Full automorphism group of generalized unitary graphs

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\section{Introduction}

Let $F_{q^2}$ be a finite field with $q^2$ elements, where $q$ is a prime power. For any $m \times n$ matrix $A = (a_{ij})$ over $F_{q^2}$, we write the transpose of $A$ by $A^t$, the rank of $A$ by $r(A)$. For any automorphism $\pi$ of $F_{q^2}$, we write $\pi(A) = (\pi(a_{ij}))$. It is well known that $F_{q^2}$ has an involutive automorphism $- : a \mapsto \bar{a} = a^q$. We often write $\tilde{A} = (\bar{a}_{ij})$, $A^* = (\tilde{A})^t$. Suppose

$$H = \begin{pmatrix}
0 & I^{(m)} \\
I^{(m)} & 0 \\
I^{(n-2m)} & 0
\end{pmatrix}.$$
The set of all $n \times n$ matrices $T$ over $\mathbb{F}_{q^2}$ satisfying $THT^* = H$ forms a group with respect to matrix multiplication, called the unitary group of degree $n$ over $\mathbb{F}_{q^2}$ with respect to $H$, denoted by $U_n(\mathbb{F}_{q^2})$.

There is a natural action of $U_n(\mathbb{F}_{q^2})$ on the $n$-dimensional row vector space $\mathbb{F}_{q^2}^n$ by the vector matrix multiplication as follows:

$$
\mathbb{F}_{q^2}^n \times U_n(\mathbb{F}_{q^2}) \rightarrow \mathbb{F}_{q^2}^n
$$

$$(\alpha, T) \mapsto \alpha T := \sigma T(\alpha).$$

$\mathbb{F}_{q^2}^n$ together with the action is called the $n$-dimensional unitary space over $\mathbb{F}_{q^2}$. A matrix representation of an $m$-dimensional subspace $P$ is a matrix whose rows form a basis of $P$. If there is no danger of confusion, we use the same symbol to denote a subspace and its matrix representation. An $m$-dimensional subspace $P$ is a subspace of type $(m, r)$ if $r(PHQ^*) = r$. In particular, subspaces of type $(m, 0)$ are called $m$-dimensional totally isotropic subspaces.

The unitary graph $U(n, q^2)$ has as its vertex set the set of 1-dimensional isotropic subspaces of $\mathbb{F}_{q^2}^n$, and two vertices $\alpha$ and $\beta$ are adjacent if and only if $\alpha \beta^* \neq 0$. Wan and Zhou [8] proved that its full isomorphism group is $P\Gamma U_n(\mathbb{F}_{q^2})$ when $n \neq 4, 5$. As a generalization of unitary graphs, Yang [10] defined the generalized unitary graph $GU_n(q^2, m)$, by taking all the $m$-dimensional totally isotropic subspaces as vertices with two vertices $P$ and $Q$ adjacent if $r(PHQ^*) = 1$ and $\dim(P \cap Q) = m - 1$.

By [9, Theorem 3.1] $GU_n(q^2, m)$ is arc-transitive. In this paper, we determine its full automorphism group.

**Theorem 1.1.** Let $m$ and $n$ be positive integers with $n - 2m \geq 4$ and $m \geq 2$. Then the full automorphism group of the generalized unitary graph $GU_n(q^2, m)$ is $P\Gamma U_n(\mathbb{F}_{q^2})$.

Godsil and Royle [1,2] initiated the study of symplectic graphs. Tang and Wan [4] determined the full automorphism group of symplectic graphs. Recently the full automorphism group of the orthogonal graphs was determined in [3,7].

### 2. Subconstituent

Let $\Gamma'$ denote the generalized unitary graph $GU_n(q^2, m)$ with the vertex set $V$. In this section we shall discuss the relationship between any two subconstituents with respect to a given vertex $M$. By the arc-transitivity of $\Gamma'$, we may choose

$$
M = \begin{pmatrix}
m & m & n - 2m \\
1 & 0 & 0
\end{pmatrix}.
$$

For nonnegative integers $r$ and $t$ with $r + t \leq m$, let

$$
S(r, t) = \{X \in V \mid r(MHX^*) = r, \dim(M \cap X) = t\}.
$$

Each vertex $X$ of $S(r, t)$ is of form

$$
X = \begin{pmatrix}
m & m & n - 2m \\
B_1 & 0 & B_3 \\
A_1 & A_2 & A_3 \\
A & 0 & 0
\end{pmatrix}
$$

$$
\begin{pmatrix}
m - (r + t) \\
r \\
t
\end{pmatrix},
$$

where $r(A) = t, r(A_2) = r, r(B_3) = m - (r + t), AA_2^* = 0, B_3B_3^* = 0, A_2B_1^* + A_3B_3^* = 0$ and $A_2A_1^* + A_1A_2^* + A_3A_3^* = 0$. 

