{3}{2} \left( \lambda - \mu_1 r^2 - \mu_2 r^2 U_L \right) - \frac{25}{4} \mu_3 p^2 q \right] U_1^{n+1} = \left( 1 - 2 \mu_3 \right) p^2 q U_3^n + \left[ 3 \lambda + \frac{3}{2} \left( 1 - 2 \mu_1 \right) r^2 - \frac{29}{4} \left( 1 - 2 \mu_3 \right) p^2 q \right] U_2^n + \left[ 2 \mu_2 - 3 \lambda - \frac{3}{2} \left( 1 - 2 \mu_1 \right) r^2 + \frac{25}{4} \left( 1 - 2 \mu_3 \right) p^2 q \right] U_1^n + \frac{1}{4} \left( 1 - 2 \mu_2 \right) r^2 \left[ -7 \left( U_1^n \right)^2 + 2 U_1^n U_2^n + 5 \left( U_2^n \right)^2 \right] + \mu_3 p^2 q U_3^{n-1} - \frac{1}{2} \left( 3 \left( \lambda - \mu_1 r^2 \right) + \frac{29}{4} \mu_3 p^2 q \right) U_2^{n-1} - \left( 1 - \frac{3}{2} \lambda + \frac{3}{2} \mu_1 r^2 - \frac{25}{4} \mu_3 p^2 q \right) U_1^{n-1} + \frac{1}{4} \mu_2 r^2 \left[ -7 \left( U_1^{n-1} \right)^2 + 2 U_1^{n-1} U_2^{n-1} + 5 \left( U_2^{n-1} \right)^2 \right] + b_1, \]  \hspace{1cm} (2.29)

for \(m = 1,\)

\[-\mu_3 p^2 q U_4^{n+1} + \left( \lambda - \mu_1 r^2 - \mu_2 r^2 U_L + 4 \mu_3 p^2 q \right) U_3^{n+1} + \left( 1 - 2 \lambda + 2 \mu_1 r^2 + 2 \mu_2 r^2 U_L - \frac{13}{4} \mu_3 p^2 q \right) U_2^{n+1} + \left( \lambda - \mu_1 r^2 - \mu_2 r^2 U_L + \frac{7}{2} \mu_3 p^2 q \right) U_1^{n+1} = \left( 1 - 2 \mu_3 \right) p^2 q U_4^n + \left[ 2 \lambda + \left( 1 - 2 \mu_1 \right) r^2 - 4 \left( 1 - 2 \mu_3 \right) p^2 q \right] U_3^n + 2 \lambda - 1 \left( 1 - 2 \mu_1 \right) r^2 + \frac{13}{4} \left( 1 - 2 \mu_3 \right) p^2 q \right] U_2^n + \left( 1 - 2 \mu_2 \right) r^2 \left[ \left( U_3^n \right)^2 - 2 \left( U_2^n \right)^2 + \left( U_1^n \right)^2 \right] + \left[ 2 \lambda + \left( 1 - 2 \mu_1 \right) r^2 - \frac{7}{2} \left( 1 - 2 \mu_3 \right) p^2 q \right] U_1^n + \mu_3 p^2 q U_4^{n-1} - \left( \lambda - \mu_1 r^2 + 4 \mu_3 p^2 q \right) U_3^{n-1} - \left( 1 - 2 \lambda + 2 \mu_1 r^2 - \frac{13}{4} \mu_3 p^2 q \right) U_2^{n-1} + \mu_3 p^2 q U_4^{n-1} - \left( \lambda - \mu_1 r^2 + 4 \mu_3 p^2 q \right) U_3^{n-1} - \left( 1 - 2 \lambda + 2 \mu_1 r^2 - \frac{13}{4} \mu_3 p^2 q \right) U_2^{n-1} + \mu_2 r^2 \left[ \left( U_3^{n-1} \right)^2 - 2 \left( U_2^{n-1} \right)^2 + \left( U_1^{n-1} \right)^2 \right] - \left( \lambda - \mu_1 r^2 + \frac{7}{2} \mu_3 p^2 q \right) U_1^{n-1}, \]  \hspace{1cm} (2.30)

for \(m = 2,\)

\[-\mu_3 p^2 q \left( U_{m+1}^{n+1} + U_{m-1}^{n+1} \right) + \left( \lambda - \mu_1 r^2 - \mu_2 r^2 U_L + 4 \mu_3 p^2 q \right) \left( U_{m+1}^{n+1} + U_{m-1}^{n+1} \right) + \left( 1 - 2 \lambda + 2 \mu_1 r^2 + 2 \mu_2 r^2 U_L - 6 \mu_3 p^2 q \right) U_{m+1}^{n+1} = \left( 1 - 2 \mu_3 \right) p^2 q \left( U_{m+2}^{n+1} + U_{m-2}^{n+1} \right) + \left[ 2 \lambda + \left( 1 - 2 \mu_1 \right) r^2 - 4 \left( 1 - 2 \mu_3 \right) p^2 q \right] \left( U_{m+1}^{n+1} + U_{m-1}^{n+1} \right).\]
\[ +2 \left[ 1 - 2\lambda - (1 - 2\mu_1) r^2 + 3 (1 - 2\mu_3) p^2 q \right] U_{m}^{n} + (1 - 2\mu_2) r^2 \left[ (U_{m+1}^{n})^2 - 2 (U_{m}^{n})^2 + (U_{m-1}^{n})^2 \right] \\
+ \mu_3 p^2 q \left( U_{m+2}^{n-1} + U_{m-2}^{n-1} - (\lambda - \mu_1 r^2 + 4\mu_3 p^2 q) \left( U_{m+1}^{n-1} + U_{m-1}^{n-1} \right) \right) \\
- \left[ 1 - 2\lambda + 2\mu_1 r^2 - 6\mu_3 p^2 q \right] U_{m}^{n-1} + \mu_2 r^2 \left[ (U_{m+1}^{n-1})^2 - 2 (U_{m}^{n-1})^2 + (U_{m-1}^{n-1})^2 \right], \] (2.31)

