On the Generalized Degrees of Freedom of the $K$-user Symmetric MIMO Gaussian Interference Channel

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Abstract

This work derives inner and outer bounds on the generalized degrees of freedom (GDOF) of the $K$-user symmetric MIMO Gaussian interference channel (IC). For the inner bound, an achievable GDOF is derived by employing a combination of treating interference as noise, zero-forcing (ZF) at the receivers, interference alignment (IA), and extending the Han-Kobayashi (HK) scheme to $K$ users, depending on the number of antennas and the INR/SNR level. An outer bound on the GDOF is derived, using a combination of the notion of cooperation and providing side information to the receivers. Several interesting conclusions are drawn from the expressions derived. For example, when $K > N/M + 1$, a combination of the HK and IA schemes performs the best among the schemes considered. However, for $\frac{N}{M} < K \leq \frac{N}{M} + 1$, HK scheme outperforms other schemes and also found to be GDOF optimal. In this case, ZF-receiving coincides with HK scheme at $\alpha = 1$ and GDOF optimal also.

I. INTRODUCTION

Interference management in multiuser interference channels has been a major focus of research in the past few years, and it is now widely recognized that straightforward methods such as orthogonalizing users in time or frequency is suboptimal. In case of interference channel (IC), every sender has an individual message to its corresponding receiver and there is no apriori knowledge how channel is shared among them. The capacity region of a two user IC remains an open problem since 30 years. The exact capacity region is not yet known even in case of two-user Gaussian IC. Only in the strong interference case, the capacity region is known [1]. In recent years, there has been increased interest in approximate characterization of capacity of wireless networks. The concept of Generalized Degrees Of Freedom (GDOF) was introduced in [2], as a means for quantifying the extent of interference management in terms of the number of free signaling dimensions in a two-user IC. Also, in a multiuser MIMO setup, the multiple antennas at the transmitters and receivers can provide additional dimensions
for signaling, which can in turn improve the GDOF performance of the IC. Characterizing the GDOF performance of a multiuser MIMO IC is therefore an important problem, and is the focus of this work.

Although there exist many methods to mitigate the effect of interference, two major approaches have emerged in the literature. The first is based on the notion of splitting the message into private and public parts (also known as the Han-Kobayashi (HK) scheme) \cite{3, 2}; and the second is based on the idea of Interference Alignment (IA) \cite{4} - \cite{6}. These schemes are based on different ideas: the former allows part of the interference to be decoded and canceled at unintended receivers, while the latter makes the interfering signals cast “overlapping shadows” \cite{6} at unintended receivers, allowing them to project the received signal in an orthogonal direction and remove the effect of interference.

The HK scheme proposed in \cite{3} was known to be the best achievable rate region in case of two-user IC. Later, in the seminal work by Etkin, Wang and Tse \cite{2}, it was shown that a simple HK scheme can achieve rate within 1 bit/s/Hz of the capacity of the channel for all values of the channel parameters. Different variants of HK-scheme can be found in \cite{7} - \cite{10}. The idea of IA was originated from the work of Maddah-Ali \textit{et al.} in \cite{4, 11}. It was subsequently used in DOF analysis of $X$ channel in \cite{5, 12}. But the notion of alignment was crystallized in the seminal work by Cadambe \textit{et al.} in \cite{6}. In case of IA, the precoding matrix is designed such that the interfering signals occupy a reduced dimension at the unintended receiver whereas the desired signal remains decodable at the intended receiver. As compared to HK scheme, as the interference is aligned into a reduced subspace, there is no such common part in this case. The idea of IA was further extended in case of $K$-user MIMO in \cite{13}. More works on IA can be found in \cite{14} - \cite{16}.

The GDOF performance of the two-user MIMO IC was characterized in \cite{8}. It was extended to the $X$-channel and the $K$-user SISO IC in \cite{17} and \cite{18}, respectively. Also, in \cite{9}, the idea of message splitting was used to derive the GDOF in a SIMO setting when $K = N + 1$, where $N$ is the number of receive antennas at each user. The idea of IA was extended to derive the Degrees Of Freedom (DOF) in a multiuser MIMO scenario in \cite{10}, when $\alpha$, the ratio of the logarithm of the Interference to Noise Ratio (INR) to the logarithm of the SNR, is equal to 1.

In this work, we make significant progress in characterizing the GDOF of $K$-user symmetric MIMO Gaussian IC in different regime of interference. We have extended HK-scheme to a multiuser MIMO scenario for symmetric case. The inner bound also takes into account other existing schemes such as IA, ZF-receiving and treating interference as noise. We have proposed three outer bounds which are valid for all the values of channel parameters and these outer bounds are simplified to obtain bound on GDOF in symmetric case. Inspired by \cite{19}, one outer bound is based on the notion of cooperation which takes in to account of different possible ways of cooperation. The other two outer bounds are based on the notion of providing side information to the receivers in a specific manner. Some useful insights are obtained from the inner bound and outer bound in case of $K$-user symmetric MIMO IC. Optimality of the scheme is also established in some cases from GDOF prospective. Many existing results in literature can be obtained as special cases from this work.

In summary, the major contributions of this paper are as follows:
1) A new outer bound is derived for the GDOF of the $K$-user symmetric MIMO Gaussian IC using a combination of user cooperation and providing noisy side information to the receivers.

2) An inner bound is derived for the symmetric IC as a combination of the HK scheme, IA, Zero-Forcing (ZF), and treating interference as noise. To the best of the authors’ knowledge, the extension of the HK scheme to the multiuser MIMO scenario presented here is new.

3) The interplay between the HK scheme and IA is explored from an achievable GDOF perspective. The conditions under which the different achievable schemes are optimal in terms of the GDOF are established.

The rest of the paper is organized as follows. In Section II, the system model and GDOF is described. In Section III, three outer bounds are derived and also simplified in case of symmetric scenario. Section IV focuses on the achievable GDOF. In Section V, some examples are considered to get better insight from the inner bound and the outer bound.

The following notations are used in the sequel. Lower case or upper case letters are used to represent scalars. Small bold faced letters represent a vector where as capital bold faced letters represent matrix. $h(.)$ represents the differential entropy whereas $I( ; )$ represents the mutual information.

II. PRELIMINARIES

A. System Model

Consider a MIMO IC with $K$ transmitter-receiver pairs, with $M$ antennas at each transmitter and $N$ antennas at each receiver. Let $H_{ji}$ represent the $N \times M$ channel gain matrix from transmitter $i$ to receiver $j$. The channel coefficients are assumed to be drawn from a continuous distribution such as the Gaussian distribution and channel is assumed to be time-varying. The received signal at the $j$-th receiver, denoted $y_j$, is modeled as

$$y_j = \sqrt{\rho^{\alpha_{ji}}} H_{jj} x_j + \sum_{i=1, i \neq j}^{K} \sqrt{\rho^{\alpha_{ji}}} H_{ji} x_i + z_j,$$

where $z_j$ is the complex symmetric Gaussian noise vector, distributed as $z_j \sim \mathcal{CN}(0, I)$ and $x_i$ is the signal transmitted by the $i$-th user, satisfying $\mathbb{E}\{x_i^H x_i\} = 1$. Also, $\rho^{\alpha_{ji}}$ represents the received signal power relative to the noise power at receiver $j$ due to the signal from user $i$. In this work, for analytical tractability, attention is restricted to the symmetric IC, where $\alpha_{jj} = 1$, and $\alpha_{ji} = \alpha, i \neq j$, for $i, j = 1, \ldots, K$ \cite{6}, \cite{8}, \cite{18}. Here, $\alpha > 0$ represents the ratio of the logarithm of the INR to the logarithm of the SNR, and is one of the key parameters that determine the GDOF. Further, for the inner bound it is assumed that $M \leq N$.

B. Generalized Degrees of Freedom

The generalized degrees of freedom (GDOF), similar to the standard degrees of freedom, introduced in \cite{2} is an asymptotic quantity, in the limit of high SNR and INR. For symmetric case, it is defined as:

$$d_{\text{sym}}(\alpha) = \frac{1}{K} \lim_{\rho \rightarrow \infty} \frac{C_{\Sigma}(\rho, \alpha)}{\log \rho},$$

where $C_{\Sigma}(\rho, \alpha)$ is the capacity of the Gaussian MIMO channel with $K$ transmit users and $N$ receive antennas, $\rho$ is the signal-to-noise ratio, and $\alpha$ is the parameter that determines the GDOF.
and \( C_S(\rho, \alpha) \) is the sum capacity of the \( K \) user IC defined above. When \( \alpha = 1 \), the GDOF reduces to the Degrees of Freedom (DOF) defined in [10]. Since channel is random, the GDOF results derived in this work hold in an almost-sure sense.

III. OUTER BOUND

In this section, three outer bounds on the sum rate of \( K \)-user MIMO Gaussian IC are stated as theorems: 3.5, 3.7, and 3.9. The outer bounds are obtained by first deriving general outer bounds on the sum rate of the \( K \)-user MIMO IC, and then specializing the general outer bounds to the case of the symmetric Gaussian MIMO IC. The simplified outer bounds are stated as lemmas 3.6, 3.8, and 3.10. Finally, overall outer bound is obtained by taking the minimum of the three outer bounds and the interference free GDOF bound of \( M \) per user.

The first outer bound is obtained by considering cooperation among subsets of users and providing partial side information to the receivers to convert the system to a MIMO Z-IC, whose capacity cannot be worse than the original MIMO IC. Then, an outer bound on the Z-IC is derived, and the minimum of the outer bounds obtained by considering all possible combinations of cooperating users forms an outer bound on the sum rate of the MIMO IC.

The second outer bound in the Theorem 3.7 is based on providing side information to the receivers in a carefully chosen manner, in the form of a noisy version of the intended message.

The third outer bound in the Theorem 3.9 is based on providing each receiver with side information comprising a noisy version of a carefully chosen part of the interference experienced by it. For the SIMO case, this bound reduces to the outer bound presented in [9].

Following lemmas are used in derivation of the outer bound.

Lemma 3.1: [8,20] Let \( R_1 \) and \( R_2 \) be \( N \times N \) covariance matrices with rank \( r_1 \) and \( r_2 \), respectively. Let \( R_1 = U_1 \Lambda_1 U_1^H \) and \( R_2 = U_2 \Lambda_2 U_2^H \) represent the EVD of \( R_1 \) and \( R_2 \), with \( U_1 \in \mathbb{C}^{N \times r_1} \) and \( U_2 \in \mathbb{C}^{N \times r_2} \). If \( \text{rank}[U_1 U_2] = \min(r_1 + r_2, N) \), then for \( \eta \geq \beta \),

\[
J_1 \triangleq \log |I_N + \rho^\beta R_1 + \rho^\beta R_2|,
= r_1 \eta \log \rho + \min(r_2, N - r_1) \beta \log \rho + O(1). \tag{3}
\]

Lemma 3.2: [21] Let \( x^n = \{x_1, x_2, \ldots, x_n\} \) and \( y^n = \{y_1, y_2, \ldots, y_n\} \) be two sequences of random vectors and let \( \hat{x}, \hat{y}, \hat{x} \) and \( \hat{y} \) be Gaussian vectors with covariance matrices satisfying

\[
\text{Cov} \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = \frac{1}{n} \sum_{i=1}^{n} \text{Cov} \begin{bmatrix} x_i \\ y_i \end{bmatrix} \leq \text{Cov} \begin{bmatrix} x^* \\ y^* \end{bmatrix},
\]

then we get the following bound

\[
h(x^n) \leq nh(\hat{x}) \leq nh(x^*),
\]

\[
h(y^n|x^n) \leq nh(\hat{y}|\hat{x}) \leq nh(y^*|x^*).
\]
Lemma 3.3: \[22\] Let $0 \leq G_1 \preceq G_2$ and $0 \preceq A$ are positive semi-definite matrices of size $n$ and for any given $\pi \in \mathbb{R}^+$, then
$$G_1 \{I + \pi G_1 AG_1\}^{-1} G_1 \preceq G_2 \{I + \pi G_2 AG_2\}^{-1} G_2. \quad (4)$$

Lemma 3.4: \[20\] Let $R_1, R_2$ and $R_3$ be $N \times N$ covariance matrices with rank $r_1, r_2$ and $r_3$ respectively. For $\eta \geq \beta \geq \gamma$,
$$J_1 \triangleq \log |I_N + \rho^\eta R_1 + \rho^\beta R_2 + \rho^\gamma R_3|,
= r_1 \eta \log \rho + \min(r_2,N-r_1) \beta \log \rho + \min(r_3,(N-r_1-r_2)^+) \beta \log \rho + O(1). \quad (5)$$

A. Outer bound based on the notion of cooperation

Theorem 3.5: The sum rate of the $K$-user MIMO Gaussian IC is upper bounded as follows:
$$\sum_{i=1}^L R_i \leq \log |I_{L_i N} + \mathbf{P}_{11} \mathbf{P}_{11}^H + \mathbf{P}_{12} \mathbf{P}_{12}^H| + \log |I_{L_2 N} + \mathbf{P}_{22}^{1/2} \left\{I_{L_2 M} + \mathbf{P}_2^{1/2} \mathbf{P}_{12} \mathbf{P}_{12}^{1/2} \right\}^{-1} \mathbf{P}_2^{1/2} \mathbf{P}_{22}^H| + \epsilon_n,$$
where $L_1 + L_2 = L \leq K$, $0 \leq L_1 \leq K$, $0 \leq L_2 \leq K$, $I_L : L \times L$ identity matrix
$$\mathbf{P}_{11} = \text{blockdiag}(H_{11}, H_{22}, \ldots, H_{L_1,L_1}), \quad \mathbf{P}_{22} = \text{blockdiag}(H_{L_1+1,L_1+1}, H_{L_1+2,L_1+2}, \ldots, H_{L,L}),$$
$$\mathbf{P}_{12} = \begin{bmatrix}
H_{1,L_1+1} & H_{1,L_1+2} & \cdots & H_{1,L} \\
H_{2,L_1+1} & H_{2,L_1+2} & \cdots & H_{2,L} \\
\vdots & \vdots & \ddots & \vdots \\
H_{L_1,L_1+1} & H_{L_1,L_1+2} & \cdots & H_{L_1,L}
\end{bmatrix},$$
$$\mathbf{P}_1 = \text{blockdiag}(P_1, P_2, \ldots, P_{L_1}), \quad \mathbf{P}_2 = \text{blockdiag}(P_{L_1+1}, P_{L_1+2}, \ldots, P_{L_2}),$$
$$\mathbf{P}_{ij} \in \mathbb{C}^{L_i N \times L_j M}, \quad H_{ij} \in \mathbb{C}^{N \times M}, \quad P_j \in \mathbb{C}^{M \times M}$$
is the input covariance matrix of $j$th user and $\mathbf{P}_j \in \mathbb{C}^{L_j M \times L_j M}$.

Proof: See Appendix A

Lemma 3.6: In the symmetric case, the per user GDOF upper bound of Theorem 3.5 can be expressed as follows:

1) When $M \leq N$ and $0 \leq \alpha \leq 1$:
$$d(\alpha) \leq \min_{L_1,L_2} \frac{1}{L} \left[ L_1 M + \min \left\{ r, L_1(N-M) \right\} \alpha + (L_2 M - r)^+ + \min \left\{ r, L_2 N - (L_2 M - r)^+ \right\} (1 - \alpha) \right],$$

2) When $M \leq N$ and $\alpha > 1$:
$$d(\alpha) \leq \min_{L_1,L_2} \frac{1}{L} \left[ r \alpha + \min \left\{ L_1 M, L_1 N - r \right\} + (L_2 M - r)^+ \right],$$

3) When $M > N$ and $0 \leq \alpha \leq 1$:
$$d(\alpha) \leq \min_{L_1,L_2} \frac{1}{L} \left[ L_1 N + \min \left\{ L_2 N, (L_2 M - r)^+ \right\} + \min \left\{ L_2 N, r \right\}, L_2 N - \min \left\{ L_2 N, (L_2 M - r)^+ \right\} (1 - \alpha) \right],$$
4) When $M > N$ and $\alpha > 1$:

$$d(\alpha) \leq \min_{L_1, L_2} \frac{1}{L} \left[ L_1 N + r(\alpha - 1) + \min \{ L_2 N, (L_2 M - r)^+ \} \right],$$

where $r = \min \{ L_2 M, L_1 N \}$.