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Let \( \Gamma_k(P) \) denote the set of vertices at distance \( k \) from \( P \), which is called the \( k \)-th subconstituent of \( \Gamma \) with respect to \( P \). Each vertex in \( S(r, t) \) is at distance \( 2m - 2t - r \) from \( M \), so

\[
\Gamma_k(M) = \bigcup_{2m - 2t - r = k} S(r, t).
\]

For convenience let \( \Gamma(P) \) denote the neighborhood of \( P \). If \( S \) is a subset of \( V \), assume that \( \Gamma(S) = \bigcup_{P \in S} \Gamma(P) \).

We will employ the matrix method in our discussion. So we fix the following notation: 0 represents the number 0, a zero vector or a zero matrix whose size can be inferred from the context; \( e_1, e_2, \ldots \) denote the row vectors of an identity matrix, the size of which can also be inferred from the context; and each \( E_{ij} \) denotes the \( m \times m \) matrix whose \((i, j)\)th entry is 1 and other entries are 0. We will use lower case Greek letters \( \alpha, \beta, \gamma, \ldots \) to denote vectors and lower case of Latin letters \( a, b, c, \ldots \) to denote elements in \( \mathbb{F}_{q^2} \).

**Lemma 2.1.** The first subconstituent \( \Gamma_1(M) \) consists of vertices of form

\[
\Gamma(i, \gamma; a, \beta) = \begin{pmatrix}
    i - 1 & 1 & m - i & i - 1 & 1 & m - i & n - 2m \\
    0 & a & 0 & 0 & 1 & \gamma & \beta \\
    1 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & -\gamma^* & I & 0 & 0 & 0 & 0 \\
\end{pmatrix},
\]

where \( a^* + a + \beta \beta^* = 0 \).

**Proof.** By (1) each vertex \( X \) in \( \Gamma_1(M) \) has the form

\[
X = \begin{pmatrix}
    \alpha_1 & \alpha_2 & \alpha_3 \\
    A & 0 & 0 \\
\end{pmatrix},
\]

where \( r(A) = m - 1, \alpha_2 \neq 0 \) and \( A\alpha_2^* = 0 \). Since \( r(A) = m - 1 \), there exists a \( T' \in \text{GL}_{m-1}(\mathbb{F}_{q^2}) \) such that

\[
T'A = \begin{pmatrix}
    I^{(i-1)} & 0 & 0 \\
    0 & -\gamma^* & I^{(m-i)} \\
\end{pmatrix}.
\]

It follows that there exists a \( T \in \text{GL}_m(\mathbb{F}_{q^2}) \) satisfying \( TX \) is of form (2). \( \square \)

For given \( i = 1, 2, \ldots, m \) and \( \gamma \in \mathbb{F}_{q^2}^{m-i} \), suppose \( \Gamma(i, \gamma) \) denotes the set of vertices with form (2). In particular, if \( i = m \), then \( \gamma = \emptyset \). By [11, Theorem 3.3] each induced subgraph on each \( \Gamma(i, \gamma) \) is connected and there is no edge between any two distinct subgraphs. Let

\[
\overline{\Gamma}(i, \gamma) = \begin{pmatrix}
    i - 1 & 1 & m - i & i - 1 & 1 & m - i & n - 2m \\
    0 & a & 0 & 0 & 0 & 0 & \beta \\
    1 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & -\gamma^* & I & 0 & 0 & 0 & 0 \\
\end{pmatrix} | \beta \neq 0, \beta \beta^* = 0 \).
\]

