A PROBLEM OF ARRANGEMENTS ON CHESSBOARDS
AND GENERALIZATIONS

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Received 14 April 1978

For a unified approach to the study of the distributions of like objects on chessboards, the
numbers defined by

\[ C(k, s; m, n) = \frac{1}{m!n!} \left[ \Delta_m^s \Delta_n^s (sw) \right]_{a=0, \ldots, 0}. \]

where \( \Delta_n \) denotes the partial (with respect to \( u \)) difference operator, are examined. This
approach facilitates the treatment of further generalizations of such problems.

1. Introduction

In [1], Carlitz, Roselle and Scoville have studied the number, \( S(k; m, n) \), of
ways of distributing \( k \) like objects into an \( m \) by \( n \) rectangle with no lines (rows or
columns) empty, by means of the definition

\[ \binom{x+y+k-1}{k} = \sum_{m=0}^{k} \binom{x}{m} \binom{y}{n} S(k; m, n). \]

On the other hand, Riordan and Stein [4] have studied the number, \( A(k; m, n) \),
of ways of distributing \( k \) like objects into an \( m \) by \( n \) rectangle with no lines
empty, when each square cell has at most one object, by means of the definition

\[ \binom{x+y}{k} = \sum_{m=0}^{k} \sum_{n=0}^{k} \binom{x}{m} \binom{y}{n} A(k; m, n). \]

It is not difficult to verify that

\[ A(k; m, n) = \frac{1}{k!} \left[ \Delta_m^u \Delta_n^v (uv) \right]_{a=0, \ldots, 0}, \]

and

\[ S(k; m, n) = \frac{(-1)^k}{k!} \left[ \Delta_m^u \Delta_n^v (-uv) \right]_{a=0, \ldots, 0}. \]
For unifying the study of these numbers and extending their properties, we introduce the numbers

\[ C(k, s; m, n) = \frac{1}{m! n!} [\Delta^m_u \Delta^n_v(suv)_{k}]_{u=0, v=0}, \]

where \( s \) is a real number. Note that the differences in this definition were divided by \( m! n! \) instead of \( k! \) by analogy to the definitions of the (usual) Stirling numbers. In addition to this unification, the numbers \( C(k, s; m, n) \) are of interest in themselves. When \( s \) is a positive or negative integer, a useful combinatorial interpretation is given for these numbers.

2. The numbers \( C(k, s; m, n) \)

From the symbolic formula

\[ E^s_i E^c_j = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(bx)(cy)}{m! n!} \Delta^m_u \Delta^n_v \]

and

\[ [E^s_i E^c_j(suv)_{k}]_{u=0, v=0} = (axy)_k, \quad s = a/bc, \]

we get

\[ (axy)_k = \sum_{m=0}^{k} \sum_{n=0}^{k} \frac{(bx)_m (cy)_n [\Delta^m_u \Delta^n_v(suv)_{k}]}{m! n!} \]

Denoting the number in brackets by

\[ C(k, s; m, n) = \frac{1}{m! n!} [\Delta^m_u \Delta^n_v(suv)_{k}]_{u=0, v=0}, \quad (2.1) \]

we have

\[ (axy)_k = \sum_{m=0}^{k} \sum_{n=0}^{k} C(k, s; m, n)(bx)_m (cy)_n, \quad s = a/bc. \quad (2.2) \]

From (2.1) we deduce immediately that

\[ C(0, s; m, n) = \delta_m \delta_n \delta_{st}, \quad C(k, s; m, 0) = C(k, s; 0, n) = 0, \quad \text{if } k > 0, \]

\[ C(k, s; m, n) = 0, \quad \text{if } k < m \text{ and/or } k < 1. \quad (2.3) \]
Using the symbolic formula

$$\Delta^m \Delta^n = \sum_{i=0}^{m} \sum_{j=0}^{n} (-1)^{m+n-i-j} \binom{m}{i} \binom{n}{j} E^i E^j$$

and (2.1), we obtain the explicit formula

$$C(k, s; m, n) = \frac{1}{m! n!} \sum_{i=0}^{m} \sum_{j=0}^{n} (-1)^{m+n-i-j} \binom{m}{i} \binom{n}{j} (sij)_k. \quad (2.4)$$