for \( m = 3, 4, \ldots, N - 2, \)
\[-\mu_3 p^2 q U_{N-3}^{n+1} + (\lambda - \mu_1 r^2 - \mu_2 r^2 U_L + 4\mu_3 p^2 q) U_{N-2}^{n+1} \]
\[+ \left( 1 - 2\lambda + 2\mu_1 r^2 + 2\mu_2 r^2 U_L - \frac{13}{2} \mu_3 p^2 q \right) U_{N-1}^{n+1} \]
\[+ \left( \lambda - \mu_1 r^2 - 2\mu_2 r^2 U_L + \frac{7}{2} \mu_3 p^2 q \right) U_{N-1}^{n+1} \]
\[= (1 - 2\mu_3) p^2 q U_{N-3}^{n} + [2\lambda + (1 - 2\mu_1) r^2 - 4 (1 - 2\mu_3) p^2 q] U_{N-2}^{n} \]
\[+ 2 \left[ 1 - 2\lambda - (1 - 2\mu_1) r^2 + \frac{13}{4} (1 - 2\mu_3) p^2 q \right] U_{N-1}^{n} + (1 - 2\mu_2) r^2 \left[ (U_{N-2}^{n})^2 \right. \\
\[-2 \left( U_{N-1}^{n} \right)^2 + \left( U_{N}^{n} \right)^2 \right] + \left( 2\lambda + (1 - 2\mu_1) r^2 - \frac{7}{2} (1 - 2\mu_3) p^2 q \right) U_{N}^{n} \]
\[+ \mu_3 p^2 q U_{N-3}^{n-1} -(\lambda - \mu_1 r^2 + 4\mu_3 p^2 q) U_{N-2}^{n-1} - \left( 1 - 2\lambda + 2\mu_2 r^2 - \frac{13}{2} \mu_3 p^2 q \right) U_{N-1}^{n-1} \]
\[+ \mu_2 r^2 \left[ (U_{N-1}^{n-1})^2 - 2 (U_{N-1}^{n-1})^2 + (U_{N-1}^{n})^2 \right] - \left( \lambda - \mu_1 r^2 + \frac{7}{2} \mu_3 p^2 q \right) U_{N-1}^{n-1} \] (2.32)

for \( m = N - 1, \) and,
\[-\mu_3 p^2 q U_{N-2}^{n+1} + \frac{1}{2} \left[ 3 (\lambda - \mu_1 r^2 - \mu_2 r^2 U_L) + \frac{29}{2} \mu_3 p^2 q \right] U_{N-1}^{n+1} \]
\[+ \left[ 1 - \frac{3}{2} (\lambda - \mu_1 r^2 - \mu_2 r^2 U_L) - \frac{25}{4} \mu_3 p^2 q \right] U_{N}^{n+1} \]
\[= (1 - 2\mu_3) p^2 q U_{N-2}^{n} + \left[ 3\lambda + \frac{3}{2} (1 - 2\mu_1) r^2 - \frac{29}{4} (1 - 2\mu_3) p^2 q \right] U_{N-1}^{n} \]
\[+ \left[ 2 - 3\lambda - \frac{3}{2} (1 - \mu_1) r^2 + \frac{25}{4} (1 - 2\mu_3) p^2 q \right] U_{N}^{n} \]
\[+ \frac{1}{4} (1 - 2\mu_2) r^2 \left[ 5 \left( U_{N-1}^{n} \right)^2 + 2 U_{N-1}^{n} U_{N}^{n} - 7 \left( U_{N}^{n} \right)^2 \right] \]
\[+ \mu_3 p^2 q U_{N-2}^{n-1} - \frac{1}{2} \left[ 3 (\lambda - \mu_1 r^2) + \frac{29}{2} \mu_3 p^2 q \right] U_{N-1}^{n-1} \]
\[- \left( 1 - \frac{3}{2} \lambda + \frac{3}{2} \mu_1 r^2 - \frac{25}{4} \mu_3 p^2 q \right) U_{N-1}^{n-1} \]
\[+ \frac{1}{4} \mu_2 r^2 \left[ 5 \left( U_{N-1}^{n-1} \right)^2 + 2 U_{N-1}^{n-1} U_{N}^{n-1} - 7 \left( U_{N}^{n-1} \right)^2 \right] + b_{N}. \] (2.33)

for \( m = N, \) otherwise the nonlinear system (2.9) is transformed to a linear system written in a matrix-vector form as
\[ A U^{n+1} = G \left( U^{n}, U^{n-1} \right) + b. \] (2.34)

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in which \( A \) is a quindiaogonal matrix of order \( N \) given by

\[
A = \begin{bmatrix}
c_1 & d_1 & g & e_N - 2 & e_N - 3 & \cdots & e_{N-2} & e_{N-3} & \cdots & g \\
d_2 & d_2 & g & e_{N-1} & e_{N-2} & \cdots & e_{N-3} & e_{N-4} & \cdots & g \\
g & e_3 & e_2 & d_3 & d_2 & \cdots & e_{N-2} & e_{N-3} & \cdots & g \\
g & e_4 & e_3 & d_4 & d_3 & \cdots & e_{N-1} & e_{N-2} & \cdots & g \\
g & e_N & e_N & e_N & e_N & \cdots & e_N & e_N & \cdots & e_N \\
\end{bmatrix},
\]

with

\[
c_1 = c_N = 1 - \frac{3}{2} \left( \lambda - \mu_1 r^2 - \mu_2 r^2 U_L \right) - \frac{25}{4} \mu_3 p^2 q,
\]

\[
c_2 = c_{N-1} = 1 - 2 \lambda + 2 \mu_1 + 2 \mu_2 r^2 U_L - \frac{13}{2} \mu_3 p^2 q,
\]

\[
d_1 = e_N = \frac{1}{2} \left[ 3 \left( \lambda - \mu_1 r^2 - \mu_2 r^2 U_L \right) + \frac{29}{2} \mu_3 p^2 q \right],
\]

\[
d_2 = e_{N-1} = \lambda - \mu_1 r^2 - \mu_2 r^2 U_L + 4 \mu_3 p^2 q,
\]

\[
d_{N-1} = e_2 = \lambda - \mu_1 r^2 - \mu_2 r^2 U_L + \frac{7}{2} \mu_3 p^2 q,
\]

\[
g = - \mu_3 p^2 q,
\]

\[
c_i = 1 - 2 \lambda + 2 \mu_1 r^2 + 2 \mu_2 r^2 U_L - 6 \mu_3 p^2 q,
\]

\[
d_i = e_i = \lambda - \mu_1 r^2 - \mu_2 r^2 U_L + 4 \mu_3 p^2 q,
\]

for \( i = 3, 4, ..., N - 2, \) and,

\[
b = [b_1, ..., b_N]^T = \left[ -3 p^2 q h \left\{ \mu_3 \left[ (u_0^{n+1})' + (u_0^{n-1})' \right] + (1 - 2 \mu_3) (u_0^n)' \right\} \right.
\]

\[
, ..., 3 p^2 q h \left\{ \mu_3 \left[ (u_1^{n+1})' + (u_1^{n-1})' \right] + (1 - 2 \mu_3) (u_1^{n+1})' \right\} \right]^T.
\]

a vector of order \( N \) with the boundary conditions involving the derivatives of the theoretical solution of the BS equation. The stability analysis of Method I is given in paragraph 2.3.2.