Then all \( \overline{\Gamma}(i, \gamma) \) form a partition of \( S(0, m - 1) \) and there is no edge between the subgraphs induced on two distinct sets.
For \( 1 \leq i < j \leq m \) and \( \eta = (\eta_1 \quad \eta_2) \), let \( \Gamma(i, j, \eta) \) be the set of vertices
\[
\begin{pmatrix}
0 & a_1 & 0 & b_1 & 0 & 0 & 1 & \eta_{11} & 0 & \eta_{12} & \delta_1 \\
0 & a_2 & 0 & b_2 & 0 & 0 & 0 & 0 & 1 & \eta_2 & \delta_2 \\
f^{(i-1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\eta_{11}^* & f^{(j-i-1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\eta_{12}^* & 0 & -\eta_{22}^* & f^{(m-j)} & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]
denoted by \( \Gamma(i, j, \eta; a_1, a_2, b_1, b_2, \delta_1, \delta_2) \), where \( \delta_1, \delta_2 \in \mathbb{F}_q^{(n-2m)} \), \( a_1^* + a_1 + \delta_1 \delta_1^* = 0, a_2^* + b_1 + \delta_1 \delta_2^* = 0 \), and \( b_2^* + b_2 + \delta_2 \delta_2^* = 0 \). All \( \Gamma(i, j, \eta) \) form a partition of \( S(2, m-2) \).

**Lemma 2.2.** (i) Let \( 0 \leq t \leq m-1 \) and \( n - 4m + 2r + 2t - 2 \geq 0 \). For each vertex \( P \) in \( S(r-1, t) \), the set \( \Gamma(P) \cap S(r, t) \) has \( q^2(2^{m-r-t}) + 1 \) vertices and is uniquely determined by vertex \( P \). In particular, \( \Gamma(P) \cap S(1, m-1) \subseteq \Gamma(i, \gamma) \) for each \( P \in \Gamma(i, \gamma) \).

(ii) For each vertex \( P \) in \( S(m-t, t) \), \( 0 \leq t \leq m-2 \), the set \( \Gamma(P) \cap S(m-(t+1), t+1) \) has \( q^{2(m-r-t)} + 1 \) vertices and is uniquely determined by vertex \( P \). In particular, for each vertex \( P = \Gamma(i, j, \eta; a_1, a_2, b_1, b_2, \delta_1, \delta_2) \in \Gamma(i, j, \eta) \) where \( \eta = (\eta_{11} \quad \eta_{12} \quad \eta_2) \), we have that \( \Gamma(P) \cap S(1, m-1) \) consists of vertices \( \Gamma(j, \eta_2; b_1, \delta_2) \) and \( \Gamma(i, \gamma; a, \beta) \), where \( a = (a_1 + ka_2) + \beta_1(b_1 + kb_2), \gamma = (\eta_{11} \quad \eta_{12} + k\eta_2), \beta = \delta_1 + k\delta_2, \) and \( k \in \mathbb{F}_q^2 \).

**Proof.** (i) By (1), \( P \) has the form
\[
P = \begin{pmatrix}
B_1 & 0 & B_3 \\
A_1 & A_2 & A_3 \\
A & 0 & 0
\end{pmatrix}
\]
where \( r(A) = t, r(A_2) = r-1, \) and \( AA_2^n = 0 \). Since \( r + t \leq m, m - r - t + 1 \) is a positive integer. So we may write
\[
P = \begin{pmatrix}
C_1 & 0 & C_3 \\
\alpha_1 & 0 & \alpha_3 \\
A_1 & A_2 & A_3 \\
A & 0 & 0
\end{pmatrix},
\]
The neighborhood of \( P \) in \( S(r, t) \) consists of vertices of form
\[
X = \begin{pmatrix}
C_1 & 0 & C_3 \\
\beta_1 & \beta_2 & \beta_3 \\
A_1 & A_2 & A_3 \\
A & 0 & 0
\end{pmatrix},
\]
where \( r(\beta_1 A_2^n) = r, A_1 B_2 = 0, A_1 \beta_2^* + A_2 \beta_1^* + A_3 \beta_3^* = 0, C_1 \beta_2^* + C_3 \beta_3^* = 0, \beta_1 \beta_2^* + \beta_2 \beta_1^* + \beta_3 \beta_3^* = 0, \) and \( \alpha_3 \beta_3^* + \alpha_1 \beta_2^* \neq 0 \). Then the number of 1-dimensional subspaces generated by \( (\beta_1, \beta_2, \beta_3) \) is
\[
\frac{q^{2(m-r-t)} - 1}{q^2 - 1} \cdot q^{2(n-4m+2r+2t-2)} (q^4 - 1) \cdot q^{2(m-r-t)} + 1,
\]
which is just the size of the neighborhood of \( P \) in \( S(r, t) \). From the discussion above \( \Gamma(P) \cap S(1, m-1) \subseteq \Gamma(i, \gamma) \).
(ii) Observe $P$ has the form