Taking $f(u, v) = (suv)_k$ and $g(u, v) = (suv - k)$ in

$$\Delta^m \Delta^n f(u, v) g(u, v) = \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} \Delta^i \Delta^j f(u, v) \Delta^{m-i} E^i \Delta^{n-j} E^j g(u, v),$$

we get

$$\Delta^m \Delta^n (suv)_{k+1} = [s(u + m)(v + n) - k] \Delta^m \Delta^n (suv)_k + sn(u + m) \Delta^m \Delta^{n-1} (suv)_k$$
$$+ sm(v + n) \Delta^{m-1} \Delta^n (suv)_k + smn \Delta^{m-1} \Delta^{n-1} (suv)_k.$$ 

Dividing both members of this equation by $m! n!$ and putting $u = 0, v = 0$, we obtain by virtue of (2.1) the following recurrence relation

$$C(k + 1, s; m, n) = (smn - k) C(k, s; m, n) + sm C(k, s; m, n - 1)$$
$$+ sn C(k, s; m - 1, n) + s C(k, s; m - 1, n - 1). \quad (2.5)$$

Using this relation and the initial conditions (2.3), we can obtain successively these numbers. Moreover, when $s$ is a positive integer, since $C(k, s; m, n) = 0$, if $k > smn$, the numbers $s^{-k} C(k, s; m, n)$ are positive integers; $s^{-k} C(k, s; m, n)$ are also positive integers when $s$ is a negative integer.

From [3, p. 43] we have

$$(suv)_k = \sum_{r=0}^{k} L(k, r)(-suv)_r \quad (2.6)$$

where $L(k, r)$ denote the Lah numbers which may be defined by

$$L(k, r) = \frac{1}{r!} [\Delta^r (-u)]_{u=0}. \quad (2.6)$$

Performing the operation $\Delta^m \Delta^n$ on both sides of (2.6), we get by virtue of (2.1) the
following relation

\[ C(k, s; m, n) = \sum_{r=0}^{k} L(k, r)C(r, -s; m, n) \]  \hspace{1cm} (2.7)

which is a generalization of Eq. (14) and its inverse in [4]. A further generalization of (2.7) may be obtained by using the relation (Eq. (1.3) in [2])

\[ (\mu uv)_k = \sum_{r=1}^{k} C(k, r, \mu/\nu)(\nu uv)_r, \]

where

\[ C(k, r, s) = \frac{1}{r!} \left[ \Delta_r'(su)_k \right]_{n-0}. \]  \hspace{1cm} (2.8)

We get

\[ C(k, \mu; m, n) = \sum_{r=0}^{k} C(k, \mu/\nu)C(r, \nu; m, n) \]  \hspace{1cm} (2.9)

which for \( \mu = s, \nu = -s \) and since \( C(k, r, -1) = L(k, r) \), reduces to (2.7).

From (2.1) and using (2.8) we obtain

\[ C(\nu, s; m, n) = \sum_{r=0}^{m} (-1)^{m-r} \binom{m}{r} C(k, n, rs), \]  \hspace{1cm} (2.10)

which along with formula (Eq. (6.17) in [2])

\[ C(k, n, rs) = \sum_{k_1! \cdots k_n!} \frac{k!}{k_1! k_2! \cdots k_n!} \binom{rs}{1}^{k_1} \binom{rs}{2}^{k_2} \cdots \binom{rs}{n}^{k_n}, \]

where the summation is over all partitions of \( k \) with \( n \) parts, gives in particular \( (s = 1) \) the formula used in [4] in connection with the “Symmetrical tables of binomial coefficients” of MacMahon.

Expanding \((\nu uv)_k\) into powers of \( uv\):

\[ (\nu uv)_k = \sum_{r=0}^{k} s(k, r)(\nu uv)^r \]

with \( s(k, r) = D^r(0)/k! \), the Stirling numbers of the first kind, and performing the
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operation $\Delta_u^m \Delta_u^n$, we get

\[
C(k, s; m, n) = \sum_{r=0}^{k} s^r s(k, r) S(r, m) S(r, n)
\]

(2.11)

with $S(r, m) = \Delta^m 0'/m!$, the Stirling numbers of the second kind. This equation reduces for $s = -1, 1$ to equations (8.11) in [1] and (10) in [4], respectively.