**Proof:** See Appendix D.

**Theorem 3.7:** The sum rate of the $K$-user MIMO Gaussian IC is upper bounded as follows:

$$R_1 + 2 \sum_{i=2}^{K-1} R_i + R_K \leq \sum_{i=1}^{K-1} \log \left| I_{N_i} + \sum_{j=1, j \neq i}^{K} H_{ij} P_j H_{ij}^H + H_{ii} P_i^{1/2} \left( I_{M_i} + P_i^{1/2} H_{i+1,i} H_{i+1,i} P_i^{1/2} \right)^{-1} P_i^{1/2} H_{ii}^H \right| + \sum_{i=2}^{K} \log \left| I_{N_i} + \sum_{j=1, j \neq i}^{K} H_{ij} P_j H_{ij}^H + H_{ii} P_i^{1/2} \left( I_{M_i} + P_i^{1/2} H_{i-1,i} H_{i-1,i} P_i^{1/2} \right)^{-1} P_i^{1/2} H_{ii}^H \right| + \epsilon_n$$

**Proof:** See Appendix C.

**Lemma 3.8:** In the symmetric case, the per user GDOF upper bound of Theorem 3.7 can be expressed as follows:

1) When $M \leq N$:

$$d(\alpha) \leq \begin{cases} 
M(1 - \alpha) + \min \{ \min (N, (K - 1)M), N - M \} & \text{if } 0 \leq \alpha \leq \frac{1}{2} \\
\min \{ \min (N, (K - 1)M), N - \min (N, (K - 1)M) \} & \text{if } \frac{1}{2} \leq \alpha \leq 1 \\
\min \{ N, (K - 1) M \} & \text{if } \alpha \geq 1
\end{cases}$$

2) When $M > N$:

$$d(\alpha) \leq \begin{cases} 
M - N + (2N - M)^+(1 - \alpha) & \text{if } 0 \leq \alpha \leq \frac{1}{2} \\
(M - N) + \min \{ N, (2N - M)^+ \} & \text{if } \frac{1}{2} \leq \alpha \leq 1 \\
N \alpha & \text{if } \alpha \geq 1
\end{cases}$$

**Proof:** See Appendix D.
Theorem 3.9: The sum rate of the $K$-user MIMO Gaussian IC is upper bounded as follows:

$$ R_1 + \sum_{i=2}^{K-1} R_i + R_K \leq \log \left| I_{N_1} + \sum_{j=2}^{K} H_{1j} P_j H_{1j}^H + H_{11} P_1^{1/2} \left( I_{M_1} + P_1^{1/2} H_{K1}^H H_{K1} P_1^{1/2} \right)^{-1} P_1^{1/2} H_{11} \right| + \sum_{i=2}^{K-1} \log \left| I_{N_i} + \mathbf{H}_{i1} P_{i1}^{1/2} \left( I_{M_{r_i}} + P_{i1}^{1/2} H_{K1}^H H_{K1} P_{i1}^{1/2} \right)^{-1} P_{i1}^{1/2} H_{i1} \right| + \mathbf{H}_{i,i+1} P_{i2}^{1/2} \left( I_{M_{r_i}} + P_{i2}^{1/2} H_{K1}^H H_{K1} P_{i2}^{1/2} \right)^{-1} P_{i2}^{1/2} H_{i,i+1} + \mathbf{H}_{i,K-1} P_{i4}^{1/2} \left( I_{M'_{r_i}} + P_{i4}^{1/2} H_{K1}^H H_{K1} P_{i4}^{1/2} \right)^{-1} P_{i4}^{1/2} H_{i,K-1} + \log \left| I_{N_K} + \sum_{j=1}^{K-1} H_{Kj} P_j H_{Kj}^H + H_{KK} P_K^{1/2} \left( I_{M_K} + P_K^{1/2} H_{1K}^H H_{1K} P_K^{1/2} \right)^{-1} P_K^{1/2} H_{KK} \right| + \epsilon_n $$

where

$$ \mathbf{H}_{i1} = \begin{bmatrix} H_{i1} & H_{i2} & \ldots & H_{ii} \end{bmatrix}, \quad \mathbf{H}_{i,i+1} = \begin{bmatrix} H_{i,i+1} & H_{i,i+2} & \ldots & H_{iK} \end{bmatrix}, $$

$$ \mathbf{H}_{K1} = \begin{bmatrix} H_{K1} & H_{K2} & \ldots & H_{Kr} \end{bmatrix}, \quad \mathbf{H}_{i,i+1} = \begin{bmatrix} H_{i,i+1} & H_{i,i+2} & \ldots & H_{iK} \end{bmatrix}, $$

$$ \mathbf{H}_{1i} = \begin{bmatrix} H_{1i} & H_{12} & \ldots & H_{1i} \end{bmatrix}, \quad \mathbf{H}_{K,i} = \begin{bmatrix} H_{K1} & H_{K2} & \ldots & H_{K,i+1} \end{bmatrix}, $$

$$ \mathbf{H}_{i,K} = \begin{bmatrix} H_{iK} & H_{i2} & \ldots & H_{ii} \end{bmatrix}, \quad \mathbf{H}_{i,K-1} = \begin{bmatrix} H_{i1} & H_{i,i+1} & \ldots & H_{i,K-1} \end{bmatrix}, $$

$$ P_{i1} = \text{blockdiag} \left( P_1 P_2 \ldots P_i \right), \quad P_{i2} = \text{blockdiag} \left( P_{i+2} P_{i+3} \ldots P_K \right), $$

$$ P_{i3} = \text{blockdiag} \left( P_K P_2 \ldots P_i \right), \quad P_{i4} = \text{blockdiag} \left( P_1 P_{i+1} \ldots P_{K-1} \right), $$

$$ M_{r_i} = \sum_{j=1}^{i} M_j, \quad M_{s_i} = \sum_{j=i+1}^{K} M_j, \quad M'_{r_i} = \sum_{j=2}^{i} M_j + M_K \quad \text{and} \quad M'_{s_i} = M_{1} + \sum_{j=i+1}^{K-1} M_j. \quad (6) $$

Proof: See Appendix F. \hfill \blacksquare

Lemma 3.10: In the symmetric case, when $\frac{N}{M} < K \leq \frac{N}{M} + 1$, the GDOF upper bound of Theorem 3.9 can be expressed as follows:

$$ d_j \leq \begin{cases} M(1-\alpha) + \frac{1}{K-1} \min \left\{ \min \left\{ N, (K-1)M \right\}, N-M \right\} \alpha & \text{if } 0 \leq \alpha \leq \frac{1}{2} \\ \frac{1}{K-1} \left[ \min \left\{ N, (K-1)M \right\} \alpha + \min \left\{ M, N - \min \left\{ N, (K-1)M \right\} \right\} (1-\alpha) \\ \right. \\ + (K-2)M(1-\alpha) \right] \leq \min \left\{ \frac{N}{K-1}, M \right\} \alpha & \text{if } \frac{1}{2} \leq \alpha \leq 1 \\ \end{cases} $$

Proof: See Appendix F. \hfill \blacksquare
IV. INNER BOUND ON THE GDOF OF $K$-USER SYMMETRIC MIMO GAUSSIAN IC

In this section, the inner bound is derived for symmetric $K$-user MIMO Gaussian IC. The achievability scheme uses a combination of HK-scheme, IA, ZF-receiving and treating interference as noise. We have extended HK-scheme to multiuser scenario depending on the number of antennas at transmitter and receiver. The extension of HK-scheme is neither unique nor straightforward. The results do not get directly extended from previous work form SISO or SIMO. We are giving one way of generalizing the HK-scheme. The extended HK-scheme is in similar flavor as in [8], [10]. It is assumed that $M \leq N$ and $KM > N$ as ZF-receiving is sufficient to achieve interference free GDOF when $KM \leq N$. Before stating the inner bounds, the following known results on the achievable DOF using IA and Zero-Forcing (ZF) reception are recapitulated.

A. Interference Alignment (IA)

In [9], Gou and Jafar used the idea of IA for a $K$-user MIMO IC. In IA, the transmit signal vectors are designed such that, at every receiver the interfering signals are aligned in a subspace, linearly independent of the desired signal subspace. The desired signals can be decoded by merely zero forcing the interference at individual receiver. It is shown that the achievable per user degrees of freedom using IA is:

$$d_{IA} = \frac{MN}{M+N}, \text{ if } N < KM$$

(7)

It is important to note that for IA, the relative strength between the signal and interference does not matter. It requires global channel knowledge at every nodes and the channel to be time varying.

B. Zero-Forcing (ZF) Receiving

The achievable degrees of freedom by ZF-receiving is given by:

$$d_{ZF} = \min\left\{M, \frac{N}{K}\right\},$$

(8)

In this case also, the relative strength between the signal and interference does not matter.

C. Treating Interference as Noise

Treating interference as noise is one of the simplest methods of dealing with interference, and may work well when the interference is weak. The following theorem summarizes the GDOF obtained by treating interference as noise.

**Theorem 4.1:** The following per user GDOF is achievable for the $K$-user symmetric MIMO Gaussian IC:

1) When $\frac{N}{M} < K \leq \frac{N}{M} + 1$:

$$d(\alpha) \geq M + \alpha(N - KM)$$

2) When $K > \frac{N}{M} + 1$:

$$d(\alpha) \geq M(1 - \alpha)$$

**Proof:** See Appendix G.
D. Han-Kobayashi (HK) Scheme

In this section, the achievable scheme is based on the notion of HK scheme. The proposed scheme uses the similar approach as in [2], [8], [9] and is extended to $K$-user MIMO IC for symmetric case. Following interference regimes are considered:

1) Strong interference case $(\alpha > 1)$
2) Moderate interference case $(1/2 < \alpha \leq 1)$ and
3) Weak interference case $(0 \leq \alpha \leq 1/2)$.

1) **Strong Interference Case** $(\alpha \geq 1)$: When the interference is strong, each receiver can decode both the unintended messages as well as the intended message. Hence, a $K$-user MAC channel is formed at each receiver, and the achievable rate region is the intersection of the $K$-MAC regions obtained. This results in the following inner bound on the per user GDOF.

**Theorem 4.2:** In the strong interference case $(\alpha \geq 1)$, the following per user GDOF is achievable by the HK scheme:

1) When $\frac{N}{M} < K \leq \frac{N}{M} + 1$:
   
   $$d(\alpha) \geq \min \left\{ M, \frac{1}{K} \left[ (K-1)M\alpha + N - (K-1)M \right] \right\}.$$

2) When $K > \frac{N}{M} + 1$:
   
   $$d(\alpha) \geq \min \left\{ M, \frac{\alpha N}{K} \right\}.$$

**Proof:** See Appendix [H].

2) **Moderate Interference Case** $(1/2 \leq \alpha \leq 1)$: In the moderate interference regime, one achievable scheme based on the HK-type message splitting is as follows. The transmitter $j$ splits its message $W_j$ into two sub-messages, a common message $W_{c,j}$ that is required to be decodable at every receiver, and a private message $W_{p,j}$ that is only required to be decodable at the desired receiver. The common message is encoded using a Gaussian code book with rate $R_{c,j}$ and power $P_{c,j}$. Similarly, the private message is encoded using a Gaussian code book with rate $R_{p,j}$ and power $P_{p,j}$. The code words are transmitted using superposition coding. Further, it is assumed that the rates are symmetric, i.e., $R_{c,j} = R_c$ and $R_{p,j} = R_p$. Also, $P_{c,j} = P_c$ and $P_{p,j} = P_p$. The power on the private and common messages satisfy the constraint $P_c + P_p = 1$. The transmitted signal $X_j$ is a superposition of private message and public message i.e. one codeword is superimposed on the other.

Similar to [8], the power in the private message is set such that it is received at the noise floor of the unintended receivers, resulting in $\text{INR}_p = 1$. Coupled with the transmit power constraint at each of the users, the SNRs of the common and private parts at the desired signal (denoted $\text{SNR}_c$ and $\text{SNR}_p$) and the INRs of the common and private parts at unintended receivers (denoted $\text{INR}_c$ and $\text{INR}_p$) are given by

$$\text{SNR}_c = \rho - \rho^{1-\alpha}, \text{SNR}_p = \rho^{1-\alpha}, \text{INR}_c = \rho^\alpha - 1, \text{INR}_p = 1.$$
The transmit covariance of the common message is assumed to be same as that of private message. The decoding order is such that the common message is decoded first, followed by the private message. While decoding the common message, all the user’s private messages are treated as noise (including its own private message). While decoding the private message, since the common messages have been decoded and their interference eliminated, the rate due to the private message is obtained by treating all remaining user’s private messages as noise.

The GDOF is contributed by both the private and public parts of the message:

\[ d(\alpha) = d_p(\alpha) + d_c(\alpha), \]

where \( d_p(\alpha) \) and \( d_c(\alpha) \) are the GDOF contributed by the private and public parts of the message, respectively. The following theorem summarizes the per user GDOF achievable by this scheme.

**Theorem 4.3:** In the moderate interference regime \( (1/2 \leq \alpha \leq 1) \), the following per user GDOF is achievable by the HK scheme:

1) When \( N/M < K \leq N/M + 1 \):

\[ d(\alpha) \geq M(1 - \alpha) + \min \left\{ \frac{N\alpha}{K}, \frac{[M(\alpha(2K - 1) - K) + N(1 - \alpha)]}{K - 1} \right\}. \]

2) When \( K > N/M + 1 \):

\[ d(\alpha) \geq M(1 - \alpha) + \min \left\{ \frac{N\alpha}{K}, \frac{[N\alpha - M(1 - \alpha)]}{K - 1} \right\}. \]

**Proof:** See Appendix I.

3) **Weak Interference Case** \( \left( 0 \leq \alpha \leq \frac{1}{2} \right) \): The received SNR and INR of the common and private message at the desired and unintended receiver is of the same form as also given in case of moderate interference scenario. The per user GDOF achieved in this case is summarized in the following theorem.

**Theorem 4.4:** In the weak interference regime \( (0 \leq \alpha \leq \frac{1}{2}) \) and when \( KM > N \), the following per user GDOF is achievable by the HK scheme:

\[ d(\alpha) \geq M(1 - \alpha) + \frac{1}{K - 1}(N - M). \]

**Proof:** See Appendix I.

**E. Achievable GDOF as a Combination of HK, IA, ZF-Receiving and Treating Interference as Noise**

The achievability is based on combination of HK-scheme, IA, ZF-receiving and treating interference as noise. Final inner bound is obtained by taking maximum of all these schemes. In this section, we determine performance of various schemes in terms of \( \alpha, K, M \) and \( N \). The achievable per user GDOF in case of IA and ZF-receiving are:

\[ d_{IA} = \frac{MN}{M + N}, \]

and \( d_{ZF} = \min \left\{ M, \frac{N}{K} \right\}. \)
1) **Low interference case** \((0 \leq \alpha \leq \frac{1}{2})\): Following two cases are considered to evaluate the performance of various schemes considered in this work:

1) \(M = N\)
2) \(M < N\)

**Case-1: \((M = N)\)**

In this case, IA, HK-scheme, treating interference as noise and ZF-receiving achieve a per user GDOF of:

- \(d_{HK}(\alpha) = M(1 - \alpha)\),
- \(d_{int-noise} = M(1 - \alpha)\),
- \(d_{IA} = \frac{M}{2}\),
- and \(d_{ZF} = \frac{M}{K}\)

From above equations it can be observed that treating interference as noise performs same as that of HK-scheme. Treating interference as noise as well as HK-scheme performs better than IA and ZF-receiving and at \(\alpha = \frac{1}{2}\), HK-scheme, IA and treating interference as noise coincide. Other schemes perform better than ZF-receiving always in this case.

**Case-2: \((M < N)\)**

In this case, HK-scheme and treating interference as noise achieve following per user GDOF as obtained from Theorem - 4.4 and 4.1:

- \(d_{HK}(\alpha) = M(1 - \alpha) + \frac{1}{K - 1}(N - M)\alpha\), \(\text{(12)}\)
- \(d_{int-noise}(\alpha) = \begin{cases} 
M + \alpha(N - KM) & \text{if } \frac{N}{M} < K \leq \frac{N}{M} + 1 \\
M(1 - \alpha) & \text{if } K > \frac{N}{M} + 1
\end{cases}\), \(\text{(13)}\)

When \(K > \frac{N}{M} + 1\), from (12) and (13) it can be observed that HK-scheme performs better than treating interference as noise. The per user GDOF achievable by HK-scheme can also be expressed as follows:

- \(d_{HK}(\alpha) = M + \frac{1}{K - 1}(N - KM)\alpha\), \(\text{(14)}\)

When \(\frac{N}{M} < K \leq \frac{N}{M} + 1\), as \(N < KM\), \(\frac{1}{K - 1}(N - KM)\alpha > (N - KM)\alpha\) and hence HK-scheme performs better than treating interference as noise in this case also.

Now we compare HK-scheme with IA:

- \(M(1 - \alpha) + \frac{1}{K - 1}(N - M)\alpha < \frac{MN}{M + N}\)
- or \(\alpha < \frac{M^2}{M(N + M) - \frac{N^2 - M^2}{K - 1}}\), \(\text{(15)}\)
Also we require:

\[
\frac{M^2}{M(N+M) - \frac{N^2 - M^2}{K-1}} < \frac{1}{2}
\]

or

\[K > 2 + \frac{N}{M}\]  \hspace{1cm} (16)

When \(K \leq 2 + \frac{N}{M}\), HK-scheme performs better than IA.

When IA is compared with treating interference as noise, then following condition is obtained:

\[
\frac{MN}{M+N} \leq M(1-\alpha)
\]

or

\[\alpha \leq \frac{M}{M+N}\]  \hspace{1cm} (17)

When \(\alpha \leq \frac{M}{M+N}\), then treating interference as noise performs better than IA whereas IA performs better than treating interference as noise when \(\alpha > \frac{M}{M+N}\).

Now we compare HK-scheme with ZF-receiving and both schemes coincide at:

\[
M(1-\alpha) + \frac{1}{K-1}(N-M)\alpha = \frac{N}{K}
\]

or

\[\alpha = \frac{K-1}{K}\]  \hspace{1cm} (18)

But we know \(\alpha \leq \frac{1}{2}\). Hence

\[
\frac{K-1}{K} \leq \frac{1}{2}
\]

or

\[K \leq 2\]  \hspace{1cm} (19)

When \(K = 2\), ZF-receiving and HK scheme coincide at \(\alpha = \frac{1}{2}\). But for \(K > 2\), HK scheme performs better than ZF-receiving. In a similar manner, it can also be shown that other schemes perform better than ZF-receiving at every value of \(\alpha\) in weak interference regime.

When \(K > 2 + \frac{N}{M}\), the per user GDOF that can be achieved in the weak interference regime is:

\[
d(\alpha) \geq \max\{d_{\text{HK}}(\alpha), d_{\text{IA}}, d_{\text{ZF}}, d_{\text{int-noise}}(\alpha)\} = \begin{cases} 
M(1-\alpha) + \frac{1}{K-1}(N-M)\alpha & \text{if } 0 \leq \alpha \leq \frac{M^2}{M(N+M) - \frac{N^2 - M^2}{K-1}} \\
\frac{NM}{N+M} & \text{if } \frac{M^2}{M(N+M) - \frac{N^2 - M^2}{K-1}} < \alpha \leq \frac{1}{2}
\end{cases}
\]

(20)

When \(K \leq 2 + \frac{N}{M}\), HK-scheme alone performs better as compared to other schemes and the per user GDOF achievable by this scheme is:

\[
d(\alpha) \geq \max\{d_{\text{HK}}(\alpha), d_{\text{IA}}, d_{\text{ZF}}, d_{\text{int-noise}}(\alpha)\} = M(1-\alpha) + \frac{1}{K-1}(N-M)\alpha
\]

(21)
2) Moderate interference case ($\frac{1}{2} \leq \alpha \leq 1$): Following two cases are considered in this regime.