### 2.4.2 Method II

This method is obtained by Taylor’s expansion of \( (U_m^{n+1})^2 \) about the \( n \)-th time level as follows

\[
(U_m^{n+1})^2 = (U_m^n)^2 + \ell \left( \frac{\partial (U_m^n)^2}{\partial t} \right) + O (\ell^2) \text{ as } \ell \to 0,
\]

which, when using backward difference approximants for the second term on the right-hand side of Eq. (2.44), it gives the following approximation

\[
(U_m^{n+1})^2 \approx \left( U_m^n \right)^2 - 2 U_m^n U_m^{n-1}.
\]

Using approximation (2.45) Eq. (2.8) leads to the following finite-difference scheme

\[
- \mu_3 p^2 q U_{3,3}^{n+1} - \frac{1}{2} \left[ 3 \left( \lambda - \mu_1 r^2 \right) + \frac{29}{2} \mu_3 p^2 q \right] U_{2,2}^{n+1} + \left( 1 - \frac{3}{2} \lambda + \frac{3}{2} \mu_1 r^2 - \frac{25}{4} \mu_3 p^2 q \right) U_1^{n+1}
\]

\[
= (1 - 2 \mu_3) p^2 q U_3^n + \left[ 3 \lambda + \frac{3}{2} (1 - 2 \mu_1) r^2 - \frac{29}{4} (1 - 2 \mu_3) p^2 q \right] U_2^n,
\]

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\[\begin{align*}
&+ \left[ 2 - 3\lambda - \frac{3}{2} (1 - 2\mu_1) r^2 + \frac{25}{4} (1 - 2\mu_3) p^2 q \right] U_1^n \\
&+ \frac{1}{4} (1 - \mu_2) r^2 \left[ -7 (U^n_1)^2 + 2U^n_1 U^n_2 + 5 (U^n_2)^2 \right] + \mu_3 p^2 q U^n_{-1} \\
&- \frac{1}{2} \left( 3 (\lambda - \mu_1 r^2) + \frac{29}{2} \mu_3 p^2 q \right) U_2^{n-1} - \left( 1 - \frac{3}{2} \lambda + \frac{3}{2} \mu_1 r^2 - \frac{25}{4} \mu_3 p^2 q \right) U_1^{n-1} \\
&+ \mu_2 r^2 \left\{ \frac{1}{4} \left[ -7 (U^{n-1}_1)^2 + 2U^{n-1}_1 U^{n-1}_2 + 5 (U^{n-1}_2)^2 \right] + \frac{1}{4} \left[ 15 (U^n_2)^2 - 21 (U^n_1)^2 - 6U^n_1 U^n_2 \right] \right\} + b_1, \\
&\text{for } m = 1, \\
&- \mu_3 p^2 q U^{n+1}_4 + (\lambda - \mu_1 r^2 + 4\mu_3 p^2 q) U^{n+1}_3 \\
&+ \left( 1 - 2\lambda + 2\mu_1 r^2 - \frac{13}{2} \mu_3 p^2 q \right) U^{n+1}_2 + \left( \lambda - \mu_1 r^2 + \frac{7}{2} \mu_3 p^2 q \right) U^{n+1}_1 \\
&= (1 - 2\mu_3) p^2 q U^n_4 + \left[ 2\lambda + (1 - 2\mu_1) r^2 - 4 (1 - 2\mu_3) p^2 q \right] U^n_3 \\
&+ 2 \left[ 1 - 2\lambda - (1 - 2\mu_1) r^2 + \frac{13}{4} (1 - 2\mu_3) p^2 q \right] U^n_2 \\
&+ (1 - \mu_2) r^2 \left[ (U^n_3)^2 - 2 (U^n_2)^2 + (U^n_1)^2 \right] + \left[ 2\lambda + (1 - 2\mu_1) r^2 - \frac{7}{2} (1 - 2\mu_3) p^2 q \right] U^n_1 \\
&+ \mu_3 p^2 q U^{n-1}_4 - (\lambda - \mu_1 r^2 + 4\mu_3 p^2 q) U^{n-1}_3 \\
&- \left( 1 - 2\lambda + 2\mu_1 r^2 - \frac{13}{2} \mu_3 p^2 q \right) U^{n-1}_2 - \left( \lambda - \mu_1 r^2 + \frac{7}{2} \mu_3 p^2 q \right) U^{n-1}_1 \\
&+ \mu_2 r^2 \left\{ 3 \left[ (U^n_3)^2 - 2 (U^n_2)^2 + (U^n_1)^2 \right] + (U^n_1)^2 \right\} \\
&- 2 (U^{n-1}_2)^2 + (U^{n-1}_1)^2 - 2 (U^n_1 U^{n-1}_1 - 2 U^n_2 U^{n-1}_2 + U^n_3 U^{n-1}_3) \right\}, \\
&\text{for } m = 2, \\
&- \mu_3 p^2 q (U^{n+1}_{m+1} + U^{n+1}_{m-2}) + (\lambda - \mu_1 r^2 + 4\mu_3 p^2 q) (U^{n+1}_{m+1} + U^{n+1}_{m-1}) \\
&+ (1 - 2\lambda + 2\mu_1 r^2 - 6\mu_3 p^2 q) U^{n+1}_m \\
&= (1 - 2\mu_3) p^2 q (U^n_{m+2} + U^n_{m-2}) + \left[ 2\lambda + (1 - 2\mu_1) r^2 - 4 (1 - 2\mu_3) p^2 q \right] (U^n_{m+1} + U^n_{m-1}) \\
&+ 2 \left[ 1 - 2\lambda - (1 - 2\mu_1) r^2 + 3 (1 - 2\mu_3) p^2 q \right] U^n_m + (1 - \mu_2) r^2 \left[ (U^n_{m+2})^2 - 2 (U^n_m)^2 + (U^n_{m-1})^2 \right] \\
&+ \mu_3 p^2 q (U^{n-1}_{m+1} + U^{n-1}_{m-2}) - (\lambda - \mu_1 r^2 + 4\mu_3 p^2 q) (U^{n-1}_{m+1} + U^{n-1}_{m-1}) \\
&+ (1 - 2\lambda + 2\mu_1 r^2 - 6\mu_3 p^2 q) U^{n-1}_m \\
&+ \mu_2 r^2 \left\{ 3 \left[ (U^n_{m+1})^2 - 2 (U^n_m)^2 + (U^n_{m-1})^2 \right] + (U^n_{m+1})^2 - 2 (U^n_m)^2 + (U^n_{m-1})^2 \right\} \\
&- 2 (U^{n-1}_{m+1} U^{n-1}_m - 2 U^n_m U^{n-1}_m + U^n_{m-1} U^{n-1}_m) \right\}, \\
&\text{for } m = 3, 4, \ldots, N - 2, \\
&- \mu_3 p^2 q U^{n+1}_{N-3} + (\lambda - \mu_1 r^2 + 4\mu_3 p^2 q) U^{n+1}_{N-2} + \left( 1 - 2\lambda + 2\mu_1 r^2 - \frac{13}{2} \mu_3 p^2 q \right) U^{n+1}_{N-1} \\
&\text{for } m = N - 1, \\
&= \frac{1}{4} \left[ -7 (U^n_1)^2 + 2U^n_1 U^n_2 + 5 (U^n_2)^2 \right] + \mu_3 p^2 q U^n_{-1} \\
&- \frac{1}{2} \left( 3 (\lambda - \mu_1 r^2) + \frac{29}{2} \mu_3 p^2 q \right) U_2^{n-1} - \left( 1 - \frac{3}{2} \lambda + \frac{3}{2} \mu_1 r^2 - \frac{25}{4} \mu_3 p^2 q \right) U_1^{n-1} \\
&+ \mu_2 r^2 \left\{ \frac{1}{4} \left[ -7 (U^{n-1}_1)^2 + 2U^{n-1}_1 U^{n-1}_2 + 5 (U^{n-1}_2)^2 \right] + \frac{1}{4} \left[ 15 (U^n_2)^2 - 21 (U^n_1)^2 - 6U^n_1 U^n_2 \right] \right\} + b_1.
\end{align*}\]
\[ + \left( \lambda - \mu_3 r^2 + \frac{7}{2} \mu_3 p^2 q \right) U_{N+1}^{n+1} = (1 - 2 \mu_3) p^2 q U_{N-3}^n + 2 \left( \lambda + (1 - 2 \mu_1) r^2 - 4 (1 - 2 \mu_3) p^2 q \right) U_{N-2}^n + 2 \left[ 1 - 2 \lambda - (1 - 2 \mu_1) r^2 + \frac{13}{4} (1 - 2 \mu_3) p^2 q \right] U_{N-1}^n + (1 - \mu_2) r^2 \left( (U_{n-2}^n)^2 - 2 (U_{n-1}^n)^2 + (U_n^2) \right) + 2 \left( \lambda + (1 - 2 \mu_1) r^2 - \frac{7}{2} (1 - 2 \mu_3) \right) U_N^n + \mu_3 p^2 q U_{n-3}^{n-1} - \left( \lambda - \mu_1 r^2 + 4 \mu_3 p^2 q \right) U_{n-2}^{n-1} - \left( 1 - 2 \lambda + 2 \mu_1 r^2 - \frac{13}{2} \mu_3 p^2 q \right) U_{n-1}^{n-1} - \left( \lambda - \mu_2 r^2 + \frac{7}{2} \mu_3 p^2 q \right) U_{n-1}^{n-1} + \mu_2 r^2 \left\{ 3 \left( (U_{n-2}^n)^2 - 2 (U_{n-1}^n)^2 + (U_n^2) \right) + (U_{n-1}^n)^2 - 2 \left( U_{n-1}^n \right)^2 + (U_n^1)^2 \right\} - 2 \left( U_{n-2}^n U_{n-2}^{n-1} - 2 U_{n-1}^n U_{n-1}^{n-1} + U_{n-1}^n U_{n-1}^{n-1} \right) \right\}, \] (2.49)