$$P = \begin{pmatrix} A_1 & 2 & A_3 \\ A & 0 & 0 \end{pmatrix} m - t,$$

where $r(A) = t$, $r(A_2) = m - t$ and $AA_2^\ast = 0$. Each vertex $X$ in $S(m - (t + 1), t + 1)$ adjacent to $P$ must have the form

$$X = \begin{pmatrix} A_1' & A_2' & A_3' \\ A & 0 & 0 \\ \alpha & 0 & 0 \end{pmatrix},$$

where $r \left( \begin{pmatrix} A \\ \alpha \end{pmatrix} \right) = t + 1$, $r(A_2') = m - (t + 1)$, $\begin{pmatrix} A \\ \alpha \end{pmatrix} (A_2')^* = 0$, and $\begin{pmatrix} A_1' & A_2' & A_3' \end{pmatrix}$ is a subspace of $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$.

Conversely, pick any vertex $X$ in $S(m - (t + 1), t + 1)$ with form (4). Rewrite

$$P = \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ A_1' & A_2' & A_3' \\ A & 0 & 0 \end{pmatrix}.$$  

From $r(A_2) = r \left( \begin{pmatrix} \beta_1 \\ A_2' \end{pmatrix} \right) = m - t$, we have $\beta_1 \notin A_2'$, which implies that $r(XP^*) = 1$. Therefore, $X$ is adjacent to $P$.

By above discussion, the neighborhood of $P$ in $S(m - (t + 1), t + 1)$ consists of all vertices of form (4). Since $\begin{pmatrix} A_1' & A_2' & A_3' \end{pmatrix}$ is determined by $A$ and $\alpha$, this neighborhood is uniquely determined by $P$; and so its size is the number of $(t + 1)$-dimensional subspaces containing a given $t$-dimensional subspace.

By [6, Corollary 1.9] the size is $\frac{q^{2(m-t)-1}}{q^t - 1}$. □

### 3. Full automorphism group

In this section we shall prove Theorem 1.1. For convenience, for each $i = 1, \ldots, m$ and $\gamma \in F_{q^2}$, write $N_{i,\gamma} = \Gamma(i, \gamma; 0, 0)$. Since $PGL_n(F_{q^2})$ acts arc-transitively on $\Gamma$, it suffices to show that the pointwise stabilizer of the set $\{M, N_{m,\theta}\}$ in $\text{Aut}(\Gamma)$ is a subgroup of $PGL_n(F_{q^2})$. We shall complete our proof by a series of lemmas.

**Lemma 3.1.** For any $\sigma \in \text{Aut}(\Gamma)_{M,N_{m,\theta}}$, there exists an element $\sigma' \in PGL_n(F_{q^2})$ such that $\sigma' \sigma \in \text{Aut}(\Gamma)_{M}$ is a setwise stabilizer of each $\Gamma(i, \gamma)$.

**Proof.** By [10, Theorem 3.3], $\sigma$ induces a permutation on the set $\Delta$ of all $\Gamma(i, \gamma)$. Then the one-to-one transformation $\hat{\sigma}$ induced by $\sigma$ on the projective space $PG(m - 1, F_{q^2})$ is equivalent to $\sigma$ on $\Delta$. By Lemma 2.2 (ii), $\sigma$ induces a permutation on the set of all $\Gamma(i, j, \eta)$, which implies that $\hat{\sigma}$ transforms lines to lines in $PG(m - 1, F_{q^2})$. By the fundamental theorem of the projective geometry [5, Theorem 2.23], there exist $\pi_0 \in \text{Aut}(F_{q^2})$ and $\tilde{T_0} \in PGL_m(F_{q^2})$ such that $\sigma_{T_0} \pi_0 \tilde{T} \sigma$ is the identity permutation on the point set of $PG(m - 1, F_{q^2})$. Pick
Then the permutation $\sigma T_0 \pi \sigma$ on the set $\Delta$ is equivalent to $\sigma T_0 \pi 0 \sigma$ on $PG(m-1, \mathbb{F}_{q^2})$. Since $\sigma T_0 \pi \sigma \in \text{Aut}(\Gamma)M$ is a setwise stabilizer of each $\Gamma(i, \gamma)$, $\sigma' = \sigma T_0 \pi 0$ is the desired element. □