- $\frac{N}{M} < K \leq \frac{N}{M} + 1$
- $K > \frac{N}{M} + 1$

Case - 1: ($\frac{N}{M} < K \leq \frac{N}{M} + 1$)

In this case, the per user GDOF achievable by HK-scheme from Theorem - 4.3 can also be expressed as:

$$d_{HK}(\alpha) = \begin{cases} M(1 - \alpha) + \frac{1}{K-1} [M \{\alpha(2K - 1) - K\} + N(1 - \alpha)] & \text{if } \frac{1}{2} \leq \alpha \leq \frac{K}{2K-1} \\ M(1 - \alpha) + \frac{N\alpha}{K} & \text{if } \frac{K}{2K-1} \leq \alpha \leq 1 \end{cases}$$

(22)

It can also be shown that HK-scheme performs better than treating interference as noise and ZF-receiving. But at $\alpha = 1$, ZF-receiving performs same as that of HK-scheme. In this case, IA is not applicable.

Case - 2: ($K > \frac{N}{M} + 1$)

In this case, HK-scheme as well as IA performs better than ZF-receiving and treating interference as noise. Only at $\alpha = 1$, HK-scheme coincides with ZF-receiving. In this regime, the achievable per GDOF in Theorem - 4.3 can also be expressed as:

$$d_{HK}(\alpha) = \begin{cases} M(1 - \alpha) + \frac{N\alpha - M(1-\alpha)}{K-1} & \text{if } \frac{1}{2} \leq \alpha \leq \frac{KM}{N+KM} \\ M(1 - \alpha) + \frac{N\alpha}{K} & \text{if } \frac{KM}{N+KM} < \alpha \leq 1 \end{cases}$$

(23)

Now we compare HK-scheme with IA. When $\alpha \leq \frac{KM}{N+KM}$, then first equation in (23) is active and consider the following equation:

$$M(1 - \alpha) + \frac{1}{K-1} [N\alpha - M(1 - \alpha)] \geq MN \frac{M}{M+N}$$

(24)

or

$$\alpha [N - M(K - 2)] \geq M \left[ \frac{N - M(K - 2)}{M + N} \right]$$

(25)

When $N = M(K - 2)$, then in (24) LHS and RHS are equal.

When $N - M(K - 2) > 0$, then following condition is obtained:

$$\alpha \geq \frac{M}{M+N}$$

(26)

In this regime, above condition is always satisfied and HK-scheme performs better than IA.

When $N - M(K - 2) < 0$, then following condition is obtained:

$$\alpha < \frac{M}{M+N}$$

(27)

In this case, it is not possible to find a $\alpha$ which satisfies the above condition and hence, IA always performs better.

When $\alpha > \frac{KM}{N+KM}$, then the second equation in (23) is active and we consider the following equation:

$$M(1 - \alpha) + \frac{N\alpha}{K} > MN \frac{M}{M+N},$$

(28)

or

$$\alpha < \frac{KM^2}{(M+N)(KM-N)}.$$
1) When $N = (K - 2)M$, it can be shown that both the points $\frac{KM}{N+KM}$ and $\frac{KM^2}{(M+N)(KM-N)}$ coincide. In the range $\frac{1}{2} \leq \alpha \leq \frac{KM}{N+KM}$, HK-scheme performs same as that of IA. When $\frac{KM}{N+KM} < \alpha \leq 1$, IA performs better than HK-scheme. The achievable per user GDOF in this case is:

$$d(\alpha) \geq \begin{cases} M(1-\alpha) + \frac{N\alpha-M(1-\alpha)}{K-1} & \text{if } \frac{1}{2} \leq \alpha \leq \frac{KM}{N+KM} \\ \frac{MN}{M+N} & \text{if } \frac{KM}{N+KM} < \alpha \leq 1 \end{cases}$$

(30)

2) When $N - M(K - 2) > 0$ or $K < 2 + \frac{N}{M}$, then we have following conditions:

a) When $\frac{1}{2} \leq \alpha \leq \frac{KM}{N+KM}$, HK-scheme performs better than IA and it achieves a per user GDOF of

$$d(\alpha) \geq M(1-\alpha) + \frac{N\alpha-M(1-\alpha)}{K-1}.$$  

(31)

b) When $\frac{KM}{N+KM} < \alpha \leq \frac{KM^2}{(M+N)(KM-N)}$, HK-scheme achieves a per user GDOF of:

$$d(\alpha) \geq M(1-\alpha) + \frac{N\alpha}{K}.$$  

(32)

The given condition requires:

$$\frac{KM}{N+KM} < \frac{KM^2}{(M+N)(KM-N)}$$

or $K < 2 + \frac{N}{M}$

(33)

It also results in the same condition of $N - M(K - 2) > 0$.

c) When $\frac{KM^2}{(M+N)(KM-N)} < \alpha \leq 1$, then the following per user GDOF is achievable:

$$d(\alpha) \geq \frac{MN}{M+N}.$$  

(34)

3) When $N - M(K - 2) < 0$ or $K > 2 + \frac{N}{M}$, IA always performs better than HK-scheme and the following per user GDOF is achievable:

$$d(\alpha) \geq \frac{NM}{N+M}.$$  

(35)

3) High interference case ($\alpha > 1$): Same cases are considered as in the previous regime.

Case-1: ($\frac{N}{M} < K \leq \frac{N}{M} + 1$)

In this case HK scheme achieves following GDOF as given in Theorem - 4.2

$$d_{HK}(\alpha) \geq \min \left\{ M, \frac{1}{K} \left[ \alpha(K-1)M + N - (K-1)M \right] \right\}$$

(36)

The HK scheme achieves the interference free GDOF at:

$$\frac{1}{K} \left[ \alpha(K-1)M + N - (K-1)M \right] = M$$

or $\alpha = \frac{M(2K-1) - N}{M(K-1)}$

(37)
Hence, (36) can also be expressed as:

\[
d_{HK}(\alpha) \geq \begin{cases} 
\frac{1}{K} \left[ \alpha(K-1)M + N - (K-1)M \right] & \text{if } 1 < \alpha < \frac{M(2K-1) - N}{M(K-1)} \\
\frac{1}{K} & \text{if } \alpha \geq \frac{M(2K-1) - N}{M(K-1)} 
\end{cases}
\]  

(38)

From the following equation, it can be noticed that HK scheme performs better than ZF-receiving always.

\[
\frac{1}{K} \left[ \alpha(K-1)M + N - (K-1)M \right] < \frac{N}{K} 
\]

or \( \alpha > 1 \), which is always true in this case 

(39)

Finally, the achievable GDOF per user in this case is:

\[
d(\alpha) \geq \max\{d_{HK}, d_{ZF}\} = \begin{cases} 
\frac{1}{K} \left[ \alpha(K-1)M + N - (K-1)M \right] & \text{if } 1 \leq \alpha \leq \frac{M(2K-1) - N}{M(K-1)} \\
\frac{1}{K} & \text{if } \alpha \geq \frac{M(2K-1) - N}{M(K-1)} 
\end{cases}
\]

(40)

Case-2: \((K \geq \frac{N}{M} + 1)\)

In this case HK scheme achieves a per user GDOF of as obtained from Theorem - 4.2

\[
d_{HK}(\alpha) \geq \min\left\{ M, \frac{\alpha N}{K} \right\}
\]

When HK scheme is compared with IA, we obtain the following result

\[
\frac{\alpha N}{K} = \frac{N M}{N + M} \\
\text{or } \alpha = \frac{K M}{N + M}
\]

(41)

At this point IA and HK scheme coincide. It can be also observed that \( \frac{K M}{N + M} > 1 \). Hence, we obtain the following result.

\[
d(\alpha) \geq \begin{cases} 
\frac{M N}{M + N} & \text{if } 1 \leq \alpha < \frac{K M}{M + N} \\
\frac{\alpha N}{K} & \text{if } \frac{K M}{M + N} \leq \alpha < \frac{K M}{N} \\
\frac{M}{N} & \text{if } \alpha \geq \frac{K M}{N}
\end{cases}
\]

(42)

For \( \alpha > 1 \), it can be clearly seen that HK-scheme always outperforms ZF-receiving. Finally, we obtain the following result.

\[
d(\alpha) \geq \max\{d_{HK}, d_{IA}, d_{ZF}\} = \begin{cases} 
\frac{M N}{M + N} & \text{if } 1 \leq \alpha < \frac{K M}{M + N} \\
\frac{\alpha N}{K} & \text{if } \frac{K M}{M + N} \leq \alpha < \frac{K M}{N} \\
\frac{M}{N} & \text{if } \alpha \geq \frac{K M}{N}
\end{cases}
\]

(43)
V. DISCUSSION ON INNER BOUND AND OUTER BOUND

First, some observations on how the inner and outer bounds on the GDOF derived above stand in relation to existing work are as follows:

1) By setting $M = 1$ and $K = N + 1$, the theorem 3.10 and the HK-type achievable scheme in Sec. IV-D reduce to the corresponding SIMO GDOF results in [9].

2) By setting $K = 2$, the inner and outer bounds derived here reduce to the corresponding two-user symmetric GDOF result in [8].

3) By setting $M = N = 1$ and $K = 2$, the inner and outer bounds derived here reduce to the corresponding GDOF results derived in [2].

4) By setting $M = N = 1$, the inner bounds derived here match with the result in [18] only in the weak interference regime. In [18] assumes the constant IC model and uses lattice coding to achieve a higher GDOF.

5) By setting $\alpha = 1$, the cooperative outer bound of Lemma 3.6 matches with the DOF outer bound in [19] for many cases of $K$, $M$, and $N$ (e.g., $K = 3$, $M = 2$, $N = 5$). Theorem 3.6 uses genie-aided message sharing in addition to cooperation, to handle the $\alpha \neq 1$ cases. The bound in [19] only requires cooperation, due to which it is lower for some values of $M$, $N$, and $K$. Due to marginal difference between these two bounds, while plotting the GDOF result, outer bound in [19] is not taken into account.

Next, some specific examples are considered to get a deeper insight on the inner bound and outer bound for different cases of $K$, $M$, $N$, and $\alpha$.

In Fig. 1, the GDOF is plotted versus $\alpha$ for $K = 3$ and $M = N = 2$. In the low interference regime, the inner bound coincides with the outer bound. In this case, treating interference as noise performs as well as the HK scheme, and the outer bound in Lemma 3.8 is active. Also, IA and ZF are suboptimal in this regime. At $\alpha = 1/2$, IA, HK and treating interference as noise all coincide. In the moderate interference regime, the flat segment is due to IA and performs better than the other schemes, and it is optimal at $\alpha = 1/2$ and 1. In terms of outer bounds, initially, the side-information based bound of Lemma 3.8 is active, and as $\alpha$ increases, the cooperative bound of Lemma 3.6 is active. In the high interference regime, IA initially performs the best, and as $\alpha$ increases, the HK scheme performs the best, and finally achieves the interference free GDOF. There exists a gap between the inner and outer bounds in the moderate and high interference regimes.

Figure 2 shows the GDOF plot for $K = 3$, $M = 2$ and $N = 3$. In the low interference regime, the HK scheme outperforms treating interference as noise. In the moderate interference regime, the HK scheme initially performs the best, while the IA scheme performs better for larger $\alpha$. In the high interference regime, the observations are similar to Fig. 1.

In Fig. 3 and 4 the GDOF is plotted versus $\alpha$ for $K = 3$, $M = 2$ and $N = 4$ and $K = 3$, $M = 2$ and $N = 5$. In these cases, the inner bound and the outer bound coincide, and HK scheme is GDOF optimal. At $\alpha = 1$, the ZF
scheme coincides with HK scheme. The side-information based outer bound of Lemma 3.10 is active in the low interference regime and in the initial part of the moderate interference regime. Also, in this case, IA and treating interference as noise are strictly suboptimal for the entire range of $\alpha$.

In Fig. 5 and 6 similar trends can be observed as in the previous cases. In case of Fig. 6 the inner bound and the outer bound coincides. In Fig. 7 the number of users is increased to $K = 6$ with $M = 2$ and $N = 5$. In contrast to the previous cases, in a small part of low interference regime, IA performs better. In the entire moderate and in the initial portion of high interference regime, IA is performing better.

Before concluding, the major insights of the work are summarized as below:

1) When $M = N$, in low interference regime treating interference as noise is GDOF optimal.
2) When $N > M$, splitting the message into private and public part helps. For $K > 3$ and $M = N$ IA always performs better than HK in moderate interference case.
3) When $K > \frac{N}{M} + 1$, neither IA or HK alone is able to perform well. Rather a combination of IA and HK perform well. In case of interference channel only IA or splitting the message into private and public part may not be sufficient. An inherent combination of these two schemes may be helpful.
4) When $\frac{N}{M} < K \leq \frac{N}{M} + 1$, HK scheme out performs other existing schemes and the scheme is GDOF optimal.
5) ZF-receiving is found to be suboptimal for all ranges of $\alpha$ when $K > \frac{N}{M} + 1$. When $\frac{N}{M} < K \leq \frac{N}{M} + 1$, ZF-receiving coincides with the HK scheme only at $\alpha = 1$ and GDOF optimal only in this case.

VI. CONCLUSION

This work derived inner and outer bounds on the GDOF of the $K$-user symmetric MIMO interference channel as a function of $\alpha = \log \text{INR}/\log \text{SNR}$. The outer bound was based on a combination of three schemes, one of which was derived using the notion of cooperation, and the two other outer bounds were based on providing partial side information at the receivers. The inner bound was derived using a combination of ZF, treating interference as noise, interference alignment, and the Han-Kobayashi scheme. Several interesting insights were obtained from the derived bounds. For example, it was found that when $M = N$, treating interference as noise performs as well as the HK scheme and outperforms both IA and the ZF bound in the weak interference regime. However, when $N > M$, treating interference as noise is always suboptimal. For $K > N/M + 1$, a combination of HK and IA performs the best in the moderate interference regime. Finally, when $N/M < K \leq N/M + 1$, HK scheme is GDOF optimal for all values of $\alpha$. In contrast to two user IC, ZF-receiving is found to be GDOF optimal at $\alpha = 1$ when $N/M < K \leq N/M + 1$.

APPENDIX

A. Proof of Theorem 3.5

Consider a $K$-user MIMO Gaussian IC with $M$ antennas at each transmitter and $N$ antennas at each receiver. The $K$-user system is divided into two disjoint groups; group-1 containing $L_1 \ (0 \leq L_1 \leq K)$ users and the group -
2 containing \( L_2 (0 \leq L_1 \leq K) \) users such that \( L_1 + L_2 = L \leq K \). In group-1, all \( L_1 \) users are allowed to cooperate among themselves but they experience interference from the second group. In group-2, all the receivers are given the messages of users \( 1, \ldots, L_1 \) by a genie. As a result, the second group does not see any interference from the users in group-1. All the users in group-2 are allowed to cooperate among themselves. Now, the modified system model becomes:

\[
\mathbf{Y}_1 = \mathbf{H}_{11} \mathbf{X}_1 + \mathbf{H}_{12} \mathbf{X}_2 + \mathbf{Z}_1, \\
\mathbf{Y}_2 = \mathbf{H}_{22} \mathbf{X}_2 + \mathbf{Z}_2, 
\]

where \( \mathbf{Y}_1 \triangleq [ \mathbf{Y}_1, \ldots, \mathbf{Y}_{L_1} ]^T, \mathbf{Y}_2 \triangleq [ \mathbf{Y}_{L_1+1}, \ldots, \mathbf{Y}_L ]^T, \mathbf{X}_1 \triangleq [ \mathbf{X}_1, \ldots, \mathbf{X}_{L_1} ]^T, \mathbf{X}_2 \triangleq [ \mathbf{X}_{L_1+1}, \ldots, \mathbf{X}_L ]^T, \mathbf{Z}_1 \triangleq [ \mathbf{Z}_1, \ldots, \mathbf{Z}_{L_1} ]^T \) and \( \mathbf{Z}_2 \triangleq [ \mathbf{Z}_{L_1+1}, \ldots, \mathbf{Z}_L ]^T \).