for \( m = N - 1 \), and,

\[
- \mu_3 p^2 q U_{n-2}^{n+1} + \frac{1}{2} \left( 3 \left( \lambda - \mu_1 r^2 \right) + \frac{29}{2} \mu_3 p^2 q \right) U_{n-1}^{n+1} + \left( 1 - \frac{3}{2} \lambda + \frac{3}{2} \mu_1 r^2 - \frac{25}{4} \mu_3 p^2 q \right) U_{n}^{n+1} = (1 - 2 \mu_3) p^2 q U_{n-2} + \left[ 3 \lambda + \frac{3}{2} (1 - 2 \mu_1) r^2 - \frac{29}{4} (1 - 2 \mu_3) p^2 q \right] U_{n}^n + \left[ 2 - 3 \lambda - \frac{3}{2} (1 - 2 \mu_1) r^2 + \frac{25}{4} (1 - 2 \mu_3) p^2 q \right] U_{n}^n + \frac{1}{4} (1 - \mu_2) r^2 \left[ \left( U_{n}^n \right)^2 + 5 U_{n-1}^n U_{n}^n - 7 \left( U_{n}^n \right)^2 \right] + \mu_3 p^2 q U_{n-2}^{n-1} - \frac{1}{2} \left( 3 \left( \lambda - \mu_1 r^2 \right) + \frac{29}{2} \mu_3 p^2 q \right) U_{n-1}^{n-1} - \left( 1 - \frac{3}{2} \lambda + \frac{3}{2} \mu_1 r^2 - \frac{25}{4} \mu_3 p^2 q \right) U_{n-1}^{n-1} + \mu_2 r^2 \left\{ \frac{1}{4} \left[ 15 \left( U_{n-1}^n \right)^2 - 6 U_{n-1}^n U_{n}^n - 21 \left( U_{n}^n \right)^2 \right] \right\} + \frac{1}{4} \left[ \left( U_{n-1}^n \right)^2 + 5 U_{n-1}^n U_{n-1}^{n-1} - 7 \left( U_{n-1}^n \right)^2 \right] - \frac{1}{2} \left( 5 U_{n-1}^n U_{n-1}^{n-1} - 7 \left( U_{n-1}^n \right)^2 + 2 U_{n-1}^n U_{n-1}^{n-1} + U_{n-1}^n U_{n-1}^{n-1} + U_{n-1}^n U_{n-1}^{n-1} \right) \right\} + 2 b_n. \] (2.50)

for \( m = N \), otherwise the nonlinear system (2.9) is transformed to a linear system written in a matrix-vector form as