**Lemma 3.2.** If $\sigma \in \text{Aut}(\Gamma)M$ is a setwise stabilizer of each $\Gamma(i, \gamma)$, then there exists an element $T \in PU_n(\mathbb{F}_{q^2})$ such that $\sigma T \sigma$ is a stabilizer of each vertex $N_{i, \gamma}$ and $\sigma T \sigma(\Gamma(i, \gamma)) = \Gamma(i, \gamma)$.

**Proof.** Combining Lemma 2.2 and [10, Lemma 2.4] we get $\sigma(S(m, 0)) = S(m, 0)$. Write $L = (0^{(m)} \ I^{(m)} \ 0^{(n-2m)})$. Then $\sigma(L) \in S(m, 0)$ has the form $(A \ I^{(m)} \ C)$, where $A + A^* + C^* = 0$. Pick

$$T = \begin{pmatrix} I^{(m)} & -C^* - A & -C \\ -C \ & I^{(m)} \ C^* \end{pmatrix} \in PU_n(\mathbb{F}_{q^2}).$$

Then $\sigma T \sigma(L) = L$. From the proof of Lemma 2.2 (ii) $N_{i, \gamma}$ is the unique vertex in $\Gamma(i, \gamma) \cap \Gamma_{m-1}(L)$. Since $\sigma T \sigma$ is a setwise stabilizer of each $\Gamma(i, \gamma)$, $\sigma T \sigma(N_{i, \gamma}) = N_{i, \gamma}$. Hence, $T$ is the desired element. □

In the remaining of this section, we always assume that $G_0$ is the set of elements $\sigma \in \text{Aut}(\Gamma)M$ such that $\sigma$ is a pointwise stabilizer of $\bigcup_{1 \leq k \leq m-1} \Gamma(k, 0) \cup \Gamma(m, 0)$ and $\sigma(\Gamma(k, j, 0)) = \Gamma(k, j, 0)$ for each $k < j$; suppose $G_1$ denotes the set of elements $\sigma \in \text{Aut}(\Gamma)M$ such that $\sigma$ is a setwise stabilizer of each $\Gamma(i, \gamma)$ and $\sigma$ fixes each vertex $N_{i, \gamma}$.

Let $e(i, \gamma) = \{M\} \cup \Gamma(i, \gamma) \cup \Gamma(i, \gamma)$. Then the induced subgraph on $e(i, \gamma)$ is isomorphic to the unitary graph $U(n-2m+2, q^2)$. If $\sigma \in \text{Aut}(\Gamma)M$ is a setwise stabilizer of each $\Gamma(i, \gamma)$, by Lemma 2.2 (i), $\sigma(\Gamma(i, \gamma)) = \Gamma(i, \gamma)$, which implies that $\sigma|_{e(i, \gamma)}$ induces an automorphism $\bar{\sigma}$ of $U(n-2m+2, q^2)$.

**Lemma 3.3.** For every element $\sigma \in G_1$, there exists an element $T \in PU_n(\mathbb{F}_{q^2})$ and an element $\pi \in \text{Aut}(\mathbb{F}_{q^2})$ such that $\sigma T \pi \sigma \in G_0$ and $\sigma T \pi \sigma(\Gamma(k, \gamma)) = \Gamma(k, \pi(\gamma))$.

**Proof.** Since $\sigma(e(i, 0)) = e(i, 0)$, by [8, Theorems 3.3 and 4.1] there exist $\pi \in \text{Aut}(\mathbb{F}_{q^2})$ and $\bar{T}_i \in PU_{n-2m+2}(\mathbb{F}_{q^2})$ such that $\sigma T \pi \bar{\sigma}$ is the identity permutation of the vertex set of $U(n-2m+2, q^2)$. The fact that $\sigma \in \text{Aut}(\Gamma)M$ and $\sigma(N_{i, 0}) = N_{i, 0}$ implies that $\bar{\sigma}$ stabilizes vertices $e_1$ and $e_2$; and so

$$\bar{T}_i = \begin{pmatrix} 1 & 1 \\ 1 & B_0 \end{pmatrix},$$

where $B_0 B_0^* = I^{(n-2m)}$. Pick

$$T_i = \begin{pmatrix} I^{(m)} & 0 \\ 0 & B_0 \end{pmatrix} \in PU_n(\mathbb{F}_{q^2}).$$