The above system model is equivalent to a two-user MIMO \( L \)-interference channel with each transmitter having \( L_1 M \) and \( L_2 M \) antennas and each receivers having \( L_1 N \) and \( L_2 N \) antennas. The outer bound derived for this modified system is an outer bound for the \( K \)-user MIMO Gaussian IC. By using Fano’s inequality, the sum rate of the modified system is upper bounded as given below:

\[
\begin{aligned}
n \sum_{i=1}^{L} R_i - n \epsilon_n & \leq I (\mathbf{X}_1^n; \mathbf{Y}_1^n) + I (\mathbf{X}_2^n; \mathbf{Y}_2^n), \\
& \leq I (\mathbf{X}_1^n; \mathbf{Y}_1^n) + I (\mathbf{X}_2^n; \mathbf{Y}_2^n, \mathbf{S}^n), \text{ where } \mathbf{S}^n = \mathbf{H}_{12} \mathbf{X}_2^n + \mathbf{Z}_1^n, \\
& \leq nh (\mathbf{Y}_1^n) - nh (\mathbf{Z}_1^n) + nh (\mathbf{Y}_2^n) - nh (\mathbf{Z}_2^n), \\
or \sum_{i=1}^{L} R_i - \epsilon_n & \leq h (\mathbf{Y}_1^n) - h (\mathbf{Z}_1^n) + h (\mathbf{Y}_2^n) - h (\mathbf{Z}_2^n), 
\end{aligned}
\]

where (a) follows from the fact that by giving side information, the mutual information does not reduce and (b) follows from the Lemma - 3.2. In the above equation * indicates that the inputs are i.i.d. Gaussian i.e. \( \mathbf{X}^* \sim CN(0, \mathbf{I}) \) and the quantities \( \mathbf{S}^*, \mathbf{Y}_1^* \) and \( \mathbf{Y}_2^* \) are the signals obtained due to Gaussian inputs. Each term in (45) are simplified as follows:

\[
\begin{aligned}
h (\mathbf{Y}_1^n) &= \log |\pi \exp [I_{L_1 N} + \mathbf{H}_{11} \mathbf{F}_1 \mathbf{H}_{11}^H + \mathbf{H}_{12} \mathbf{F}_2 \mathbf{H}_{12}^H]|, 
\end{aligned}
\]

\[
\begin{aligned}
h (\mathbf{Y}_2^n | \mathbf{S}^n) &= \log |\pi \exp \Sigma_{\mathbf{Y}_2 | \mathbf{S}^n}|, 
\end{aligned}
\]

where

\[
\Sigma_{\mathbf{Y}_2 | \mathbf{S}^n} = \mathbb{E} \left[ \mathbf{Y}_2 \mathbf{Y}_2^H \right] - \mathbb{E} \left[ \mathbf{Y}_2 \mathbf{S}^H \right] \left[ \mathbf{S} \mathbf{S}^H \right]^{-1} \mathbb{E} \left[ \mathbf{S} \mathbf{Y}_2^H \right], 
\]

\[
= I_{L_2 N} + \mathbf{H}_{22} \mathbf{F}_2^{1/2} \{ I_{L_2 M} + \mathbf{P}_2^{1/2} \mathbf{H}_{12} \mathbf{H}_{12}^H \mathbf{P}_2^{1/2} \}^{-1} \mathbf{P}_2^{1/2} \mathbf{H}_{22}. 
\]

Equation (47) is obtained by using the fact that Gaussian distribution maximizes conditional differential entropy for a given covariance constraint. Equation (48) is obtained by using Woodbury matrix identity [23], which is stated
below:

\[(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + D A^{-1}B)^{-1} DA^{-1}.\]

The conditional differential entropy in (47) reduces to:

\[
h\left(\mathbf{Y}_2 | \mathbf{S}\right) = \log \left| \pi e \left[I_{L_2N} + \mathbf{H}_{22} \mathbf{P}_2^{1/2} \left\{ I_{L_2M} + \mathbf{P}_2^{1/2} \mathbf{H}_{12}^H \mathbf{P}_2^{1/2} \right\}^{-1} \mathbf{P}_2^{1/2} \mathbf{H}_{22}^H \right] \right|. \tag{49}\]

From (46) and (49), the sum rate bound in (45) becomes:

\[
\sum_{i=1}^{L} R_i - \epsilon_n \leq \log \left| I_{L_1N} + \mathbf{H}_{11} \mathbf{P}_1 \mathbf{H}_{11}^H + \mathbf{H}_{12} \mathbf{P}_2 \mathbf{H}_{12}^H \right| + \log \left| I_{L_2N} + \mathbf{H}_{22} \mathbf{P}_2^{1/2} \left\{ I_{L_2M} + \mathbf{P}_2^{1/2} \mathbf{H}_{12}^H \mathbf{P}_2^{1/2} \right\}^{-1} \mathbf{P}_2^{1/2} \mathbf{H}_{22}^H \right|. \tag{50}\]

In (50), \(L_1\) users in group - 1 are allowed to cooperate where as in the second group \(L_2\) users are allowed to cooperate. As there are different ways of cooperating, the minimum sum rate obtained out of all possible ways of cooperation is the best outer bound on the sum rate in this case. Since the users have different power constraint and users see different SNR and INR, the optimization problem becomes a formidable task. However for the symmetric case, a simplified solution exists which is given in Lemma 3.6.

**B. Proof of Lemma 3.6**

In the symmetric case, the system model in (44) reduces to:

\[
\mathbf{Y}_1 = \sqrt{\rho} \mathbf{H}_{11} \mathbf{X}_1 + \sqrt{\rho^2} \mathbf{H}_{12} \mathbf{X}_2 + \mathbf{Z}_1,
\]

\[
\mathbf{Y}_2 = \sqrt{\rho} \mathbf{H}_{22} \mathbf{X}_2 + \mathbf{Z}_2. \tag{51}\]

Under the symmetric assumption, the sum rate in (49) is bounded as follows:

\[
\sum_{i=1}^{L} R_i - \epsilon_n \leq \log \left| I_{L_1N} + \rho \mathbf{H}_{11} \mathbf{P}_1 \mathbf{H}_{11}^H + \rho \mathbf{H}_{12} \mathbf{P}_2 \mathbf{H}_{12}^H \right| + \log \left| I_{L_2N} + \rho \mathbf{H}_{22} \mathbf{P}_2^{1/2} \left\{ I_{L_2M} + \rho \mathbf{P}_2^{1/2} \mathbf{H}_{12}^H \mathbf{P}_2^{1/2} \right\}^{-1} \mathbf{P}_2^{1/2} \mathbf{H}_{22}^H \right|,
\]

\[
\leq \log \left| I_{L_1N} + \rho \mathbf{H}_{11} \mathbf{H}_{11}^H + \rho \mathbf{H}_{12} \mathbf{H}_{12}^H \right| + \log \left| I_{L_2N} + \rho \mathbf{H}_{22} \left\{ I_{L_2M} + \rho \mathbf{H}_{12}^H \mathbf{H}_{12} \right\}^{-1} \mathbf{H}_{22}^H \right|. \tag{52}\]

Equation (52) is obtained by using Lemma - 3.3 and using the fact that \(\log |\cdot|\) is a monotonically increasing function on the cone of positive definite matrices.

Consider the following term in (52):

\[
I_{L_2N} + \rho \mathbf{H}_{22} \left\{ I_{L_2M} + \rho \mathbf{H}_{12}^H \mathbf{H}_{12} \right\}^{-1} \mathbf{H}_{22}^H \tag{d}\]

\[
= I_{L_2N} + \rho \mathbf{H}_{22} \left\{ I_{L_2M} + \rho \mathbf{U}_{12} \mathbf{U}_{12}^H \right\}^{-1} \mathbf{H}_{22}^H,
\]

\[
= I_{L_2N} + \rho \mathbf{H}_{22} \left\{ I_{L_2M} + \rho \mathbf{\Sigma}_{12} \right\}^{-1} \mathbf{H}_{22}^H, \text{ where } \mathbf{H}_{22} = \mathbf{H}_{22} \mathbf{U}_{12},
\]

\[
= I_{L_2N} + \rho \mathbf{H}_{22} \left\{ I_{L_2M} + \rho \mathbf{\Sigma}_{12} \right\}^{-1} \mathbf{H}_{22}^H, \text{ where } \mathbf{\Sigma}_{12} = \sum_{r}^{r} \begin{bmatrix} 0 & 0 \\ 0 & (L_2M - r)^+ \end{bmatrix}, \text{ and } r = \min \{L_2M, L_1N\}. \tag{e}\]
\[
\begin{align*}
&= \mathbf{I}_{L_2N} + \rho \begin{bmatrix} \bar{H}_{12}^{(a)} & \bar{H}_{12}^{(b)} \\ \end{bmatrix} \begin{bmatrix} (\mathbf{I}_r + \rho \Sigma_r)^{-1} & 0 \\ 0 & \mathbf{I}_{(L_2M-r)^+} \end{bmatrix} \begin{bmatrix} \bar{H}_{12}^{(a)} & \bar{H}_{12}^{(b)} \end{bmatrix}^H,
\end{align*}
\]

where (d) follows by taking EVD of \( \bar{H}_{12}^H \bar{H}_{12} \), \( \bar{U}_{12} \in \mathbb{C}^{L_2M \times L_2M} \), (e) \( \Sigma_r \) contains non-zero singular values of \( \bar{H}_{12}^H \bar{H}_{12} \) and \( \mathbf{0}_{(L_2M-r)^+} \) is a zero matrix of dimension \((L_2M-r)^+ \times (L_2M-r)^+\) and when \( L_2M \leq r \), the zero matrix \( \mathbf{0}_{(L_2M-r)^+} \) does not exist, (f) \( \bar{H}_{22} \) is partitioned into two sub-matrices \( \bar{H}_{22}^{(a)} \) and \( \bar{H}_{22}^{(b)} \) of dimensions \( L_2N \times r \) and \( L_2N \times (L_2M-r)^+ \), respectively.

Using (53), the outer bound in (52) becomes:

\[
\sum_{i=1}^{L} R_i - \epsilon_n \leq \log \left| \mathbf{I}_{L_2N} + \rho \bar{U}_{11} \bar{U}_{11}^H + \rho \bar{U}_{12} \bar{U}_{12}^H \right|
+ \log \left| \mathbf{I}_{L_2N} + \rho \bar{H}_{22}^{(a)} (\mathbf{I}_r + \rho \Sigma_r)^{-1} \bar{H}_{22}^{(a)} + \rho \bar{H}_{22}^{(b)} \mathbf{I}_{(L_2M-r)^+} \bar{H}_{22}^{(b)} \right|
+ \log \left| \mathbf{I}_{L_2N} + \rho \bar{H}_{22}^{(a)} \Sigma_r^{-1} \bar{H}_{22}^{(a)} + \rho \bar{H}_{22}^{(b)} \mathbf{I}_{(L_2M-r)^+} \bar{H}_{22}^{(b)} \right| + O(1) \tag{54}
\]

The above equation is simplified further depending on the values of \( M, N \) and \( \alpha \).

**Case 1 (M \leq N and 0 \leq \alpha \leq 1):**

Under this condition, the outer bound in (54) by using Lemma - 3.1 becomes:

\[
\sum_{i=1}^{L} R_i \leq r_{11} \log \rho + \min \left\{ r_{12}, L_1N - r_{11} \right\} \alpha \log \rho + r_{22}^{(b)} \log \rho
+ \min \left\{ r_{22}^{(a)}, L_2N - r_{22}^{(b)} \right\} (1 - \alpha) \log \rho + O(1), \tag{55}
\]

where \( r_{ij} = \text{rank}(\bar{H}_{ij}) \), \( r_{22}^{(a)} = \text{rank}(\bar{H}_{22}^{(a)}) \) and \( r_{22}^{(b)} = \text{rank}(\bar{H}_{22}^{(b)}) \). As the channel coefficients are drawn from a continuous distribution such as Gaussian, the channel matrices are full rank with probability one. The outer bound in (55) reduces to following form:

\[
\sum_{i=1}^{L} R_i \leq L_1M \log \rho + \min \left\{ r, L_1N - L_1M \right\} \alpha \log \rho + (L_2M - r)^+ \log \rho
+ \min \left\{ \min \left\{ L_2N, r \right\}, L_2N - (L_2M-r)^+ \right\} (1 - \alpha) \log \rho + O(1),
= L_1M \log \rho + \min \left\{ r, L_1N - L_1M \right\} \alpha \log \rho + (L_2M - r)^+ \log \rho
+ \min \left\{ r, L_2N - (L_2M-r)^+ \right\} (1 - \alpha) \log \rho + O(1). \tag{56}
\]

Hence, the sum GDOF is upper bounded as given below:

\[
d_i + \ldots + d_k \leq L_1M + \min \left\{ r, L_1(N - M) \right\} \alpha + (L_2M - r)^+ \min \left\{ r, L_2N - (L_2M-r)^+ \right\} (1 - \alpha). \tag{57}
\]
As $L$ users are chosen among $K$ users in $\binom{K}{L}$ different ways and each user appears in $\binom{K-1}{L-1}$ of these ways. By adding all inequalities like (57), the sum GDOF is upper bounded as:

$$d(\alpha) \leq \frac{1}{L} \left[ L_1 M + \min \{ r, L_1 (N - M) \} \alpha + (L_2 M - r)^+ + \min \{ r, L_2 N - (L_2 M - r)^+ \} (1 - \alpha) \right].$$

By taking minimum of (58) over all possible values of $L_1$ and $L_2$ results in Lemma 3.6(1).

**Case - 2 ($M \leq N$ and $\alpha > 1$):**

The outer bound in (54) is simplified to following form by using Lemma 3.1

$$\sum_{i=1}^{L} R_i \leq \min \{ L_1 N, L_2 M \} \alpha \log \rho + \min \{ L_1 M, L_1 N - \min \{ L_1 N, L_2 M \} \} \log \rho + (L_2 M - r)^+ \log \rho + O(1).$$

By following the same steps as in the previous case, the per user GDOF is upper bounded as given below:

$$d(\alpha) \leq \frac{1}{L} \left[ \min \{ L_1 N, L_2 M \} \alpha + \min \{ L_1 M, L_1 N - \min \{ L_1 N, L_2 M \} \} + (L_2 M - r)^+ \right].$$

By taking minimum of (60) over all possible values of $L_1$ and $L_2$ results in Lemma 3.6(2).

**Case - 3 ($M > N$ and $0 \leq \alpha \leq 1$):**

When $M > N$ and $0 \leq \alpha \leq 1$, the sum rate in (54) reduces to following form by using Lemma 3.1

$$\sum_{i=1}^{L} R_i \leq L_1 N \log \rho + \min \{ L_2 N, (L_2 M - r)^+ \} \log \rho + \min \{ \min \{ L_2 N, r \}, L_2 N - \min \{ L_2 N, (L_2 M - r)^+ \} \} (1 - \alpha) \log \rho + O(1)$$

By following the same steps as in case - 1, the per user GDOF is upper bounded as given below by using (61):

$$d(\alpha) \leq \frac{1}{L} \left[ L_1 N + \min \{ L_2 N, (L_2 M - r)^+ \} + \min \{ \min \{ L_2 N, r \}, L_2 N - \min \{ L_2 N, (L_2 M - r)^+ \} \} (1 - \alpha) \right].$$

By taking minimum of (62) over all possible values of $L_1$ and $L_2$ results in Lemma 3.6(3).

**Case - 4 ($M > N$ and $\alpha \geq 1$):**

Under this condition the outer bound in (54) is simplified to following form by using Lemma 3.1

$$\sum_{i=1}^{L} R_i \leq r \alpha \log \rho + \min \{ L_1 N, L_1 N - r \} \log \rho + \min \{ L_2 N, (L_2 M - r)^+ \} \log \rho + O(1)$$

$$= r \alpha \log \rho + (L_1 N - r) \log \rho + \min \{ L_2 N, (L_2 M - r)^+ \} \log \rho + O(1)$$

By following the same steps as in case-1, the per user GDOF is upper bounded as given below by using (63):

$$d_i(\alpha) \leq \frac{1}{L} \left[ L_1 N + r(\alpha - 1) + \min \{ L_2 N, (L_2 M - r)^+ \} \right].$$

By taking minimum of (64) over all possible values of $L_1$ and $L_2$ results in Lemma 3.6(4). This completes the proof.
C. Proof of Theorem 3.7

The signal received at receiver $i$ is:

$$Y_i = H_{ii}X_i + \sum_{j=1,j\neq i}^{K} H_{ij}X_j + Z_j. \quad (65)$$

Define the following quantity

$$S_{j,B} = \sum_{i \in B} H_{ji}X_i + Z_j, \quad (66)$$

where $B \subseteq \{1, 2, \ldots, K\}$ is the set of users.

The rate of first user is upper bounded as follows:

$$nR_1 \leq (l) I(X_1^n; Y_1^n) + n\epsilon_n,$$

$$\leq (m) I(X_1^n; Y_1^n, S_{2,1}^n) + n\epsilon_n,$$

$$\leq (n) h(S_{2,1}^n) - h(Z_2^n) + h(Y_1^n|S_{2,1}^n) - h(Y_1^n|S_{2,1}^n, X_1^n) + n\epsilon_n,$$

$$\leq (o) h(S_{2,1}^n) - h(Z_2^n) + h(Y_1^n|S_{2,1}^n) - h(S_{1,2}^n) + n\epsilon_n, \quad (67)$$

where (l) follows due to Fano’s inequality; (m) results as the mutual information does not reduce by giving side information; (n) follows from chain rule of mutual information and (o) results due to the fact that by conditioning differential entropy can not increase and the last differential entropy term is conditioned on $X_i, i = 1, \ldots, K$ and $i \neq 2$.