\[ \bar{A} U_{n+1} = \tilde{C} \left( U_n, U_{n-1} \right) + b_n. \] (2.51)

in which \( \bar{A} \) is a quindiaogonal matrix of order \( N \) given by

\[ \bar{A} = \begin{bmatrix}
\bar{c}_1 & \bar{d}_1 & g \\
\bar{c}_2 & \bar{d}_2 & g \\
g & \bar{c}_3 & \bar{d}_3 & g \\
\vdots & \ddots & \ddots & \ddots & g \\
g & \bar{c}_{N-2} & \bar{d}_{N-2} & g \\
g & \bar{c}_{N-1} & \bar{d}_{N-1} & g \\
g & \bar{c}_N
\end{bmatrix}, \] (2.52)
with
\[
\tilde{c}_1 = \tilde{c}_N = 1 - \frac{3}{2} \left( \lambda - \mu_1 r^2 \right) - \frac{25}{4} \mu_3 p^2 q, \tag{2.53}
\]
\[
\tilde{c}_2 = \tilde{c}_{N-1} = 1 - 2\lambda + 2\mu_1 r^2 - \frac{13}{2} \mu_3 p^2 q, \tag{2.54}
\]
\[
\tilde{d}_1 = \tilde{c}_N = \frac{1}{2} \left[ 3 \left( \lambda - \mu_1 r^2 \right) + \frac{29}{2} \mu_3 p^2 q \right], \tag{2.55}
\]
\[
\tilde{d}_2 = \tilde{c}_{N-1} = \lambda - \mu_1 r^2 + 4\mu_3 p^2 q, \tag{2.56}
\]
\[
\tilde{d}_{N-1} = \tilde{c}_2 = \lambda - \mu_1 r^2 + \frac{7}{2} \mu_3 p^2 q, \tag{2.57}
\]
\[
\tilde{c}_i = 1 - 2\lambda + 2\mu_1 r^2 - 6\mu_3 p^2 q, \tag{2.58}
\]
for \( i = 3, 4, \ldots, N-2 \) and \( b \) defined by Eq. (2.43).

For the stability analysis of Method II consider \( U_B^n \) to be a constant typical value of \( U_m^n, U_m^{n-1} \) for \( m = 1, 2, \ldots, N \). Then Eq. (2.48) leads to Eq. (2.12) with \( U_B^n \) instead of \( U_B \), so the stability analysis of Method II is similar to that developed in paragraph 2.3.2.

### 3 Numerical results

BS equation was solved numerically with initial time \( t_0 = 0, L_0 = -80, L_1 = 100 \) and initial displacement defined by Eq. (1.3), \( U(x, 0) = u(x, 0); x \in (L_0, L_1) \), that is the numerical solution is set to be equal to the theoretical solution for \( t_0 = 0 \). For the numerical solution at the first time step \( t = \ell \), the approximation
\[
U(x, \ell) = u(x, 0) + \ell \frac{\partial u}{\partial t} + \ell^2 \frac{\partial^2 u}{\partial t^2} + O(\ell^3) \text{ as } \ell \to 0. \tag{3.1}
\]

\( x \in (L_0, L_1) \) was used, where the first partial derivative was obtained by Eq. (1.5) for the single and Eq. (1.6) for the double-soliton waves respectively, while the second one by using finite-difference replacements.

Let the error, \( e = L_{\infty} \), be the value of \( |u_m^n - U_m^n| \) for \( m = 1, 2, \ldots, N \) with maximum modulus at time level \( t = n\ell \) for \( n = 0, 1, \ldots, L_2 \) the familiar norm and use \( e = \epsilon_n = e \times 100/|u_m^n| \) to denote the corresponding percentage relative error.

To investigate the effect of the parameters \( \lambda \) and \( \mu_i; i = 1, 2, 3 \) at the numerical solution of the BS equation the interval \([0, 1]\) was discretized into 10 subintervals each of width \( h = 0.1 \) for each of the parameters \( \mu_i; i = 1, 2, 3 \), while, following the stability restrictions for the parameter \( \lambda \) the interval \([0, 0.24]\) into 4 subintervals each of width \( h^* = 0.06 \). Using the mentioned discretization Eq. (1.2) was solved to appropriate times with the restrictions
\[
e < 1 \text{ and } \epsilon < 10^3. \tag{3.2}
\]
for both the BB and GB equations.

#### 3.1 Numerical results for the BB equation

According to Eqs. (1.5) – (1.6) the theoretical solution \( u \) is real if \( b_i > -\frac{1}{2} A_i; i = 1, 2 \). We choose for our experiments \( b_i = \frac{1}{2} \) and so \( c_i = \pm \left( 1 + \frac{2}{9} A_i \right)^{1/2}; i = 1, 2 \), giving in theory real solutions for \( A_i \geq -\frac{3}{2}; i = 1, 2 \).
3.1.1 Single-soliton

The constant of linearization \( U_\ell \) was taken to be \( U_\ell = \max_{m=1,2,\ldots,N} \{ u^0_m \} \approx 0.359E + 00 \). In Tables 1-2 is given the numerical solution of Eq. (1.2) to times \( t = 1.2, 36 \) and 72 with theoretical parameters \( A = 0.369, \ x_1^0 = 0, \) velocity \( c = 1.11624 \) and space step \( h = 4 \). For the time step it was preferred for our experiments the value \( \ell = 0.01 \), which satisfies the stability and convergence conditions and represents the minimum value for all space steps used. With respect to \( \epsilon \) and for long time periods it is deduced from Tables 1-2, that more accurate results were obtained using Method I (see also Fig. 1 where the continuous curve shows \( \epsilon \) from time \( t = 0 \) to \( t = 72 \) for Method I). In Table 3 it is examined the behavior of Method I at the numerical solution of Eq. (1.2) for various values of \( h \) and \( \ell \). Obviously the refinement of \( \ell \) improved the accuracy, which did not occur with the relevant refinition of \( h \).

The surface on the left-hand side of Fig. 2 shows the numerical solution \( U \) using Method I, when \( A = 0.369, \ b = 0.5, \ c = 1.11624, \ x_1^0 = 0, \ h = 4, \ \ell = 0.01 \) from time \( t = 0 \) to \( t = 72 \) and parameters \( \lambda = 0, \mu_1 = 0, \mu_2 = 0.7 \) and \( \mu_3 = 1 \), while on the right-hand side the relevant theoretical solution \( u \).