Since the permutation $\sigma T \pi \sigma$ on $e(i, 0)$ is equivalent to $\sigma T \pi \bar{\sigma}$ on $U(n-2m+2, q^2)$. Write $\sigma_1 = \sigma T \pi \sigma$, then $\sigma_1 \in \text{Aut}(\Gamma)M$ is a pointwise stabilizer of $\Gamma(i, 0)$ such that $\sigma_1(\Gamma(k, \gamma)) = \Gamma(k, \pi(\gamma))$.

For any $j \neq i$, without loss of generality we may assume that $j > i$. Since $\sigma_1(\Gamma(j, 0)) = \Gamma(j, 0)$, $\sigma_1(e(j, 0)) = e(j, 0)$. From the choice of $T_i$ we know $\sigma_1(N_{j, 0}) = N_{j, 0}$. So there exist $\pi_j \in \text{Aut}(\mathbb{F}_{q^2})$ and
such that $\sigma_7\pi_j\sigma_1$ is pointwise stabilizer on $\mathop{e}(j,0)$.

In the following we shall show that $B_j = I^{(n-2m)}$ and $\pi_j$ is the identity of $\text{Aut}(\mathbb{F}_q^2)$.

By Lemma (2.2) (ii), each vertex in $\Gamma(i,j,0)$ is uniquely determined by $q^2 + 1$ vertices in

$$\Gamma(j,0) \cup \bigcup_{b \in \mathbb{F}_q^2} \Gamma(i,(0^{(j-i-1)},b,0^{(m-j)})).$$

Since $\sigma_1(\Gamma(j,0)) = \Gamma(j,0)$ and $\sigma_1(\Gamma(i,(0^{(j-i-1)},b,0^{(m-j)}))) = \Gamma(i,(0^{(j-i-1)},\pi(b),0^{(m-j)}))$, we have $\sigma_1(\Gamma(i,j,0)) = \Gamma(i,j,0)$. For any vertex $P = \Gamma(i,0; a_p, \beta_p)$, the common neighborhood in $\Gamma(i,j,0)$ of $P$ and $N_{j,0}$ is

$$S_1 = \left\{ \Gamma(i,j,0; a_p, x, -x^*, 0, \beta_p, 0) \bigg| x \in \mathbb{F}_q^2 \right\}.$$

By $\sigma_1(P) = P$ and $\sigma_1(N_{j,0}) = N_{j,0}$ we have $\sigma_1(S_1) = S_1$. Since

$$S_2 = \Gamma(S_1) \cap \Gamma(i,(0^{(j-i-1)},1,0^{(m-j)}))$$

are $\sigma_1(\Gamma(i,(0^{(j-i-1)},1,0^{(m-j)}))) = \Gamma(i,(0^{(j-i-1)},1,0^{(m-j)}))$ and $\sigma_1(S_1) = S_1$ we have $\sigma_1(S_2) = S_2$. It follows that $\sigma_{B_j}\pi_j(\beta_p) = \beta_p$ for any $\beta_p \in \mathbb{F}_q^{(n-2m)}$. If we choose $\beta_p$ to be $e_1$, $e_2$, ..., $e_{n-2m}$ respectively, then $B_j = I^{(n-2m)}$. If we choose $\beta_p$ to be $ce_1$, where $c \in \mathbb{F}_q^2$, then $\pi_j(c) = c$ and $\pi_j$ is the identity of $\text{Aut}(\mathbb{F}_q^2)$. Thus $\sigma_1 = \sigma_7\pi \in (\text{Aut}(\Gamma))_M$ is a pointwise stabilizer on $\Gamma(i,0) \cup \Gamma(j,0)$. Repeating the discussion above for each $\Gamma(k,0), T = T_j$ and $\pi \in \text{Aut}(\mathbb{F}_q^2)$ are the desired elements. $\square$

**Lemma 3.4.** The element $\pi$ in Lemma 3.3 is the identity of $\text{Aut}(\mathbb{F}_q^2)$.

**Proof.** For any given $\sigma$ in Lemma 3.3, by Lemmas 3.3 there exists a $T \in \text{PU}_n(\mathbb{F}_q^2)$ such that $\sigma_1 = \sigma_7\pi \sigma \in G_0$ and $\sigma_1(\Gamma(k,\gamma)) = \Gamma(k,\pi(\gamma))$ for each $1 \leq k \leq m$ and $\gamma \in \mathbb{F}_q^{(m-k)}$.