Similarly, the rate of the $K$th user is upper bounded as follows:

$$nR_K \leq I(X_K^n; Y_K^n) + n\epsilon_n,$$

$$\leq h(S_{K-1,K}^n) - h(Z_{K-1}^n) + h(Y_K^n|S_{K-1,K}^n) - h(S_{K,K-1}^n) + n\epsilon_n. \quad (68)$$

The rate of users $i = 2, 3, \ldots, K-1$ are upper bounded as follows:

$$nR_i \leq I(X_i^n; Y_i^n) + n\epsilon_n,$$

$$\leq I(X_i^n; Y_i^n, S_{i-1,i}^n) + n\epsilon_n,$$

$$= h(S_{i-1,i}^n) - h(Z_{i-1}^n) + h(Y_i^n|S_{i-1,i}^n) - h(S_{i,i+1}^n) + n\epsilon_n. \quad (69)$$

Again, the rate of users $i = 2, 3, \ldots, K-1$ can also be bounded as given below:

$$nR_i \leq I(X_i^n; Y_i^n) + n\epsilon_n,$$

$$\leq I(X_i^n; Y_i^n, S_{i+1,i}^n) + n\epsilon_n,$$

$$\leq h(S_{i+1,i}^n) - h(Z_{i+1}^n) + h(Y_i^n|S_{i+1,i}^n) - h(S_{i,i-1}^n) + n\epsilon_n. \quad (70)$$
Summing all the inequalities in (67), (68), (69) and (70), the sum rate is bounded as follows:

\[
R_1 + 2 \sum_{i=2}^{K-1} R_i + R_K \leq I(X^n_i; Y^n_i) + 2 \sum_{i=2}^{K-1} I(X^n_i; Y^n_i) + I(X_K^n; Y^n_K),
\]

or \( R_1 + 2 \sum_{i=2}^{K-1} R_i + R_K - \epsilon_n \leq \sum_{i=1}^{K-1} h(Y^n_i | S^n_{i+1,i}) + \sum_{i=2}^{K-1} h(Y^n_i | S^n_{i-1,i}) - h(Z_1^n) - 2n \sum_{i=2}^{K-1} h(Z_i) - nh(Z_K), \)

The conditional differential entropy terms in (71) are simplified as given below. First consider a particular \( i \) in the first summation term in (71):

\[
h(Y^n_i | S^n_{i+1,i}) = \log \pi e \Sigma_{Y_i} | S^n_{i+1,i}, \]

where

\[
\Sigma_{Y_i} | S^n_{i+1,i} = E [Y^n_i Y^n_i^H] - E [Y^n_i S^n_{i+1,i}] E [S^n_{i+1,i} S^n_{i+1,i}^H]^{-1} E [S^n_{i+1,i} Y^n_i^H].
\]

The individual terms in (73) are obtained as follows:

\[
E [Y^n_i Y^n_i^H] = I_{N_i} + H_{ii} P_i H_{ii}^H + \sum_{j=1, j \neq i}^{K} H_{ij} P_j H_{ij}^H,
\]

\[
E [Y^n_i S^n_{i+1,i}] = H_{ii} P_i H_{ii+1,i},
\]

\[
E [S^n_{i+1,i} S^n_{i+1,i}^H] = I_{N_i} + H_{i+1,i} P_i H_{i+1,i}^H,
\]

\[
E [S^n_{i+1,i} Y^n_i^H] = H_{i+1,i} P_i H_{ii}^H.
\]

Hence, (73) becomes:

\[
\Sigma_{Y_i} | S^n_{i+1,i} = I_{N_i} + \sum_{j=1, j \neq i}^{K} H_{ij} P_j H_{ij}^H + H_{ii} P_i^{1/2} \left( I_{M_i} + P_i^{1/2} H_{i+1,i} H_{i+1,i} P_i^{1/2} \right)^{-1} P_i^{1/2} H_{ii}^H.
\]

Finally we obtain:

\[
h(Y^n_i | S^n_{i+1,i}) = \log \pi e \left[ I_{N_i} + \sum_{j=1, j \neq i}^{K} H_{ij} P_j H_{ij}^H + H_{ii} P_i^{1/2} \left( I_{M_i} + P_i^{1/2} H_{i+1,i} H_{i+1,i} P_i^{1/2} \right)^{-1} P_i^{1/2} H_{ii}^H \right],
\]

(74)

In a similar manner, it can also be shown that:

\[
h(Y^n_i | S^n_{i-1,i}) = \log \pi e \left[ I_{N_i} + \sum_{j=1, j \neq i}^{K} H_{ij} P_j H_{ij}^H + H_{ii} P_i^{1/2} \left( I_{M_i} + P_i^{1/2} H_{i-1,i} H_{i-1,i} P_i^{1/2} \right)^{-1} P_i^{1/2} H_{ii}^H \right].
\]

(75)
Finally, the sum rate is upper bounded by using (74) and (75) in (71):

\[
R_1 + 2 \sum_{i=2}^{K-1} R_i + R_K \\
\leq \sum_{i=1}^{K-1} \log |N_i| + \sum_{j=1, j \neq i}^{K} H_{ij} P_j H_{ij}^H + H_{ii} P_i^{1/2} \left( I_{M_i} + P_i^{1/2} H_{i+1,i}^H H_{i+1,i} P_i^{1/2} \right)^{-1} P_i^{1/2} H_{ii}^H + \epsilon_n \tag{76}
\]

The proof is complete.

**D. Proof of Lemma 3.8**

In the symmetric case, the signal received at the \(i\)th receiver is expressed as:

\[
Y_i = \sqrt{\rho} H_{ii} X_i + \sqrt{\rho}\sum_{j=1, j \neq i}^{K} H_{ij} X_j + Z_j.
\]

For the symmetric case, (76) becomes:

\[
R_1 + 2 \sum_{i=2}^{K-1} R_i + R_K \\
\leq \sum_{i=1}^{K-1} \log |N_i| + \rho^\alpha \sum_{j=1, j \neq i}^{K} H_{ij} H_{ij}^H + \rho^{1-\alpha} H_{ii} \left( H_{i+1,i}^H H_{i+1,i} \right)^{-1} H_{ii}^H \right| + \\
\sum_{i=2}^{K} \log |N_i| + \rho^\alpha \sum_{j=1, j \neq i}^{K} H_{ij} H_{ij}^H + \rho^{1-\alpha} H_{ii} \left( H_{i-1,i}^H H_{i-1,i} \right)^{-1} H_{ii}^H \right| + \mathcal{O}(1). \tag{77}
\]

The outer bound is simplified further under the following cases using Lemma 3.1.

**Weak Interference Case** \(0 \leq \alpha \leq \frac{1}{2}\): In this case, (77) becomes:

\[
R_1 + 2 \sum_{i=2}^{K-1} R_i + R_K \\
\leq \sum_{i=1}^{K-1} [(1 - \alpha)M \log \rho + \min \{ \min (N, (K-1)M), N-M \} \alpha \log \rho] + \\
\sum_{i=2}^{K} [(1 - \alpha)M \log \rho + \min \{ \min (N, (K-1)M), N-M \} \alpha \log \rho] + \mathcal{O}(1),
\]

or

\[
R_i \leq (1 - \alpha)M \log \rho + \min \{ \min (N, (K-1)M), N-M \} \alpha \log \rho + \mathcal{O}(1). \tag{78}
\]

The per user GDOF is upper bounded as follows:

\[
d(\alpha) \leq (1 - \alpha)M + \min \{ \min (N, (K-1)M), N-M \} \alpha. \tag{79}
\]

**Moderate Interference Case** \(\frac{1}{2} \leq \alpha \leq 1\): In this regime, (77) reduces to the following form:

\[
R_1 + 2 \sum_{i=2}^{K-1} R_i + R_K \\
\leq \sum_{i=1}^{K-1} \alpha \min \{ (K-1)M, N \} + \min \{ M, N - \min (N, (K-1)M) \} (1 - \alpha) \log \rho + \\
\sum_{i=2}^{K} \alpha \min \{ (K-1)M, N \} + \min \{ M, N - \min (N, (K-1)M) \} (1 - \alpha) \log \rho + \mathcal{O}(1),
\]
or \( R_i \leq \alpha \min \{ (K-1)M, N \} + \min \{ M, N - \min (N, (K-1)M) \} (1- \alpha) \log \rho + O(1) \).

Hence, per user GDOF is upper bounded as follows:

\[
d(\alpha) \leq \alpha \min \{ (K-1)M, N \} + \min \{ M, N - \min (N, (K-1)M) \} (1- \alpha).
\]

High Interference Case \((\alpha \geq 1)\): In this regime, (77) becomes

\[
R_1 + 2 \sum_{i=2}^{K-1} R_i + R_K \leq \sum_{i=1}^{K-1} \min \{ N, (K-1)M \} \alpha \log \rho + \sum_{i=2}^{K} \min \{ N, (K-1)M \} \alpha \log \rho + O(1),
\]

or \( R_i \leq \min \{ N, (K-1)M \} \alpha + O(1) \).

The symmetric GDOF is upper bounded as:

\[
d(\alpha) \leq \min \{ N, (K-1)M \} \alpha.
\]

Case - 2 \((M > N)\):

When \( M > N \), (76) is simplified as follows:

\[
R_1 + \sum_{i=2}^{K-1} R_i + R_K
\]

\[
\leq \sum_{i=1}^{K-1} \log \{ I_N + \rho^\alpha \sum_{j=1, j \neq i}^{K} H_{ij} H_{ij}^H + \rho H_{ii} (I_M + \rho^\alpha U_{i+1,i} \Sigma_{i+1,i} U_{i+1,i}^H)^{-1} H_{ii}^H \} + \epsilon_n
\]

\[
\sum_{i=2}^{K} \log \{ I_N + \rho^\alpha \sum_{j=1, j \neq i}^{K} H_{ij} H_{ij}^H + \rho H_{ii} (I_M + \rho^\alpha U_{i-1,i} \Sigma_{i-1,i} U_{i-1,i}^H)^{-1} H_{ii}^H \} + \epsilon_n
\]

\[
\sum_{i=1}^{K} \log \{ I_N + \rho^\alpha \sum_{j=1, j \neq i}^{K} H_{ij} H_{ij}^H + \rho \tilde{H}_{ii} (I_M + \rho^\alpha \left[ \begin{array}{cc} \Sigma_{N}^{i+1,i} & 0 \\ 0 & 0_{M-N} \end{array} \right] )^{-1} \tilde{H}_{ii}^H \} + \epsilon_n
\]

\[
\sum_{i=1}^{K} \log \{ I_N + \rho^\alpha \sum_{j=1, j \neq i}^{K} H_{ij} H_{ij}^H + \rho^\alpha \tilde{H}_{ii} (I_M + \rho^\alpha \left[ \begin{array}{cc} \Sigma_{N}^{i-1,i} & 0 \\ 0 & 0_{M-N} \end{array} \right] )^{-1} \tilde{H}_{ii}^H \} + \epsilon_n
\]

\[
\sum_{i=1}^{K} \log \{ I_N + \rho^\alpha \sum_{j=1, j \neq i}^{K} H_{ij} H_{ij}^H + \rho \tilde{H}_{ii} (I_M + \rho^\alpha \left[ \begin{array}{cc} \Sigma_{N}^{i+1,i} & 0 \\ 0 & 0_{M-N} \end{array} \right] )^{-1} \tilde{H}_{ii}^H \} + \epsilon_n
\]

\[
\sum_{i=1}^{K} \log \{ I_N + \rho^\alpha \sum_{j=1, j \neq i}^{K} H_{ij} H_{ij}^H + \rho^\alpha \tilde{H}_{ii} (I_M + \rho^\alpha \left[ \begin{array}{cc} \Sigma_{N}^{i-1,i} & 0 \\ 0 & 0_{M-N} \end{array} \right] )^{-1} \tilde{H}_{ii}^H \} + \epsilon_n
\]

where, (q) is obtained by taking EVD of \( H_{ij}^H H_{ij} \) and \( U_{ij} \in \mathbb{C}^{M \times M} \) and \( \Sigma_{ij} \in \mathbb{C}^{M \times M} \); (r) \( \tilde{H}_{ii} = H_{ii} U_{ij} \) and \( \Sigma_{N}^{ij} \) contains \( N \) non-zero singular values of \( H_{ij}^H H_{ij} \); and (s) \( \tilde{H}_{ii} \) is partitioned in to \( \tilde{H}_{ii}^{(a)} \) and \( \tilde{H}_{ii}^{(b)} \) matrices of dimension \( N \times N \) and \( N \times (M-N) \), respectively.
Equation (83) is simplified further based on the value of $\alpha$.

**Weak Interference Case** ($0 \leq \alpha \leq \frac{1}{2}$):

Consider a specific $i$ in (83):

\[
\log \left| I_N + \rho^\alpha \sum_{j=1, j \neq i}^K H_{ij} H_{ij}^H + \rho^{1-\alpha} \tilde{H}_{ii}^{(a)} \left( \sum_{N+1,i}^{N+1,i} \right)^{-1} \tilde{H}_{ii}^{(a)H} + \rho \tilde{H}_{ii}^{(b)} I_{M-N} \tilde{H}_{ii}^{(b)H} \right|,
\]

\[
\overset{(t)}{=} (M - N) \log \rho + \min \left\{ N, \{ N - (M - N) \}^+ \right\} (1 - \alpha) \log \rho + \min \left\{ \min \left\{ (K - 1)M, N \right\}, \{ N - (M - N) - N \}^+ \right\} \alpha \log \rho + O(1),
\]

\[
= (M - N) \log \rho + (2N - M)^+(1 - \alpha) \log \rho + O(1),
\]

where (t) is obtained by using Lemma 3.4.

From (83) and (84), the outer bound becomes:

\[
R_i \leq (M - N) \log \rho + (2N - M)^+(1 - \alpha) \log \rho + O(1)
\]

In weak interference case, the per user GDOF is upper bounded as:

\[
\tilde{d}_i(\alpha) \leq M - N + (2N - M)^+(1 - \alpha)
\]

**Moderate Interference Case** ($\frac{1}{2} \leq \alpha \leq 1$):

Consider a specific $i$ in (83):

\[
\log \left| I_N + \rho^\alpha \sum_{j=1, j \neq i}^K H_{ij} H_{ij}^H + \rho^{1-\alpha} \tilde{H}_{ii}^{(a)} \left( \sum_{N+1,i}^{N+1,i} \right)^{-1} \tilde{H}_{ii}^{(a)H} + \rho \tilde{H}_{ii}^{(b)} I_{M-N} \tilde{H}_{ii}^{(b)H} \right|,
\]

\[
\overset{(u)}{=} (M - N) \log \rho + \min \left\{ N, \{ N - (M - N) \}^+ \right\} \alpha \log \rho + \min \left\{ \min \left\{ (K - 1)M, N \right\}, \{ N - (M - N) - \min \left\{ (K - 1)M, N \right\} \}^+ \right\} (1 - \alpha) \log \rho + O(1),
\]

\[
= (M - N) \log \rho + \min \left\{ N, \{ 2N - M \}^+ \right\} \alpha \log \rho + O(1),
\]

where (u) is obtained by using Lemma 3.4.

From (83) and (87), the outer bound becomes:

\[
R_i \leq (M - N) \log \rho + \min \left\{ N, \{ 2N - M \}^+ \right\} \alpha \log \rho + O(1)
\]

In moderate interference case, per user GDOF is upper bounded as follows:

\[
\tilde{d}(\alpha) \leq M - N + \min \left\{ N, \{ 2N - M \}^+ \right\} \alpha
\]

**High Interference Case** ($\alpha \geq 1$):
In this case, the outer bound in (83) simplifies as follows:

\[
\sum_{i=1}^{K} R_i \leq \sum_{i=1}^{K-1} \log \left| I_N + \rho^\alpha \sum_{j=1, j \neq i}^{K} H_{ij} H_{ij}^H + \rho \tilde{H}_{ii}^{(b)} I_{M-N} \tilde{H}_{ii}^{(b)H} \right| + \\
\sum_{i=2}^{K} \log \left| I_N + \rho^\alpha \sum_{j=1, j \neq i}^{K} H_{ij} H_{ij}^H + \rho \tilde{H}_{ii}^{(b)} I_{M-N} \tilde{H}_{ii}^{(b)H} \right| + \mathcal{O}(1)
\]

(90)

where (v) is obtained by using Lemma 3.1.

The per user GDOF in case of high interference is upper bounded as follows:

\[ d(\alpha) \leq N\alpha \]

(91)

But the outer bound in this case exceeds the interference free GDOF i.e. \( N \) as \( \alpha \geq 1 \). Hence, this outer bound is not useful when \( \alpha \geq 1 \).

By combining (86), (89) and (91) results in Lemma 3.8. This completes the proof.

E. Proof of Theorem - 3.9

The signal received at receiver \( j \) is:

\[
Y_j = H_{ij} X_j + \sum_{j=1, j \neq i}^{K} H_{ji} X_i + Z_j.
\]

(92)

The transmitter and receiver pair \( i \) is assumed to have \( M_i \) and \( N_i \) antennas, respectively. Now define the following quantity as defined in the previous Theorem 3.7 as follows:

\[
S_{j,B} = \sum_{i \in B} H_{ji} X_i + Z_j.
\]

(93)

Here \( B \) is the set of transmitters. Let \( A = \{1, 2, \ldots, K\} \) be the set of all transmitters. \( A - B \) is the complement of \( B \) in \( A \). Following the procedure given in [9], the sum rate is bounded as follows.

\[
n \left[ R_1 + \sum_{i=2}^{K-1} R_i + R_K \right] - n\epsilon_n \leq nh(Y_1^*, S_{K,1}^*) + nh(Y_K^*, S_{1,K}^*) + n \sum_{i=2}^{K-1} h(Y_i^*, S_K^*,\{1,2,\ldots,i\}, S_i^*, A-\{1,2,\ldots,i\}) + \\
n \sum_{i=2}^{K-1} h(Y_i^*, S_1^*,\{K,2,\ldots,i\}, S_K^*, A-\{K,2,3,\ldots,i\}) - nh(Z_1) - n \sum_{i=2}^{K-1} h(Z_i) - nh(Z_K).
\]

(94)
The individual terms in (94) are evaluated as described below. The first term \( h\left( Y_1^*|S_{K,1}^* \right) \) in (94) is similar to the evaluation of conditional differential entropy in the proof of Theorem 3.7. On simplification, first term becomes:

\[
h\left( Y_1^*|S_{K,1}^* \right) = \log \left| \pi e \left( I_{N_1} + \sum_{j=2}^{K} H_{ij} P_j H_j^H + H_{11} P_1^{1/2} \left( I_{M_1} + P_1^{1/2} H_{K1} H_{K1} H_{11} P_1^{1/2} \right)^{-1} P_1^{1/2} H_{11}^H \right) \right| .
\]

(95)

Now consider the term \( h\left( Y_K^*|S_{1,K}^* \right) \) which on simplification becomes:

\[
h\left( Y_K^*|S_{1,K}^* \right) = \log \left| \pi e \left( I_{N_K} + \sum_{j=2}^{K} H_{Kj} P_j H_{Kj}^H + H_{KK} P_K^{1/2} \left( I_{M_K} + P_K^{1/2} H_{K1} H_{1K} H_{1K} P_K^{1/2} \right)^{-1} P_K^{1/2} H_{KK}^H \right) \right| .
\]

(96)

Now consider the term \( h\left( Y_i^*|S_{K,1,\ldots,i}^*, S_{1,A-\{1,\ldots,i\}}^* \right) \). In this case,

\[
Y_i^* = H_i X_i^* + \sum_{j=1, j \neq i}^{K} H_{ij} X_j^* + Z_i,
\]

\[
S_{K,1,\ldots,i}^* = \sum_{j \in \{1, \ldots, i\}} H_{Kj} X_j^* + Z_K,
\]

and \( S_{1,A-\{1,\ldots,i\}}^* = \sum_{j \in A-\{1,\ldots,i\}} H_{1j} X_j^* + Z_1 \).