### Table 1  Single-soliton Method I: Results for BB with \( h = 4 \) and \( \ell = 0.01 \), when the theoretical parameters are \( A = 0.369, \ b = 0.5, \ c = 1.11624 \) and \( x_1^0 = 0 \).

<table>
<thead>
<tr>
<th>( t )</th>
<th>( \ell )</th>
<th>( \lambda )</th>
<th>( \mu_1 )</th>
<th>( \mu_2 )</th>
<th>( \mu_3 )</th>
<th>( \epsilon )</th>
<th>( L_2 )</th>
<th>type of error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.70460E - 02</td>
<td>0.86522E - 02</td>
<td>0.21E + 01</td>
<td>min</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.96313E - 02</td>
<td>0.12654E - 01</td>
<td>0.29E + 01</td>
<td>max</td>
</tr>
<tr>
<td>0.06</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.20842E - 02</td>
<td>0.35581E - 01</td>
<td>0.85E + 00</td>
<td>min</td>
</tr>
<tr>
<td>0.12</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.76754E - 02</td>
<td>0.10830E - 01</td>
<td>0.84E + 01</td>
<td>max</td>
</tr>
<tr>
<td>36</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.50145E - 01</td>
<td>0.75642E - 01</td>
<td>0.34E + 02</td>
<td>min</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.81682E - 01</td>
<td>0.16111E + 00</td>
<td>0.22E + 02</td>
<td>max</td>
</tr>
<tr>
<td>72</td>
<td>0</td>
<td>0</td>
<td>0.2</td>
<td>0</td>
<td>0.53540E - 01</td>
<td>0.10225E + 00</td>
<td>0.39E + 02</td>
<td>min</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.11220E + 00</td>
<td>0.22465E + 00</td>
<td>0.31E + 02</td>
<td>max</td>
</tr>
</tbody>
</table>

### Table 2  Single-soliton Method II: Results for BB with \( h = 4 \) and \( \ell = 0.01 \), when the theoretical parameters are \( A = 0.369, \ b = 0.5, \ c = 1.11624 \) and \( x_1^0 = 0 \).

<table>
<thead>
<tr>
<th>( t )</th>
<th>( \ell = 0.01 )</th>
<th>( \lambda )</th>
<th>( \mu_1 )</th>
<th>( \mu_2 )</th>
<th>( \mu_3 )</th>
<th>( \epsilon )</th>
<th>( L_2 )</th>
<th>type of error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.70460E - 02</td>
<td>0.86522E - 02</td>
<td>0.21E + 01</td>
<td>min</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.70466E - 02</td>
<td>0.86529E - 02</td>
<td>0.21E + 01</td>
<td>max</td>
</tr>
<tr>
<td>0.06</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.20842E - 02</td>
<td>0.35581E - 02</td>
<td>0.85E + 00</td>
<td>min</td>
</tr>
<tr>
<td>0.12</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.76754E - 02</td>
<td>0.10830E - 01</td>
<td>0.84E + 01</td>
<td>max</td>
</tr>
<tr>
<td>36</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.50145E - 01</td>
<td>0.75642E - 01</td>
<td>0.34E + 02</td>
<td>min</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.50159E - 01</td>
<td>0.75657E - 01</td>
<td>0.34E + 02</td>
<td>max</td>
</tr>
<tr>
<td>72</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0.57117E - 01</td>
<td>0.82586E - 01</td>
<td>0.42E + 02</td>
<td>min</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0.57139E - 01</td>
<td>0.82630E - 01</td>
<td>0.42E + 02</td>
<td>max</td>
</tr>
</tbody>
</table>
Table 3  Single-soliton Method I: Results for BB at time \( t = 7.2 \) with \( \lambda = 0 \), when the theoretical parameters are \( A = 0.369, b = 0.5, c = 1.11624 \) and \( x_0^1 = 0 \).

<table>
<thead>
<tr>
<th>( h )</th>
<th>( \ell )</th>
<th>( \mu_1 )</th>
<th>( \mu_2 )</th>
<th>( \mu_3 )</th>
<th>( e )</th>
<th>( L_2 )</th>
<th>( \epsilon )</th>
<th>( \text{type of error} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.01</td>
<td>0</td>
<td>0.7</td>
<td>1</td>
<td>0.13021E + 00</td>
<td>0.32390E + 00</td>
<td>0.42E + 02</td>
<td>min</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
<td>0.19201E + 00</td>
<td>0.37533E + 00</td>
<td>0.59E + 02</td>
<td>max</td>
</tr>
<tr>
<td>0.001</td>
<td>0.8</td>
<td>0.7</td>
<td>0.2</td>
<td></td>
<td>0.13003E + 00</td>
<td>0.32412E + 00</td>
<td>0.40E + 02</td>
<td>min</td>
</tr>
<tr>
<td>0.3</td>
<td>0</td>
<td>0.7</td>
<td></td>
<td></td>
<td>0.19201E + 00</td>
<td>0.37538E + 00</td>
<td>0.59E + 02</td>
<td>max</td>
</tr>
<tr>
<td>3</td>
<td>0.01</td>
<td>0</td>
<td>0.5</td>
<td>1</td>
<td>0.13576E + 00</td>
<td>0.37676E + 00</td>
<td>0.42E + 02</td>
<td>min</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td>0.21966E + 00</td>
<td>0.46573E + 00</td>
<td>0.68E + 02</td>
<td>max</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td>0.25198E + 00</td>
<td>0.61186E + 00</td>
<td>0.69E + 02</td>
<td>min</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td>0.25199E + 00</td>
<td>0.61187E + 00</td>
<td>0.69 + 02</td>
<td>max</td>
</tr>
</tbody>
</table>

Fig. 1  BB Single-soliton: the continuous curve shows the error \( e \) using Method I from time \( t = 0 \) to \( t = 7.2 \), when \( A = 0.369, b = 0.5, c = 1.11624 \) and parameter values \( \lambda = 0, \mu_1 = 0, \mu_2 = 0.7 \) and \( \mu_3 = 1 \), while the dashed one using Method II with \( \lambda = 0, \mu_1 = 0, \mu_2 = 0 \) and \( \mu_3 = 0 \).