Some special cases of the numbers $C(k, s; m, n)$ are worth noting: Putting $n = 1$ in (2.1) and using (2.8) we get

\[
C(k, s; m, 1) = C(k, m, s).
\]

If we put $n = k$ we get

\[
C(k, s; m, k) = s^k S(k, m).
\]

Finally, the following limiting expressions can be easily verified, for example, by using the limiting properties for the numbers $C(k, n, r)$ (see [2]) and (2.10):

\[
\lim_{s \to +\infty} s^{-k} C(k, s; m, n) = \lim_{s \to -\infty} s^{-k} C(k, s; m, n) = S(k, m) S(k, n),
\]

\[
\lim_{s \to -0} s^{-n} C(k, s; m, n) = s(k, n) S(m, n), \quad m \geq n,
\]

\[
\lim_{s \to +0} s^{-n} C(k, s; m, n) = s(k, n) S(m, n), \quad m \geq n.
\]

3. Generating functions

Consider first the generating function

\[
G_k(x, y) = \sum_{m=0}^{k} \sum_{n=0}^{k} C(k, s; m, n)(x, m)(y, n).
\]

From (2.2) with $b = c = 1$ we find

\[
G_k(x, y) = (sxy)_k.
\]

(3.1)

Therefore the generating function

\[
G(x, y, t) = \sum_{k=0}^{\infty} \sum_{m=0}^{k} \sum_{n=0}^{k} C(k, s; m, n)(x, m)(y, n)t^k/k!
\]

(3.2)
is given by
\[ G(x, y, t) = (1 + t)^{xy}. \] (3.3)

Also the generating functions
\[ G_{m,n}(t) = \sum_{k=0}^{\infty} C(k, s; m, n) t^k / k! \]
and
\[ F_{k,m}(y) = \sum_{n=0}^{k} C(k, s; m, n) (y)_n, \]
are given by
\[ G_{m,n}(t) = \left[ \Delta^m \Delta^n (1 + t)^{suy} \right]_{t=0} \]
\[ = \sum_{r=0}^{n} (-1)^r \binom{n}{r} [(1 + t)^{sy} - 1]^r, \] (3.4)
and
\[ F_{k,m}(y) = \frac{1}{k!} \left[ D^k \Delta^m (1 + t)^{suy} \right]_{t=0} \]
\[ = C(k, m, sy), \] (3.5)
respectively.

4. Applications

Consider a rectangular array of \( m \) by \( n \) square cells and suppose that each cell has \( s \) different compartments. In this case the enumerator of a single cell for the unrestricted occupancy by like objects is \((1 - t)^s\) and for the restricted occupancy—of at most one object in a compartment—is \((1 + t)^s\). Hence the enumerator for \( xy \) unlike cells is \((1 - t)^{sxv}\) and \((1 + t)^{sxv}\), respectively and from (3.2) and (3.3) it follows that the number of ways of distributing \( k \) like objects into an \( m \) by \( n \) rectangular array of cells, with \( s \) different compartments each, and with no lines (rows or columns of cells) empty is given by \( m! n! C(k, s; m, n)/k! \) for the restricted occupancy and by \( m! n! (-1)^k C(k, s; m, n)/k! \) for the unrestricted occupancy.
We may arrive at the same conclusion by using the inclusion-exclusion principle and Eq. (2.10) if we observe that the number of ways of distributing \( k \) like objects into \( n \) cells, with \( rs \) different compartments each and with no cell empty, is given by

\[
\frac{n!}{k!} C(k, n, rs) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \binom{rsj}{k}
\]

for the restricted occupancy, and by

\[
\frac{n!}{k!} (-1)^k C(k, n, -rs) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \binom{rsj + k - 1}{k}
\]

for the unrestricted occupancy.

The above combinatorial interpretation of the numbers \( C(k, s; m, n) \) when \( s \) is a positive or negative integer motivated the following problem which is a generalization of the problem of distributions of like objects into a rectangular array of square cells: Consider a rectangular (in three dimensions) array of \( m \times n \times r \) cubic cells. Using the inclusion-exclusion principle we conclude that the number of ways of distributing \( k \) like objects into this array of cubic cells with no column empty in any of the three directions, is equal to

\[
\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} \frac{m! n!}{k!} C(k, s; m, n)
\]

for the restricted occupancy—and at most one object in a cell—and to

\[
\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} \frac{m! n!}{k!} (-1)^k C(k, -s; m, n)
\]

for the unrestricted occupancy. These two numbers are equal to

\[
m! n! r! C(k, 1; m, n, r)/k! \quad \text{and} \quad m! n! r! (-1)^k C(k, -1; m, n, r)/k!
\]

respectively, where

\[
C(k, s; m, n, r) = \frac{1}{m! n! r!} \left[ \Delta''''_{\omega}(suvw)_{k} \right]_{u=v=w=n}
\]

The study of these numbers is similar to that of \( C(k, s; m, n) \).
References