The conditional differential entropy becomes:

\[
h\left( Y_i^*|S_{K,1,\ldots,i}^*, S_{1,A-\{1,\ldots,i\}}^* \right) = \log \left| \pi e \Sigma Y_i^*|s_{K,1,\ldots,i}^*, s_{1,A-\{1,\ldots,i\}}^* \right| ,
\]

(97)

where,

\[
\Sigma Y_i^*|s_{K,1,\ldots,i}^*, s_{1,A-\{1,\ldots,i\}}^* = E\left[ Y_i^* Y_i^{*H} \right] - E\left[ Y_i^* S^{*H} \right] E\left[ S^* S^{*H} \right]^{-1} E\left[ S^* Y_i^{*H} \right],
\]

(98)

The output at receiver \( i \) can also be expressed as follows.

\[
Y_i^* = \overline{H}_{i1} X_1 + \overline{H}_{i,i+1} X_2 + Z_i,
\]

(99)

where \( \overline{X}_1 = \begin{bmatrix} X_1^* & X_2^* & \ldots & X_i^* \end{bmatrix}^T \), \( \overline{X}_2 = \begin{bmatrix} X_{i+1}^* & X_{i+2}^* & \ldots & X_K^* \end{bmatrix}^T \).

The other quantities \( \overline{H}_{11} \) and \( \overline{H}_{i,i+1} \) are defined as before. The two side information terms can also be expressed as follows.

\[
S_{K,1,\ldots,i} = \overline{H}_{K1} \overline{X}_1 + Z_K,
\]

(100)

\[
S_{1,A-\{1,\ldots,i\}} = \overline{H}_{1,i+1} \overline{X}_2 + Z_i.
\]

(101)

Now consider the evaluation of individual terms in (98):

\[
E\left[ Y_i^* Y_i^{*H} \right] = I_{N_i} + \overline{H}_{i1} \overline{P}_1 \overline{H}_{11} + \overline{H}_{i,i+1} \overline{P}_2 \overline{H}_{i,i+1},
\]

(102)

\[
E\left[ Y_i^* S^{*H} \right] = \begin{bmatrix} \overline{H}_{i1} \overline{P}_1 \overline{H}_{K1} & \overline{H}_{i,i+1} \overline{P}_2 \overline{H}_{i,i+1} \end{bmatrix}.
\]

(103)
where \( M_{ri} = \sum_{j=1}^{i} M_j \) and \( M_{si} = \sum_{j=i+1}^{K} M_j \).

Hence, (98) simplifies to following form:

\[
\sum Y_i^r | S^*_{K,\{1,2,\ldots, i\}}, S^*_{1, A-\{1,2,\ldots, i\}} = I_{N_i} + H_{i1} P^{1/2}_{i1} \left\{ I_{M_{ri}} + P^{1/2}_{i1} H_{iK} H_{iK} P^{1/2}_{i1} \right\}^{-1} P^{1/2}_{i1} H_{i1} + \\
H_{i, i+1} P^{1/2}_{i2} \left\{ I_{M_{ri}} + P^{1/2}_{i2} H_{i1} H_{i1} P^{1/2}_{i2} \right\}^{-1} P^{1/2}_{i2} H_{i, i+1}.
\]

(106)

Hence, (97) becomes

\[
h \left( Y_i^r | S^*_{K,\{1,2,\ldots, i\}}, S^*_{1, A-\{1,2,\ldots, i\}} \right) = \log \left| \pi e \left\{ I_{N_i} + H_{i1} P^{1/2}_{i1} \left\{ I_{M_{ri}} + P^{1/2}_{i1} H_{iK} H_{iK} P^{1/2}_{i1} \right\}^{-1} P^{1/2}_{i1} H_{i1} + \\
H_{i, i+1} P^{1/2}_{i2} \left\{ I_{M_{ri}} + P^{1/2}_{i2} H_{i1} H_{i1} P^{1/2}_{i2} \right\}^{-1} P^{1/2}_{i2} H_{i, i+1} \right\} \right|
\]

(107)

In a similar manner, it can be shown that:

\[
h \left( Y_i^r | S^*_{K,\{1,2,\ldots, i\}}, S^*_{K, A-\{1,2,\ldots, i\}} \right) = \log \left| \pi e \left\{ I_{N_i} + H_{iK} P^{1/2}_{iK} \left\{ I_{M_{ri}} + P^{1/2}_{iK} H_{i1} H_{i1} P^{1/2}_{iK} \right\}^{-1} P^{1/2}_{iK} H_{iK} + \\
H_{i, K-1} P^{1/2}_{iK} \left\{ I_{M_{ri}} + P^{1/2}_{iK} H_{i1} H_{i1} P^{1/2}_{iK} \right\}^{-1} P^{1/2}_{iK} H_{i, K-1} \right\} \right|
\]

(108)

where \( M_{ri}' = \sum_{j=2}^{i} M_j + M_K \) and \( M_{si}' = M_1 + \sum_{j=i+1}^{K-1} M_j \).

By combining (95), (96), (107) and (108) results in Theorem 3.9

**F. Proof of Lemma - 3.10**

For symmetric case, the sum rate outer bound in Theorem 3.9 reduces to following form:

\[
R_1 + \sum_{i=2}^{K-1} R_i + R_K \leq \log \left| I_{N_i} + \rho^\alpha \sum_{j=2}^{K} H_{ij} H_{ij}^H + \rho H_{i1} \left\{ I_{M_{ri}} + \rho^\alpha H_{iK} H_{iK} \right\}^{-1} H_{i1}^H \right| + \\
\sum_{i=2}^{K-1} \log \left| I_{N_i} + H_{i1} \left\{ I_{M_{ri}} + H_{iK}^H H_{iK} \right\}^{-1} H_{i1}^H + H_{i, i+1} \left\{ I_{M_{ri}} + H_{i1, i+1} H_{i1, i+1} \right\}^{-1} H_{i, i+1} \right| + \\
\sum_{i=2}^{K-1} \log \left| I_{N_i} + H_{iK} \left\{ I_{M_{ri}} + H_{iK}^H H_{iK} \right\}^{-1} H_{iK}^H + H_{i, K-1} \left\{ I_{M_{ri}} + H_{iK, i+1} H_{iK, i+1} \right\}^{-1} H_{i, K-1} \right| + \\
\log \left| I_{N_K} + \sum_{j=1}^{K-1} H_{Kj} H_{Kj}^H + H_{KK} \left\{ I_{MK} + H_{KK}^H K_H K_H \right\}^{-1} H_{KK}^H \right| + \epsilon_n,
\]

(109)
In a similar way, it can be shown that:

\[ \Pi_{i,1} = \left[ \sqrt{\rho^\alpha} H_{i,1} \sqrt{\rho^\alpha} H_{i,2} \cdots \sqrt{\rho^\alpha} H_{i,i} \right], \quad \Pi_{i,i+1} = \left[ \sqrt{\rho^\alpha} H_{i,i+1} \sqrt{\rho^\alpha} H_{i,i+2} \cdots \sqrt{\rho^\alpha} H_{i,K} \right], \]
\[ \Pi_{K,i} = \left[ \sqrt{\rho^\alpha} H_{K,1} \sqrt{\rho^\alpha} H_{K,2} \cdots \sqrt{\rho^\alpha} H_{K,i} \right], \quad \Pi_{i,i+1} = \left[ \sqrt{\rho^\alpha} H_{i,i+1} \sqrt{\rho^\alpha} H_{i,i+2} \cdots \sqrt{\rho^\alpha} H_{i,K} \right], \]
\[ \Pi_{1,i} = \left[ \sqrt{\rho^\alpha} H_{1,1} \sqrt{\rho^\alpha} H_{1,2} \cdots \sqrt{\rho^\alpha} H_{1,i} \right], \quad \Pi_{K,i+1} = \left[ \sqrt{\rho^\alpha} H_{K,1} \sqrt{\rho^\alpha} H_{K,2} \cdots \sqrt{\rho^\alpha} H_{K,K-1} \right], \]
\[ \Pi_{i,K} = \left[ \sqrt{\rho^\alpha} H_{i,K} \sqrt{\rho^\alpha} H_{i,2} \cdots \sqrt{\rho^\alpha} H_{i,i} \right], \quad \Pi_{i,K-1} = \left[ \sqrt{\rho^\alpha} H_{i,i+1} \sqrt{\rho^\alpha} H_{i,i+2} \cdots \sqrt{\rho^\alpha} H_{i,K-1} \right]. \]  

(110)

Equation (109) is obtained by using Lemma 3.3.

Now consider the following term in (109):

\[
\log \left| I_{N_i} + \Pi_{i,1} \left\{ I_{M_{i,i}} + \Pi_{K,i}^H \Pi_{K,i} \right\}^{-1} \Pi_{i,1} + \Pi_{i,i+1} \left\{ I_{M_{i,i}} + \Pi_{L,i+1}^H \Pi_{L,i+1} \right\}^{-1} \Pi_{i,i+1} \right|.
\]

(111)

In the above equation, consider the following term:

\[
\Pi_{i,1} \left\{ I_{M_{i,i}} + \Pi_{K,i}^H \Pi_{K,i} \right\}^{-1} \Pi_{i,1}^H
\]
\[= \rho^\alpha \left[ H_{i,1} H_{i,2} \cdots \sqrt{\rho^{1-\alpha} H_{i,i}} \right] \left\{ I_{M_{i,i}} + \rho^\alpha \left[ H_{K,1} H_{K,2} \cdots H_{K,i} \right]^H \left[ H_{K,1} H_{K,2} \cdots H_{K,i} \right] \right\}^{-1} \]
\[\left[ H_{i,1} H_{i,2} \cdots \sqrt{\rho^{1-\alpha} H_{i,i}} \right]^H. \]  

(112)

In a similar way, it can be shown that:

\[
\Pi_{i,i+1} \left\{ I_{M_{i,i}} + \Pi_{L,i+1}^H \Pi_{L,i+1} \right\}^{-1} \Pi_{i,i+1}^H
\]
\[= \rho^\alpha \left[ H_{i,i+1} H_{i,i+2} \cdots H_{i,K} \right] \left\{ I_{M_{i,i}} + \rho^\alpha \left[ H_{1,1} H_{1,1} H_{1,1} \cdots H_{1,K} \right]^H \left[ H_{1,1} H_{1,1} H_{1,1} \cdots H_{1,K} \right] \right\}^{-1} \]
\[\left[ H_{i,i+1} H_{i,i+2} \cdots H_{i,K} \right]^H. \]  

(113)

From (112) and (113), (111) becomes:

\[
\log \left| I_{N_i} + \rho^\alpha \left[ H_{i,1} H_{i,2} \cdots \sqrt{\rho^{1-\alpha} H_{i,i}} \right] \left\{ I_{M_{i,i}} + \rho^\alpha \left[ H_{K,1} H_{K,2} \cdots H_{K,i} \right]^H \left[ H_{K,1} H_{K,2} \cdots H_{K,i} \right] \right\}^{-1} \left[ H_{i,1} H_{i,2} \cdots \sqrt{\rho^{1-\alpha} H_{i,i}} \right]^H \right|
\]
\[+ \rho^\alpha \left[ H_{i,i+1} H_{i,i+2} \cdots H_{i,K} \right] \left\{ I_{M_{i,i}} + \rho^\alpha \left[ H_{1,1} H_{1,1} H_{1,1} \cdots H_{1,K} \right]^H \left[ H_{1,1} H_{1,1} H_{1,1} \cdots H_{1,K} \right] \right\}^{-1} \left[ H_{i,i+1} H_{i,i+2} \cdots H_{i,K} \right]^H \]
\[= \log \left| I_{N_i} + \left[ H_{i,1} H_{i,2} \cdots \sqrt{\rho^{1-\alpha} H_{i,i}} \right] \left[ H_{i,1} H_{i,2} \cdots \sqrt{\rho^{1-\alpha} H_{i,i}} \right]^H + \mathcal{O}(1), \right|
\]
\[= \log \left| I_{N_i} + \rho^\alpha H_{i,i} H_{i,i}^H \right| + \mathcal{O}(1), \]  

(114)

where (u) is obtained by using the fact that the terms containing inverses are independent of \( \alpha \) and are invertible when \( \frac{N}{M} < K \leq \frac{N}{M} + 1 \) and (v) is obtained by taking the constant terms in to \( \mathcal{O}(1) \) approximation.

In a similar manner it can be shown that:

\[
\log \left| I_{N_i} + \Pi_{i,K} \left\{ I_{M_{i,i}} + \Pi_{L,i}^H \Pi_{L,i} \right\}^{-1} \Pi_{i,K}^H + \Pi_{i,K-1} \left\{ I_{M_{i,i}} + \Pi_{L,K,i+1}^H \Pi_{L,K,i+1} \right\}^{-1} \Pi_{i,K-1}^H \right|
\]
\[= \log \left| I_{N_i} + \rho^\alpha H_{i,i} H_{i,i}^H \right| + \mathcal{O}(1). \]  

(115)
Using (114) and (115), the sum rate bound in (109) reduces to the following form:

\[
R_1 + \sum_{i=2}^{K-1} R_i + R_K \leq \log |I_{N_1} + \rho^{\alpha} \sum_{j=2}^{K} H_{1j}H_{1j}^H + \rho^{1-\alpha} H_{11} \{H_{K1}^H H_{1K}^{-1} H_{11}^H | + \]
\[
\sum_{i=2}^{K-1} \log |I_{N_i} + \rho^{1-\alpha} H_{ii}H_{ii}^H| + \sum_{i=2}^{K-1} \log |I_{N_i} + \rho^{1-\alpha} H_{ii}H_{ii}^H| + \]
\[
\log |I_{N_K} + \sum_{j=1}^{K-1} H_{Kj}H_{Kj}^H + \rho^{1-\alpha} H_{KK} \{H_{1K}^H H_{1K}^{-1} H_{KK}^H | + \epsilon_n. \tag{116}
\]

The outer bound in (116) is simplified further based on the values of \(\alpha\).

**Weak interference case** \(0 \leq \alpha \leq \frac{1}{2}\):

In this case, the sum rate bound in (116) is simplified to the following form:

\[
R_1 + \sum_{i=2}^{K-1} R_i + R_K
\leq M(1-\alpha) \log \rho + \min \{(N, (K-1)M), N-M\} \alpha \log \rho + \sum_{i=2}^{K-1} M(1-\alpha) \log \rho + \]
\[
\sum_{i=2}^{K-1} M(1-\alpha) \log \rho + M(1-\alpha) \log \rho + \min \{(N, (K-1)M), N-M\} \alpha \log \rho + O(1)
\]
\[
= [2M(1-\alpha) + 2 \min \{(N, (K-1)M), N-M\} \alpha + 2(K-2)M(1-\alpha)] \log \rho + O(1). \tag{117}
\]

The above equation is obtained by using Lemma 3.1.

The individual rate is upper bounded as follows:

\[
R_j \leq M(1-\alpha) \log \rho + \frac{1}{K-1} \min \{(N, (K-1)M), N-M\} \alpha \log \rho + O(1). \tag{118}
\]

Hence the per user GDOF is upper bounded as given below:

\[
d(\alpha) \leq M(1-\alpha) + \frac{1}{K-1} \min \{(N, (K-1)M), N-M\} \alpha. \tag{119}
\]

**Moderate interference case** \(\frac{1}{2} \leq \alpha \leq 1\):

In this case, (116) simplifies to the following form.

\[
R_1 + \sum_{i=2}^{K-1} R_i + R_K
\leq 2\alpha \min \{N, (K-1)M\} \log \rho + 2 \min \{M, N - \min \{N, (K-1)M\}\} (1-\alpha) \log \rho + 2(K-2)M(1-\alpha) \log \rho + O(1).
\]

The individual rate is bounded as follows:

\[
2(K-2)R_j \leq 2\alpha \min \{N, (K-1)M\} \log \rho + 2 \min \{M, N - \min \{N, (K-1)M\}\} (1-\alpha) \log \rho + 2(K-2)M(1-\alpha) \log \rho + O(1),
\]

or

\[
R_j \leq \frac{1}{K-1} [\alpha \min \{N, (K-1)M\} \log \rho + \min \{M, N - \min \{N, (K-1)M\}\} (1-\alpha) \log \rho + (K-2)M(1-\alpha) \log \rho] + O(1). \tag{120}
\]
The achievable rate in (125) is simplified under the following cases:

$$d(\alpha) \leq \frac{1}{K-1} \left[ \alpha \min \{ N, (K-1)M \} + \min \{ M, N - \min \{ N, (K-1)M \} \} (1-\alpha) + (K-2)M(1-\alpha) \right].$$

\text{(121)}

**High interference case** ($\alpha \geq 1$):

In the high interference case, the sum rate bound in (116) becomes:

$$R_1 + \sum_{i=2}^{K-1} R_i + R_K \leq \log \left| I + \rho^\alpha \sum_{j=2}^{K} H_{ij} P_i H_{ij}^H \right| + \log \left| I + \rho^\alpha \sum_{j=1}^{K-1} H_{Kj} H_{Kj}^H \right| + \mathcal{O}(1),$$

or $R_j \leq \min \left\{ \frac{N}{K-1}, M \right\} \alpha \log \rho + \mathcal{O}(1).$  \text{(122)}

The per user GDOF is upper bounded as follows:

$$d(\alpha) \leq \min \left\{ \frac{N}{K-1}, M \right\} \alpha. \text{(123)}$$

Now by combining (119), (121) and (122) results in Lemma - 3.10.