3.1.2 Double-soliton

For the double-soliton case there were considered the following two cases
- two waves of equal amplitudes \( A_1 = A_2 = 0.369 \) moving towards each other with velocities \( c_1 = -c_2 = 1.11624 \) placed at the initial positions \( x_0^1 = 0 \) and \( x_0^2 = -50 \) from time \( t = 0 \) to \( t = 36 \) with \( U_B \approx 0.369E + 00 \). The numerical results for both Methods I-II are given in Tables 4-5 respectively from it is seen that both methods have given almost the same numerical results (see also Fig. 3). On the left-hand side of Fig. 4 it is shown the interaction of the waves as it is deduced using Method II, while on the right-hand side the interaction using the relevant theoretical solutions. It is easily deduced from the first figure that the waves have restored their original shape after their interaction.

- with different amplitudes \( A_1 = 0.2, A_2 = 0.4 \) placed at the initial positions \( x_0^1 = 0 \) and \( x_0^2 = -50 \) moving towards each other with velocities \( c_1 = 1.0645 \) and \( c_2 = -1.1255 \) respectively from time \( t = 0 \) to \( t = 36 \) with \( U_B \approx 0.310E + 00 \). The numerical results are given in Table 6 from which it is also deduced that both Methods I-II have given almost the same numerical results (see also Fig. 5). Again on the left-hand side of Fig. 6 it is given
c = 1

It was examined only the single-soliton wave. The constant of linearization shape after their interaction. Obviously the waves have restored their original shape after their interaction.

Table 4 Double-soliton Method I: Results for BB with h = 4 and \( \ell = 0.01 \), when the theoretical parameters are \( A_1 = A_2 = 0.369, b_1 = b_2 = 0.5, c_1 = -c_2 = 1.11624 \) and \( x^0 = 0, x^2 = -50 \).

<table>
<thead>
<tr>
<th>( t )</th>
<th>( \lambda )</th>
<th>( \mu_1 )</th>
<th>( \mu_2 )</th>
<th>( \mu_3 )</th>
<th>( \epsilon )</th>
<th>( L_2 )</th>
<th>( \epsilon )</th>
<th>type of error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.70460E−02</td>
<td>0.99053E−02</td>
<td>0.21E+01</td>
<td>min</td>
</tr>
<tr>
<td>0.06</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.49212E−02</td>
<td>0.85108E−02</td>
<td>0.25E+01</td>
<td>min</td>
</tr>
<tr>
<td>36</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0.47721E−01</td>
<td>0.10111E+00</td>
<td>0.64E+02</td>
<td>min</td>
</tr>
</tbody>
</table>

Table 5 Double-soliton Method II: Results for BB with h = 4 and \( \ell = 0.01 \), when the theoretical parameters are \( A_1 = A_2 = 0.369, b_1 = b_2 = 0.5, c_1 = -c_2 = 1.11624 \), \( x^0 = 0 \) and \( x^2 = -50 \).

<table>
<thead>
<tr>
<th>( t )</th>
<th>( \lambda )</th>
<th>( \mu_1 )</th>
<th>( \mu_2 )</th>
<th>( \mu_3 )</th>
<th>( \epsilon )</th>
<th>( L_2 )</th>
<th>( \epsilon )</th>
<th>type of error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.70460E−02</td>
<td>0.99053E−02</td>
<td>0.21E+01</td>
<td>min</td>
</tr>
<tr>
<td>0.06</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.49212E−02</td>
<td>0.85108E−02</td>
<td>0.25E+01</td>
<td>min</td>
</tr>
<tr>
<td>36</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0.47721E−01</td>
<td>0.10111E+00</td>
<td>0.64E+02</td>
<td>min</td>
</tr>
</tbody>
</table>

3.2 Numerical results for the GB equation

It was examined only the single-soliton wave. The constant of linearization \( U_B \) was taken to be \( U_B = \max_{m=1,2,\ldots,N} u^0_m \approx -0.152E - 24 \). Eq. (1.2) was solved for both Methods I-II (see Tables 7 and 8) to times \( t = 1.2, 36 \).
Fig. 3  BB Double-soliton: the curve shows the error $e$ using Methods I-II from time $t = 0$ to $t = 36$, when $A_1 = A_2 = 0.369, b_1 = b_2 = 0.5, c_1 = -c_2 = 1.11624, x_1^0 = 0, x_2^0 = -50$ and parameter values $\lambda = 0, \mu_1 = 1, \mu_2 = 0$ and $\mu_3 = 1$.

Fig. 4  BB Double-soliton: the surface on Fig. 4a shows the numerical solution $U$ for BB using Method II, when $A_1 = A_2 = 0.369, b_1 = b_2 = 0.5, c_1 = -c_2 = 1.11624, x_1^0 = 0, x_2^0 = -50$ from time $t = 0$ to $t = 36$ and parameter values $\lambda = 0, \mu_1 = 1, \mu_2 = 0$ and $\mu_3 = 1$, while on Fig. 4b the relevant theoretical solution $u$.

Table 6  Double soliton: Results for BB with $h = 4$ and $\ell = 0.01$, when the theoretical parameters are $A_1 = 0.2, A_2 = 0.4, b_1 = b_2 = 0.5, c_1 = 1.0645, c_2 = -1.1255, x_1^0 = 0$ and $x_2^0 = -50$ with parameter $\lambda = 0$ at time $t = 36$.  

<table>
<thead>
<tr>
<th>Method</th>
<th>$\mu_1$</th>
<th>$\mu_2$</th>
<th>$\mu_3$</th>
<th>$\epsilon$</th>
<th>$L_2$</th>
<th>$\epsilon$</th>
<th>type of error</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0.48594 $E-01$</td>
<td>0.92567 $E-01$</td>
<td>0.58 $E+02$</td>
<td>min</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.48600 $E-01$</td>
<td>0.95261 $E-01$</td>
<td>0.58 $E+02$</td>
<td>max</td>
</tr>
<tr>
<td>II</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0.48594 $E-01$</td>
<td>0.92567 $E-01$</td>
<td>0.58 $E+02$</td>
<td>min</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0.48600 $E-01$</td>
<td>0.95262 $E-01$</td>
<td>0.58 $E+02$</td>
<td>max</td>
</tr>
</tbody>
</table>

and 72 with theoretical parameters $A = 0.369, x_1^0 = 0, b = -0.5$, velocity $c = 0.8683$, space step $h = 4$ and time step $\ell = 0.01$. With respect to $\epsilon$ and for long time periods it is deduced from Tables 7-8 that more accurate
Fig. 5  BB Double-soliton: the curve shows the error $e$ using Methods I-II from time $t = 0$ to $t = 36$, when $A = 0.369$, $b_1 = b_2 = 0.5$, $c_1 = 1.0644$, $c_2 = -1.1255$, $x_1 = 0$, $x_2 = -50$ and parameter values $\lambda = 0$, $\mu_1 = 1$, $\mu_2 = 0$ and $\mu_3 = 1$.

results were obtained using Method I (see also Fig. 7 where the continuous curve shows $e$ from time $t = 0$ to $t = 72$ using Method I). It is seen in Fig. 8 that no spurious oscillations are observed when $t < 36$.