The common neighborhood in $\Gamma(i,j,0)$ of $N_{j,0}$ and $\Gamma(j,0;0,e_1)$ is

$$S_3 = \left\{ \Gamma(i,j,0;0,a,-a^*,0,0,e_1) \bigg| a \in \mathbb{F}_q^2 \right\};$$

the common neighborhood in $\Gamma(i,j,0)$ of $\Gamma(i,0;0,be_1)$ and $N_{j,0}$ is

$$S_4 = \left\{ \Gamma(i,j,0;0,a,-a^*,0,be_1) \bigg| a \in \mathbb{F}_q^2 \right\}.$$

Since $\sigma_1(\Gamma(i,j,0)) = \Gamma(i,j,0)$ and $\sigma_1$ is a pointwise stabilizer of $\{M\} \cup \Gamma(j,0) \cup \Gamma(i,0)$, we have $\sigma_1(S_3) = S_3$ and $\sigma_1(S_4) = S_4$.

Let $S_5 = \Gamma(S_3) \cap \Gamma(i,(0^{(j-i-1)},b,0^{(m-j)}))$ and $S_6 = \Gamma(S_4) \cap \Gamma(i,(0^{(j-i-1)},b,0^{(m-j)}))$. By computation, we obtain

$$S_5 = S_6 = \left\{ \Gamma(i,(0^{(j-i-1)},b,0^{(m-j)});ba - b^*a*,be_1) \bigg| a \in \mathbb{F}_q^2 \right\}.$$

Let $S_7 = \Gamma(S_3) \cap \Gamma(i,(0^{(j-i-1)},\pi(b),0^{(m-j)}))$ and $S_8 = \Gamma(S_4) \cap \Gamma(i,(0^{(j-i-1)},\pi(b),0^{(m-j)}))$. Then

$$S_7 = \left\{ \Gamma(i,(0^{(j-i-1)},\pi(b),0^{(m-j)});\pi(b)a - (\pi(b))^*a^*,\pi(b)e_1) \bigg| a \in \mathbb{F}_q^2 \right\}.$$
\[ S_8 = \left\{ \Gamma(i, (0^{(j-i-1)}, \pi(b), 0^{(m-j)}); \pi(b)a - (\pi(b))^*a^*, be_1) \mid a \in \mathbb{F}_{q^2} \right\}. \]

Since \( \sigma_1(\Gamma(i, (0^{(j-i-1)}, b, 0^{(m-j)})) = \Gamma(i, (0^{(j-i-1)}, \pi(b), 0^{(m-j)})) \), we have \( \sigma_1(S_5) \subseteq S_7 \) and \( \sigma_1(S_6) \subseteq S_8 \), which implies \( S_7 \subseteq S_8 \) by the fact \( S_5 = S_6 \). Then be_1 = \( \pi(b)e_1 \), i.e., \( b = \pi(b) \) for any \( b \in \mathbb{F}_{q^2} \). Hence \( \pi \) is the identity automorphism of \( \mathbb{F}_{q^2} \). □

**Lemma 3.5.** If \( \sigma \in \text{Aut}(\Gamma)_M \) is a setwise stabilizer of each \( \Gamma(i, \gamma) \), there exists an element \( T \in PU_n(\mathbb{F}_{q^2}) \) such that \( \sigma_T \sigma \) is a pointwise stabilizer of \( \{M\} \cup S(1, m - 1) \).

**Proof.** By Lemmas 3.2–3.4, there exists a \( T \in PU_n(\mathbb{F}_{q^2}) \) such that \( \sigma_T \sigma \in G_0 \cap G_1 \). Repeating the discussion for \( \Gamma(j, 0) \) in Lemma 3.3, \( \sigma_T \sigma \) is a pointwise stabilizer on each \( \Gamma(k, \gamma) \). Hence \( T \) is the desired element. □

Combining Lemmas 2.2, 3.1 and 3.5, the proof of Theorem 1.1 is completed by induction.

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**References**

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