**G. Proof of Theorem 4.1**

When interference is treated as noise, the rate achieved by the individual user in case of MIMO Gaussian IC is:

$$R_i \geq \log \left| I + \left( I + \rho^\alpha \sum_{j=1, j \neq i}^{K} H_{ij} P_j H_{ij}^H \right) \rho H_{ii} P_i H_{ii}^H \right|^{-1},$$

$$= \log \left| I + \rho H_{ii} P_i H_{ii}^H + \rho^\alpha \sum_{j=1, j \neq i}^{K} H_{ij} P_j H_{ij}^H \right| - \log \left| I + \rho^\alpha \sum_{j=1, j \neq i}^{K} H_{ij} P_j H_{ij}^H \right|,$n

$$= r \log \rho + \min \left\{ r', N - r \right\} \alpha \log \rho - \alpha r' \log \rho + \mathcal{O}(1),$$ \text{(124)}

where $r = \text{rank}(H_{ii} P_i H_{ii}^H)$ and $r' = \text{rank} \left( \sum_{j=1, j \neq i}^{K} H_{ij} P_j H_{ij}^H \right)$. The above equation is obtained by using Lemma 3.1. As the input covariance matrix is considered to be full rank, (124) becomes:

$$R_i \geq M \log \rho + \min \left\{ \min \{ (K-1)M, N \}, N - M \right\} \alpha \log \rho - \alpha \min \{ (K-1)M, N \} \log \rho + \mathcal{O}(1).$$ \text{(125)}

The achievable rate in (125) is simplified under the following cases:

1) $\frac{N}{M} < K \leq \frac{N}{M} + 1$
2) $K > \frac{N}{M} + 1$

1) Case - 1 ($\frac{N}{M} < K \leq \frac{N}{M} + 1$) : Under this condition, it can be observed that:

$$\min \{ (K-1)M, N \} = (K-1)M.$$ \text{(126)}

The rate in (125) reduces to following form:

$$R_i \geq M \log \rho + \min \{ (K-1)M, N - M \} \alpha \log \rho - \alpha (K-1)M \log \rho + \mathcal{O}(1),$$

$$= M \log \rho + (N - M) \alpha \log \rho - \alpha (K-1)M \log \rho + \mathcal{O}(1).$$ \text{(127)}
Above equation is obtained by using the fact that given:

\[ N < KM, \]

or \[ N - M < KM - M = (K - 1)M. \] (128)

The per user GDOF that can be achieved in this case is:

\[ d(\alpha) \geq M + (N - M)\alpha - \alpha(K - 1)M, \]

\[ = M + \alpha(N - KM), \quad i = 1, 2, \ldots, K. \] (129)

2) Case -2 \( K > \frac{N}{M} + 1 \): Under this condition, it can be observed that:

\[ \min \{ (K - 1)M, N \} = N. \] (130)

Hence, the rate in (125) becomes:

\[ R_i \geq M \log \rho + \min \{ N, N - M \} \alpha \log \rho - N \alpha \log \rho + \mathcal{O}(1), \]

\[ = M(1 - \alpha) \log \rho + \mathcal{O}(1). \] (131)

The achievable per user GDOF in this case is:

\[ d(\alpha) \geq M(1 - \alpha). \] (132)

By combining (129) and (132) results in Theorem 4.1.

H. Proof of Theorem - 4.2

Due to the symmetry of the problem, it is sufficient to consider the GDOF achieved by any particular user, say user 1. Also consider the user subset \( S \subseteq \{2, \ldots, K\} \), and let \( S' \triangleq S \cup \{1\} \), i.e., \( S \) is a subset of users excluding user 1, while \( S' \) always includes user 1. The number of users in the set \( S \) is denoted by \( |S| \leq K - 1 \) and number of users in \( S' \) is \( |S| + 1 \). The achievable GDOF is obtained under following two conditions:

1) \( \frac{N}{M} < K \leq \frac{N}{M} + 1 \),
2) \( K > \frac{N}{M} + 1 \).

1) Case -1 \( \left( \frac{N}{M} < K \leq \frac{N}{M} + 1 \right) \):

Now, using the MAC channel formed at the receiver of user 1 with the signals from the user set \( S \), the achievable sum rate is bounded as:

\[ \sum_{j \in S} R_j \leq \log |I + \rho^\alpha \sum_{j \in S} H_{1j} P_j H_{1j}^H|, \]

or \[ R_j \leq M\alpha \log \rho + \mathcal{O}(1) \] (133)
Similarly, using the MAC channel formed at the receiver of user 1 with the signals from the user set $S'$, the achievable sum rate is bounded as:

$$\sum_{j \in S'} R_j \leq \log |I + \rho H_{11} P H_{11}^H + \rho \alpha \sum_{j \in S} H_{1j} P_j H_{1j}^H|,$$

$$= |S|M \alpha \log \rho + \min \{M, N - |S|M\} \log \rho + O(1). \quad (134)$$

Above equation is obtained by using Lemma 3.1 and simplified further based on following conditions.

When $\min \{M, N - |S|M\} = N - |S|M$, then following condition is obtained:

$$N - |S|M \leq M,$$

or $N \leq 1 + |S| \leq \frac{N}{M} + 1, \quad \therefore K \leq \frac{N}{M} + 1 \quad (135)$

When the condition in (135) is satisfied, (134) reduces to following form:

$$\sum_{j \in S'} R_j \leq |S|M \alpha \log \rho + (N - |S|M) \log \rho + O(1),$$

or $R_j \leq \frac{|S|M(\alpha - 1) + N}{1 + |S|} \log \rho + O(1). \quad (136)$

The above equation is minimized when $|S| = |S|_{\text{max}} = K - 1$ as $K \leq \frac{N}{M} + 1$. Hence, (136) becomes:

$$R_j \leq \frac{(K - 1)M(\alpha - 1) + N}{K} \log \rho + O(1). \quad (137)$$

When $\min \{M, N - |S|M\} = M$, then following condition is obtained:

$$M \leq N - |S|M,$$

or $1 + |S| \leq \frac{N}{M}. \quad (138)$

When above condition is satisfied, (134) becomes:

$$\sum_{j \in S'} R_j \leq |S|M \log \rho + M \log \rho + O(1),$$

or $R_j \leq M \log \rho + O(1). \quad (139)$

The achievable rate is obtained by taking minimum of (133), (137) and (139). It can be observed that (133) becomes superfluous given (139). Finally, by taking minimum of (137) and (139) results in Theorem 4.2 (1).

2) Case - 2: $(K > \frac{N}{M} + 1)$: Now, using the MAC channel formed at the receiver of user 1 with the signals from the user set $S$, the achievable sum rate is:

$$\sum_{j \in S} R_j \leq \log |I + \rho \alpha \sum_{j \in S} H_{1j} P_j H_{1j}^H|,$$

$$= \min \{|S|M, N\} \alpha \log \rho + O(1). \quad (140)$$

If $\min \{|S|M, N\} = |S|M$, then the above equation simplifies to:

$$R_j \leq \alpha M \log \rho + O(1) \quad (141)$$
If \( \min \{|S|, N\} = N \), then (140) simplifies to:

\[
R_j \leq \frac{N\alpha}{|S|} \log \rho + O(1), \text{ where } |S| \leq K - 1.
\]  

(142)

Similarly, using the MAC channel formed at the receiver of user 1 with the signals from the user set \( S' \), the achievable sum rate is bounded as:

\[
\sum_{j \in S'} R_j \leq |S| \alpha \log \rho + \min\{M, N - |S|\} \log \rho + O(1),
\]  

(143)

Again, using Lemma 3.1 above simplifies to:

\[
\sum_{j \in S'} R_j \leq \min\{|S|, M\} \alpha \log \rho + \min\{M, N - |S|\} \log \rho + O(1),
\]  

(144)

Above equation is simplified further under following cases.

**Case - a:**

If \( \min\{|S|, M\} = |S| \), then we have:

\[
|S| \leq N, \\
\text{or } |S| \leq \frac{N}{M}.
\]

(145)

When the above condition is satisfied, (131) becomes:

\[
\sum_{j \in S'} R_j \leq |S| \alpha \log \rho + \min\{M, N - |S|\} \log \rho + O(1),
\]  

(146)

If \( \min\{M, N - |S|\} = N - |S| \), then we have the following condition:

\[
N - |S| \leq M, \\
\text{or } \frac{N}{M} - 1 \leq |S|.
\]

(147)

From (145) and (146) we obtain the following condition:

\[
\frac{N}{M} - 1 \leq |S| \leq \frac{N}{M}.
\]

(148)

When the condition in (147) is satisfied, (144) becomes:

\[
\sum_{j \in S'} R_j \leq |S| \alpha \log \rho + (N - |S|) \log \rho + O(1),
\]

\[
\text{or } R_j \leq \frac{|S| \alpha - 1 + N}{1 + |S|} \log \rho + O(1).
\]

(149)

The above equation is required to be minimized for largest value of \( |S| \) which satisfies the condition in (148).

When \( \min\{M, N - |S|\} = M \), then following condition is obtained:

\[
M \leq N - |S|, \\
\text{or } |S| \leq \frac{N}{M} - 1.
\]

(150)
From (145) and (150), following condition is deduced:

\[ |S| \leq \frac{N}{M} - 1 < \frac{N}{M}. \]  

(151)

Under this condition, (146) becomes:

\[ \sum_{j \in S'} R_j \leq |S| M \alpha \log \rho + M \log \rho + O(1), \]

or

\[ R_j \leq \frac{|S| \alpha + 1}{|S| + 1} M \log \rho + O(1). \]  

(152)

The above equation is required to be minimized for \( |S| \leq \frac{N}{M} - 1 \) and \( |S| = 0 \) minimizes this equation. This results in the following equation:

\[ R_j \leq M \log \rho + O(1) \]  

(153)

**Case b:**

When \( (|S|, M, N) = N \), then we have the following condition:

\[ N \leq |S| M \]

or \( |S| M \geq N \)  

(154)

Under this condition, (144) reduces to:

\[ \sum_{j \in S'} R_j \leq N \alpha \log \rho + \min\{M, N - N\} \log \rho + O(1), \]

or

\[ R_j \leq \frac{N \alpha}{|S| + 1} \log \rho + O(1), \]  

(155)

Above equation is minimized when \( |S| \) takes the maximum value provided condition in (154) is also satisfied. As \( K > \frac{N}{M} + 1 \), the maximum value of \( |S| \) which satisfies the condition in (154) is \( S_{\text{max}} = K - 1 \). Hence, (155) becomes:

\[ R_j \leq \frac{N \alpha}{K} \log \rho + O(1) \]  

(156)

Finally taking minimum of (141), (142), (149), (153) and (156) the achievable GDOF is obtained as described below.

Given (153), the equation in (141) becomes superfluous as \( \alpha > 1 \). Similarly given (156), the equation in (142) is redundant. It can also be observed that (149) is redundant given (156). It can be proved as follows:

Equation (149) is minimized when the value of \( |S| \) is chosen to be maximum along with it satisfies the condition in (148). When \( |S| = \frac{N}{M} \), (149) reduces to following form.

\[ R_j \leq \frac{N(\alpha - 1) + N}{N + 1} \log \rho + O(1) \]

\[ = \frac{N \alpha}{M + 1} \log \rho + O(1) \]  

(157)

As \( K > \frac{N}{M} + 1 \), given (156), the rate in (149) becomes superfluous. When \( \frac{N}{M} \) is not an integer, then the largest integer value of \( |S| \) is chosen so that condition in (148) is satisfied and the above argument still remains valid. Finally by taking minimum of (153) and (156) results in Theorem 4.2(2).
First we calculate the rate obtained due to the private part of the message. As the private message is decoded last, the rate of the private message is obtained by treating all remaining private user’s message as noise. Due to symmetry of the problem, it is sufficient to consider only one particular user. The rate achieved by the private part is

\[
R_p \leq \log |I + \left( I + \sum_{j=1, j \neq i}^{K} H_{j}P_{j}H_{j}^{H} \right)^{-1} \rho^{1-\alpha} H_{jj}P_{j}H_{jj}^{H} | \\
= M(1-\alpha) \log \rho + O(1)
\] (158)

Following two cases are considered to obtain the achievable GDOF by common part of the message:

1) \( \frac{N}{M} < K \leq \frac{N}{M} + 1 \),

2) \( K > \frac{N}{M} + 1 \).

1) Case - 1: \( \left( \frac{N}{M} < K \leq \frac{N}{M} + 1 \right) \): For the common part, the different subset of users are considered as in Appendix - H. Now consider the set \( S' \subseteq \{1, 2, \ldots, K \} \), where user - 1 is always included in the subset. Due to symmetry of the problem, we consider a particular user -1. Each common message need to be decodable at every receiver. Hence, user-1 should be able to decode other user’s common message along with its intended common message. While decoding the common message, it should treat all other users private message as noise including its private message. The common message forms a MAC channel at Receiver-1. Similarly, \( K - 1 \) MAC channel will be formed at every receiver. The achievable rate is intersection of \( K \) such MAC regions. But due to symmetry of the problem, it is sufficient to consider only one specific receiver. The achievable rate due to the signals from \( S' \) is:

\[
\sum_{j \in S'} R_{c,j} \leq \log \left| I + \left( I + \sum_{j \in S} H_{1j}P_{j}H_{1j}^{H} + \rho^{1-\alpha} H_{11}P_{1}H_{11}^{H} + \rho^{1-\alpha} H_{jj}P_{j}H_{jj}^{H} \right)^{-1} \sum_{j \in S'} P_{c,j} H_{1j}P_{j}H_{1j}^{H} \right| 
\] (159)

Here

\[
P_{c,j} = \begin{cases} 
\rho^{\alpha} - 1 & \text{if } j \neq 1 \\
\rho - \rho^{1-\alpha} & \text{if } j = 1 
\end{cases}
\] (160)

Equation (159) becomes:

\[
\sum_{j \in S'} R_{c,j} \leq \log \left| I + \sum_{j \in S} H_{1j}P_{j}H_{1j}^{H} + \rho^{1-\alpha} H_{11}P_{1}H_{11}^{H} + \rho^{1-\alpha} H_{jj}P_{j}H_{jj}^{H} \right| - \log \left| I + \rho^{1-\alpha} H_{11}P_{1}H_{11}^{H} \right| ,
\]

or

\[
\sum_{j \in S'} R_{c,j} \leq \alpha M \log \rho + \min \{|S|M, N - M \} \alpha \log \rho + O(1).
\] (161)

Equation (161) is obtained by using Lemma 3.1 and \( |S|M \leq (K - 1)M \leq N \). Above equation is simplified under following cases.
Case - a:

When \( \min \{|S|M, N - M\} = N - M \), then we have the following condition:

\[
N - M \leq |S|M,
\]

or \( N \leq (1 + |S|)M \). (162)

Above equation results when \( |S| = K - 1 \) and (161) becomes:

\[
\sum_{j \in S} R_{c,j} \leq M \log \rho + (N - M)\alpha \log \rho + O(1)
\]

or \( R_{c,j} \leq \frac{N\alpha}{1 + |S|} \log \rho + O(1) \)

\[
= \frac{N\alpha}{K} \log \rho + O(1) \quad (: |S| = K - 1)
\]

(163)

Case - b:

When \( \min \{|S|M, N - M\} = |S|M \), then it results in following condition:

\[
(1 + |S|)M \leq N
\]

(164)

The above condition is satisfied when \( |S| < K - 1 \) and (161) simplifies to:

\[
\sum_{j \in S} R_{c,j} \leq \alpha M \log \rho + |S|M \alpha \log \rho + O(1),
\]

or \( R_{c,j} \leq \alpha M \log \rho + O(1) \). (165)

Now consider the user subset \( S \subseteq \{2, \ldots, K\} \). A MAC channel is formed at receiver - 1 due to the signals from users in \( S \). The achievable sum rate in this case is:

\[
\sum_{j \in S} R_{c,j} \leq \log \left| I + \left( I + \sum_{j \in S} H_{1j} P_j H_{ij}^H + \rho^{1-\alpha} H_{11} P_1 H_{11}^H \right)^{-1} \sum_{j \in S} H_{1j} P_j H_{ij}^H \right|
\]

\[
= |S|M \alpha \log \rho + \min \{M, N - |S|M\} (1 - \alpha) \log \rho - M(1 - \alpha) \log \rho + O(1),
\]

\[ (: \alpha \geq 1 - \alpha) \quad \text{(166)} \]

Equation (166) is simplified under following cases:

Case - a:

When \( \min \{M, N - |S|M\} = N - |S|M \), then following condition is obtained:

\[
N - |S|M \leq M
\]

or \( N \leq (1 + |S|)M \) (167)

The above condition is satisfied when \( |S| = K - 1 \) and (166) reduces to:

\[
\sum_{j \in S} R_{c,j} \leq |S|M \alpha \log \rho + (N - |S|M)(1 - \alpha) \log \rho + O(1),
\]

or \( R_{c,j} \leq \frac{1}{K - 1} [M \{\alpha (2K - 1) - K\} + N(1 - \alpha)] \log \rho + O(1). \) (168)
Case - b:

When \( \min \{M, N - |S|M\} = M \), it results in following condition:

\[
M \leq N - |S|M,
\]

or \( (1 + |S|)M \leq N \). \( (169) \)

Under this condition, \( (166) \) reduces to following form:

\[
\sum_{j \in S} R_{c,j} \leq |S|M \log \rho + M(1 - \alpha) \log \rho - M(1 - \alpha) \log \rho + O(1),
\]

\[
= |S|M \alpha \log \rho + O(1),
\]

or \( R_{c,j} \leq \alpha M \log \rho + O(1). \) \( (170) \)

The achievable rate is obtained by taking minimum of \( (163), (165), (168) \) and \( (170) \). As \( N < KM \), \( (165) \) and \( (170) \) becomes superfluous given \( (163) \). The achievable GDOF by common part of the message is:

\[
d_c(\alpha) \geq \min \left\{ \frac{N\alpha}{K}, \frac{1}{K - 1} \left[ M \{ \alpha (2K - 1) - K \} + N(1 - \alpha) \right] \right\}
\] \( (171) \)

The total GDOF achievable by private part and common part together is:

\[
d(\alpha) \geq d_p(\alpha) + d_c(\alpha)
\]

\[
= M(1 - \alpha) + \min \left\{ \frac{N\alpha}{K}, \frac{1}{K - 1} \left[ M \{ \alpha (2K - 1) - K \} + N(1 - \alpha) \right] \right\}
\] \( (172) \)

This completes the proof for Theorem 4.3(1).