4 Conclusions

In this paper we have presented a parametric finite-difference scheme defined by Eqs. (2.4)–(2.7) for the numerical solution of the BS equation in one dimension. The resulting three-time level scheme was analyzed for stability and convergence in subsections 2.3.2-2.3.3 respectively and the necessary intervals arising from this analysis for both the time and the space steps were evaluated.

Then in subsection 2.4 two linearized numerical schemes based the first on an arbitrary value of the unknown solution $U^{n+1}$ and the second on Taylor’s expansion of the nonlinear term of BS equation were applied. Both schemes were analyzed for stability and convergence. The resulting linear schemes were applied successively to known from the bibliography schemes (see Bratsos [3], [4]). It has been deduced from the experiments that there is an improvement in the accuracy of the numerical solution for both the single and the double-soliton waves.
Table 7  Single-soliton Method I: Results for GB with $h = 4$ and $\ell = 0.01$ when the theoretical parameters are $A = 0.369$, $b = -0.5$, $c = 0.8683$ and $x_1^0 = 0$.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$\lambda$</th>
<th>$\mu_1$</th>
<th>$\mu_2$</th>
<th>$\mu_3$</th>
<th>$\epsilon$</th>
<th>$L_2$</th>
<th>$\epsilon$</th>
<th>type of error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.2</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0.75076$E - 02$</td>
<td>0.10445$E - 01$</td>
<td>0.22$E + 01$</td>
<td>min</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.17172$E - 01$</td>
<td>0.21104$E - 01$</td>
<td>0.50$E + 01$</td>
<td>max</td>
</tr>
<tr>
<td>0.12</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0.26994$E - 02$</td>
<td>0.37089$E - 02$</td>
<td>0.12$E + 01$</td>
<td>min</td>
<td></td>
</tr>
<tr>
<td>0.06</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0.16532$E - 01$</td>
<td>0.20597$E - 01$</td>
<td>0.48$E + 01$</td>
<td>max</td>
<td></td>
</tr>
<tr>
<td>36</td>
<td>0</td>
<td>0</td>
<td>0.6</td>
<td>1</td>
<td>0.14149$E + 00$</td>
<td>0.25140$E + 00$</td>
<td>0.40$E + 02$</td>
<td>min</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0.19968$E + 00$</td>
<td>0.33226$E + 00$</td>
<td>0.56$E + 02$</td>
<td>max</td>
<td></td>
</tr>
<tr>
<td>72</td>
<td>0</td>
<td>0</td>
<td>0.7</td>
<td>1</td>
<td>0.13021$E + 00$</td>
<td>0.32390$E + 00$</td>
<td>0.40$E + 02$</td>
<td>min</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0.19201$E + 00$</td>
<td>0.37533$E + 00$</td>
<td>0.59$E + 02$</td>
<td>max</td>
<td></td>
</tr>
</tbody>
</table>

Table 8  Single-soliton Method II: Results for GB with $h = 4$ and $\ell = 0.01$ when the theoretical parameters are $A = 0.369$, $b = -0.5$, $c = 0.8683$ and $x_1^0 = 0$.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$\lambda$</th>
<th>$\mu_1$</th>
<th>$\mu_2$</th>
<th>$\mu_3$</th>
<th>$\epsilon$</th>
<th>$L_2$</th>
<th>$\epsilon$</th>
<th>type of error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.2</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0.17171$E - 01$</td>
<td>0.21104$E - 01$</td>
<td>0.50$E + 01$</td>
<td>min</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.17171$E - 01$</td>
<td>0.21104$E - 01$</td>
<td>0.50$E + 01$</td>
<td>max</td>
</tr>
<tr>
<td>0.12</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0.15681$E - 01$</td>
<td>0.19955$E - 01$</td>
<td>0.45$E + 01$</td>
<td>min</td>
<td></td>
</tr>
<tr>
<td>0.06</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0.16532$E - 01$</td>
<td>0.20597$E - 01$</td>
<td>0.48$E + 01$</td>
<td>max</td>
<td></td>
</tr>
<tr>
<td>36</td>
<td>0</td>
<td>0</td>
<td>0.2</td>
<td>0</td>
<td>0.19968$E + 00$</td>
<td>0.33225$E + 00$</td>
<td>0.56$E + 02$</td>
<td>min</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0.2</td>
<td>0.19968$E + 00$</td>
<td>0.33225$E + 00$</td>
<td>0.56$E + 02$</td>
<td>max</td>
</tr>
<tr>
<td>72</td>
<td>0</td>
<td>0</td>
<td>0.2</td>
<td>0</td>
<td>0.19201$E + 00$</td>
<td>0.37533$E + 00$</td>
<td>0.59$E + 02$</td>
<td>min</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0.19201$E + 00$</td>
<td>0.37533$E + 00$</td>
<td>0.59$E + 02$</td>
<td>max</td>
<td></td>
</tr>
</tbody>
</table>

This is obvious especially at the numerical solution of the GB equation. Finally, it should be noticed that even if both methods have given approximately the same numerical results, the first one is more accurate.

The methods can be easily extended to solve the Boussinesq equation in $(2+1)$ dimensions and this is a work under preparation which is going to be sent for publishing very soon.

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Fig. 7 GB Single-soliton: the continuous curve shows the error $e$ using Method I from time $t = 0$ to $t = 72$, when $A = 0.369$, $b = -0.5$, $c = 0.8683$, $x_0^3 = 0$ and parameter values $\lambda = 0$, $\mu_1 = 0$, $\mu_2 = 0.7$ and $\mu_3 = 1$, while the dashed one using Method II with $\lambda = 0$, $\mu_1 = 0$, $\mu_2 = 0$ and $\mu_3 = 0$.

Fig. 8 GB Single-soliton: The surface on Fig. 8a shows the numerical solution $-U$ using Method I, when $A = 0.369$, $b = -0.5$, $c = 0.8683$, $x_0^3 = 0$, $h = 4$, $\ell = 0.01$ from time $t = 0$ to $t = 72$ and parameters $\lambda = 0$, $\mu_1 = 0$, $\mu_2 = 0.7$ and $\mu_3 = 1$, while on Fig. 8b the relevant theoretical solution $-u$. 
References