2) Case - 2 \( (K > \frac{N}{M} + 1) \) :: We consider the set \( S' \subseteq \{1, 2, \ldots, K\} \) as in Case - 1. The achievable rate in this case using \( (159) \) and \( (160) \) simplifies to:

\[
\sum_{j \in S'} R_{c,j} \leq \log |I + \rho H_{11} P_1 H_{11}^H| + \rho \alpha \sum_{j \in S} H_{1j} P_j H_{1j}^H| - \log |I + \rho^{1-\alpha} H_{11} P_1 H_{11}^H| + O(1),
\]

\[
= M \log \rho + \min \{ \min \{ N, |S|M \}, N - M \} \alpha \log \rho - M(1 - \alpha) \log \rho + O(1). \] \( (173) \)

Above equation is simplified under following cases.

Case - a:

When \( \min\{N, |S|M\} = N \), then following condition is obtained:

\[
N \leq |S|M,
\]

or \( \frac{N}{M} \leq |S| \). \( (174) \)

The maximum value of \( |S| \) which satisfies this condition is \( K - 1 \). Under this condition, \( (173) \) becomes:

\[
\sum_{j \in S'} R_{c,j} \leq M \log \rho + \min \{ N, N - M \} \alpha \log \rho - M(1 - \alpha) \log \rho + O(1),
\]

or \( R_{c,j} \leq \frac{N\alpha}{1 + |S|} \log \rho + O(1). \) \( (175) \)
Above equation is minimized when \(|S|\) takes its maximum value i.e. \(|S| = K - 1\) and (175) becomes:

\[
R_{c,j} \leq \frac{N\alpha}{K}\log \rho + \mathcal{O}(1).
\]

(176)

**Case - b:**

When \(\min\{N, |S| M\} = |S| M\), it results in following condition:

\[
|S| M \leq N
\]

or \(|S| \leq \frac{N}{M}\) \hspace{1cm} (177)

Equation (173) becomes:

\[
\sum_{j \in S'} R_{c,j} \leq M \log \rho + \min\{|S|M, N - M\} \alpha \log \rho - M(1 - \alpha) \log \rho + \mathcal{O}(1).
\]

(178)

When \(\min\{|S|M, N - M\} = N - M\), then we have:

\[
\frac{N}{M} - 1 \leq |S|.
\]

(179)

From (177) and (179), we have:

\[
\frac{N}{M} - 1 \leq |S| \leq \frac{N}{M},
\]

\[
\frac{N}{M} \leq |S| + 1 \leq \frac{N}{M} + 1.
\]

(180)

The achievable rate in this case is:

\[
\sum_{j \in S'} R_{c,j} \leq M \log \rho + (N - M)\alpha \log \rho - M(1 - \alpha) \log \rho + \mathcal{O}(1),
\]

\[
= N\alpha \log \rho + \mathcal{O}(1),
\]

or \(R_{c,j} \leq \frac{N\alpha}{1 + |S|} \log \rho + \mathcal{O}(1)\). \hspace{1cm} (181)

The above equation is minimized by taking largest integer value of \(|S|\) which satisfies the condition in (180).

When \(\min\{|S|M, N - M\} = |S| M\), then we have:

\[
1 + |S| \leq \frac{N}{M}.
\]

(182)

From (177) and (182), following condition is deduced:

\[
|S| < 1 + |S| \leq \frac{N}{M}.
\]

(183)

Under this condition, (178) becomes:

\[
\sum_{j \in S'} R_{c,j} \leq \alpha M \log \rho + |S|M\alpha \log \rho + \mathcal{O}(1),
\]

\[
= \alpha(1 + |S|)M \log \rho + \mathcal{O}(1),
\]

or \(R_{c,j} \leq \alpha M \log \rho + \mathcal{O}(1)\). \hspace{1cm} (184)
Now consider the user set $S \subseteq \{2, 3, \ldots, K\}$. As this user set forms a MAC channel at receiver - 1, following rate equation is obtained:

$$
\sum_{j \in S} R_{c,j} \leq \log |I + \rho^\alpha \sum_{j \in S} H_{1j}P_jH_{1j}^H + \rho^{1-\alpha}H_{11}P_1H_{11}^H| - \log |I + \rho^{1-\alpha}H_{11}P_1H_{11}^H|,
$$

$$
= \min \{N, |S|M\} \alpha \log \rho + \min \{M, N - \min (N, |S|M)\} (1 - \alpha) \log \rho - M(1 - \alpha) \log \rho + O(1).
$$

(185)

Above equation is simplified under following cases.

Case - a

When $\min \{N, |S|M\} = |S|M$, then we have the following condition:

$$
|S| \leq \frac{N}{M}
$$

(186)

When the above condition is satisfied, (185) becomes:

$$
\sum_{j \in S} R_{c,j} \leq |S|M\alpha \log \rho + \min \{M, N - |S|M\} (1 - \alpha) \log \rho - M(1 - \alpha) \log \rho + O(1).
$$

(187)

Above equation is further simplified by considering following cases.

When $\min \{M, N - |S|M\} = N - |S|M$, the following condition is obtained:

$$
N - |S|M \leq M
$$

or

$$
\frac{N}{M} - 1 \leq |S|
$$

(188)

From (185) and (188), following condition is obtained:

$$
\frac{N}{M} - 1 \leq |S| \leq \frac{N}{M}
$$

(189)

When the above condition is satisfied, (187) becomes:

$$
\sum_{j \in S} R_{c,j} |S|M\alpha \log \rho + (N - |S|M)(1 - \alpha) \log \rho - M(1 - \alpha) \log \rho + O(1),
$$

or

$$
R_{c,j} \leq M(2\alpha - 1) \log \rho + \frac{(N - M)(1 - \alpha)}{|S|} \log \rho + O(1).
$$

(190)

The above equation is required to be minimized by taking largest possible value of $|S|$, which also satisfies the condition in (189). Equation (190) is minimized by taking $|S| = \frac{N}{M}$ and (190) becomes:

$$
R_{c,j} \leq M(2\alpha - 1) \log \rho + \frac{(N - M)(1 - \alpha)}{\frac{N}{M}} \log \rho + O(1),
$$

$$
= \frac{N\alpha - M(1 - \alpha)}{\frac{N}{M}} + O(1).
$$

(191)

When $\min \{M, N - |S|M\} = M$, then it results in following condition:

$$
|S| \leq \frac{N}{M} - 1
$$

(192)
From (185) and (192), following condition is obtained:

\[ |S| \leq \frac{N}{M} - 1 < \frac{N}{M} \]  \hspace{1cm} (193)

Under this condition, (187) becomes:

\[ R_{c,j} \leq |S| M \alpha \log \rho + M(1 - \alpha) \log \rho - M(1 - \alpha) \log \rho + O(1), \]

or \( R_{c,j} \leq \alpha M \log \rho + O(1). \)  \hspace{1cm} (194)

**Case - b:**

When \( \min \{ N, |S| M \} = N \), then following condition is obtained:

\[ \frac{N}{M} \leq |S| \]  \hspace{1cm} (195)

When the above equation is satisfied, (185) becomes:

\[ R_{c,j} \leq \frac{N \alpha - M(1 - \alpha)}{|S|} \log \rho + O(1). \]  \hspace{1cm} (196)

Above equation is minimized when \( |S| \) takes the maximum value and the maximum value of \( |S| \) which satisfies the condition in (195) is \( K - 1 \). As a result, (196) becomes:

\[ R_{c,j} \leq \frac{N \alpha - M(1 - \alpha)}{K - 1} \log \rho + O(1). \]  \hspace{1cm} (197)

The achievable rate by the common part of the message is obtained by taking minimum of (176), (181), (184), (191), (194) and (197). It can be observed that (184) and (194) are unnecessary given (176) as \( N < KM \). As \( K > \frac{N}{M} + 1 \), (191) is redundant given (197). Also, (181) is unnecessary given (176). Now the achievable GDOF obtained by the common part of the message is:

\[ d_c(\alpha) \geq \min \left\{ \frac{N \alpha}{K}, \frac{N \alpha - M(1 - \alpha)}{K - 1} \right\}. \]  \hspace{1cm} (198)

The per user GDOF achievable in this case is:

\[ d(\alpha) \geq d_p(\alpha) + d_c(\alpha) = M(1 - \alpha) + \min \left\{ \frac{N \alpha}{K}, \frac{N \alpha - M(1 - \alpha)}{K - 1} \right\}. \]  \hspace{1cm} (199)

This completes the proof for Theorem 4.3(2).

**J. Proof of Theorem - 4.4**

The rate achieved by the private part is same as that in case of moderate interference case and is given by:

\[ R_{p,j} \leq M(1 - \alpha) \log \rho + O(1). \]  \hspace{1cm} (200)

In order to obtain the rate for common part same procedure is followed as described in the moderate interference case. Following two cases are considered:

1) \( \frac{N}{M} < K \leq \frac{N}{M} + 1 \),

2) \( K > \frac{N}{M} + 1 \).
1) Case - 1 \( \left( \frac{N}{M} < K \leq \frac{N}{M} + 1 \right) \): First consider the MAC channel formed at receiver - 1 due to the users in \( S \subseteq \{1, 2, \ldots, K\} \). The sum rate constraint in this case is:

\[
\sum_{j \in S} R_{c,j} \leq \log |I + \rho^{\alpha} \sum_{j \in S} H_{1j} P_j H_{1j}^H + \rho^{1-\alpha} H_{11} P_1 H_{11}^H| - \log |I + \rho^{1-\alpha} H_{11} P_1 H_{11}^H|,
\]

where

\[
= \min \{|S|, M, N - M\} \alpha \log \rho + O(1), (\because K \leq \frac{N}{M} + 1).
\]

(201)

When \( \min \{|S|, M, N - M\} = N - M \), then following condition is obtained:

\[
N - M \leq |S| \leq M
\]

or

\[
N \leq (1 + |S|)M
\]

(202)

This is possible when \(|S| = K - 1\) and (201) becomes:

\[
R_{c,j} \leq \frac{N - M}{K - 1} \alpha \log \rho + O(1), (\because |S| = K - 1).
\]

(203)

When \( \min \{|S|, M, N - M\} = |S|M \), then following condition is obtained:

\[
(1 + |S|)M \leq N.
\]

(204)

Above condition results when \(|S| < K - 1\) and under this condition (201) reduces to:

\[
R_{c,j} \leq \alpha M \log \rho + O(1).
\]

(205)

Now consider the user set \( S' = \{1\} \cup \{S\} \), where user-1 is always included. The sum rate constraint for common part of the message:

\[
\sum_{j \in S'} R_{c,j} \leq \log |I + \rho H_{11} P_1 H_{11}^H + \rho^{\alpha} \sum_{j \in S} H_{1j} P_j H_{1j}^H| - \log |I + \rho^{1-\alpha} H_{11} P_1 H_{11}^H| + O(1),
\]

where

\[
= \alpha M \log \rho + \min \{|N, |S|M\}, N - M\} \alpha \log \rho + O(1).
\]

(206)

As \( K \leq \frac{N}{M} + 1 \), we have \((K - 1)M \leq N\) or \(|S|M \leq N\). Equation (206) further simplifies to:

\[
\sum_{j \in S'} R_{c,j} \alpha M \log \rho + \min \{|N, |S|M\}, N - M\} \alpha \log \rho + O(1).
\]

(207)

When \( \min \{|S|, M, N - M\} = N - M \), then following condition is obtained:

\[
N \leq (1 + |S|)M
\]

(208)

Above condition is satisfied when \(|S| = K - 1\) and (207) becomes:

\[
R_{c,j} \leq \frac{N \alpha}{1 + |S|} \log \rho + O(1).
\]

(209)

Above is minimized when \(|S|\) takes it maximum value i.e. \( K - 1 \) and (209) becomes:

\[
R_{c,j} \leq \frac{N \alpha}{K} \log \rho + O(1).
\]

(210)

When \( \min \{|S|, M, N - M\} = |S|M \), then we have following condition:

\[
(1 + |S|)M \leq N.
\]

(211)
For this condition to be satisfied, \(|S| < K - 1\) and (207) becomes:

\[ R_{c,j} \leq \alpha M \log \rho + O(1). \] (212)

The achievable rate by common part of the message is obtained by taking minimum of (203), (205), (210) and (212). With some algebraic manipulation, it can be shown that given (203) all the remaining equations become superfluous. The achievable GDOF due to common part of the message is:

\[ d_c(\alpha) \geq \frac{N - M}{K - 1} \alpha. \] (213)

The per user GDOF achievable in this case is:

\[ d(\alpha) \geq M(1 - \alpha) + \frac{N - M}{K - 1} \alpha. \] (214)

2) Case - 2: \(K > \frac{N}{M} + 1\): As in the previous case, first consider the MAC channel formed at receiver - 1 due to the users in \(S \subseteq \{1, 2, \ldots, K\}\). The sum rate constraint in this case becomes:

\[ \sum_{j \in S} R_{c,j} \leq \min \{ \min \{N, |S|M\}, N - M\} \alpha \log \rho + O(1). \] (215)

When \(\min \{N, |S|M\} = N\), then we have:

\[ N \leq |S|M, \] (216)

Under this condition, (215) reduces to:

\[ \sum_{j \in S} R_{c,j} \leq \min \{N, N - M\} \alpha \log \rho + O(1), \]

or \(\leq \frac{N - M}{|S|} \alpha \log \rho + O(1). \) (217)

Above equation is minimized when \(|S| = |S|_{max} = K - 1\) and (217) becomes:

\[ R_{c,j} \leq \frac{N - M}{K - 1} \alpha \log \rho + O(1). \] (218)

When \(\min \{N, |S|M\} = |S|M\), then we have:

\[ |S| \leq \frac{N}{M}. \] (219)

Now, (215) becomes:

\[ \sum_{j \in S} R_{c,j} \leq \min \{|S|M, N - M\} \alpha \log \rho + O(1). \] (220)

If \(\min \{|S|M, N - M\} = N - M\), then we obtain the following condition:

\[ \sum_{j \in S} R_{c,j} \leq (N - M)\alpha \log \rho + O(1), \]

or \(R_{c,j} \leq \frac{N - M}{K - 1} \alpha \log \rho + O(1). \) (\(\because\) for \(|S| = K - 1\), RHS is minimized) (221)
When \( \min \{|S|M, N - M\} = |S|M \), then following condition is obtained:

\[
(1 + |S|)M \leq N. \tag{222}
\]

Under this condition, (215) becomes:

\[
R_{c,j} \leq \alpha M \log \rho + O(1). \tag{223}
\]

Now consider the user set \( S' = 1US \), where user-1 is always included. By following the same procedure as in the previous case, following equation is obtained:

\[
\sum_{j \in S'} R_{c,j} \leq \alpha M \log \rho + \min\left\{ \min\{N, |S|M\}, N - M \right\} \alpha \log \rho + O(1), \tag{224}
\]

When \( \min\{N, |S|M\} = |S|M \), then following condition is obtained:

\[
\sum_{j \in S'} R_{c,j} \leq \alpha M \log \rho + \min\{ |S|M, N - M \} \alpha \log \rho + O(1). \tag{225}
\]

By following the same procedure as in previous case above equation is simplified further and following rate constraints are obtained under following conditions:

- When \( \frac{N}{M} - 1 \leq |S| \leq \frac{N}{M} \), then we have:
  \[
  R_{c,j} \leq \frac{N\alpha}{1 + |S|} \log \rho + O(1). \tag{226}
  \]

  Above equation is minimized when \( |S| \) takes its maximum value i.e. \( |S| = \frac{N}{M} \) and (226) becomes:

  \[
  R_{c,j} \leq \frac{N\alpha}{1 + \frac{N}{M}} \log \rho + O(1). \tag{227}
  \]

- When \( |S| < 1 + |S| \leq \frac{N}{M} \), (225) reduces to:
  \[
  R_{c,j} \leq \alpha M \log \rho + O(1). \tag{228}
  \]

- When \( N \leq |S|M \), then (225) becomes:
  \[
  R_{c,j} \leq \frac{N\alpha}{K} \log \rho + O(1). \tag{229}
  \]

Finally, the achievable rate by common part of the message is obtained by taking minimum of (218), (221), (223), (227), (228) and (229). Given (229), (227) becomes redundant as \( K > \frac{N}{M} + 1 \). Given (218) and (221), (223) and (229) becomes redundant. Finally, the GDOF achievable by the common part of the message:

\[
d_c(\alpha) \geq \frac{N - M}{K - 1} \alpha. \tag{230}
\]

The per user GDOF achievable in this case is:

\[
d(\alpha) \geq M(1 - \alpha) + \frac{N - M}{K - 1} \alpha. \tag{231}
\]

The above equation is same as that in previous case. This completes the proof.
Fig. 1. GDOF for $K = 3$ user Interference Channel with $M = N = 2$

Fig. 2. GDOF for $K = 3$ user Interference Channel with $M = 2$ and $N = 3$
Fig. 3. GDOF for $K = 3$ user Interference Channel with $M = 2$ and $N = 4$

Fig. 4. GDOF for $K = 3$ user Interference Channel with $M = 2$ and $N = 5$
Fig. 5. GDOF for $K = 4$ user Interference Channel with $M = 3$ and $N = 7$

Fig. 6. GDOF for $K = 4$ user Interference Channel with $M = 3$ and $N = 9$
Fig. 7. GDOF for $K = 6$ user Interference Channel with $M = 2$ and $N = 5$

REFERENCES