Phase plot manifestations in globally coupled maps: effects of scale

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Phase plots of coupled maps have shown turbulent, semi-ordered, intermittent and ordered behaviour as a function of control parameters for networks of uniform scale. We investigate the dependence of coupled map dynamics on scale, i.e. number of neurons. We compare results for globally coupled maps (GCMs) and ones with minor lesions, employing a new chaotic function that we propose. The behaviours are described in terms of phase transformations involving emergence, disappearance, migration, or engulfment of phases.

Keywords: Complex systems; Chaos; Non-linear dynamics; Brain activity

1. Introduction

Coupled maps have been used extensively as a tool to study dynamic synchronization in complex systems. Dynamic synchrony in these systems shows a variety of relevant phenomena, such as spatiotemporal intermittency, chaotic itinerancy and self-organized criticality (Kaneko 1985, 1989, Fujimoto and Kaneko 2003). In particular, the presence of chaotic itinerancy and spatiotemporal intermittency has made coupled maps a useful tool in the study of brain dynamics (Tsuda 2001, van Leeuwen and Simionescu 2002). Studies of phase plots of such maps summarize the dynamics of the whole system over a range of parameters on which it operates. Usually in these studies, the scale of the system has been kept uniform (Kaneko 1989), where by 'scale' we mean the number of nodes in the network. This article aims to study the effect of scale on the phase plots of globally coupled maps (GCMs). Our study involves scaling the network at appropriate instances of numbers of nodes and, alongside, monitoring the evolution of the corresponding phase plot. The relevance of small-scale models of brain processes depends, to a large extent, on the tacit assumption that effects are robust across scales. We therefore expect this study to yield valuable information on how the phases manifest themselves (form, disappear, or exhibit transitions) when the network is scaled.
In order to study the robustness of the phase plot of the GCMs to perturbations in connectivity the same exercise was carried out over a randomly lesioned GCM (lesioned networks are those with connection density less than unity). The experiment was meant to verify whether or not the phases exhibit a graceful degradation, as one might expect. This would also reveal how the phase plot would behave in the limit. We shall use the phrase ‘phase transformation’ to indicate how the various phases emerge, disappear, migrate, or engulf as the network is scaled. Studies such as these will enable us to assess the utility of GCMs in the study of large-scale complex systems such as the brain.

2. Design

The experiment was conducted on a GCM where each link was assigned uniform connection strength. The map is described by the following equation:

\[ x_i(n + 1) = (1 - \varepsilon) f(x_i(n)) + \frac{\varepsilon}{N} \sum_{j=1}^{N} f(x_j(n)), \quad (1) \]

where \( x \) denotes a state vector with \( x_i \) denoting its components, \( \varepsilon \) denotes coupling strength, \( N \) denotes the total number of nodes in the lattice, \( f \) is the activation function, while \( n \) denotes the iteration. For activation function \( f \) we introduced a new one-parameter function, which we describe next:

\[ f(x) = \cos(\lambda \cos^{-1}(x)), \quad (2) \]

where \( f: I \rightarrow I \) is chaotic on \( I \) in the sense of Guckenheimer and Holmes (1986), see also Devaney (1989), whenever \( \lambda \in S_2 \), where \( S_2 = (-\infty, -1) \cup (1, \infty) \) and \( I = [-1, 1] \).

The definition of chaos by Guckenheimer and Holmes (1986) requires that there exist a hyperbolic invariant set on which the transformation is topologically conjugate to a subshift of finite type. According to Guckenheimer and Holmes (1986), a map \( f: M \rightarrow M \) is topologically transitive on \( M \) if for every pair of open sets \( U \) and \( V \) in \( M \) there exists a \( k \) such that \( f^k(U) \cap V \neq \emptyset \).

A map \( f: M \rightarrow M \) is sensitively dependent on initial conditions if there exists \( \delta > 0 \) such that for every neighbourhood \( U \) of \( x \) there exist a \( y \) and an \( n > 0 \) such that:

\[ |f^n(x) - f(y)| > \delta. \quad (3) \]

Two maps, \( f: M \rightarrow M \) and \( g: N \rightarrow N \), are topologically conjugate if there exists a homeomorphism \( h: M \rightarrow N \) such that the following composition rule holds: \( g \circ h = h \circ f \) (on domains as indicated in the mappings).

The map defined by \( g: S^1 \rightarrow S^1 \), which projects the unit circle \( S^1 \) on to itself according to \( \theta \rightarrow \lambda \theta \) with \( \lambda \in S_2 \), is chaotic on \( S^1 \). To see this we give \( S^1 \) the usual topology as a subset of \( R^2 \) and consider any open set about a point \( x \) of \( S^1 \). Upon iteration, as \( \lambda \in S_2 \), the open set would continue to expand and eventually a stage would be reached at which point the whole circle would be covered by the iterate. Topological transitivity follows from this. The map is sensitive to initial conditions as, following the notation in the definition, there would always exist a \( \delta > 0 \) and an \( n \) for every pair of closely spaced points such that condition (3) is true. Moreover, periodic orbits are clearly dense on \( S^1 \) because we can always align a regular polygon of an appropriate number of sides to \( x \) such that at least one of the vertices occurs in its neighbourhood for arbitrary-sized neighbourhoods of \( x \).
Next, we claim that the projection map on to abscissa of any element of \( S^1 \), \( p: S^1 \to I \) defined by means of the formula \( x = p(\theta) \) with \( p(\theta) = \cos(\theta) \), is continuous on the domains. To see this we give \( S^1 \) the usual topology and \( I \) a subspace topology from the set of real numbers. At most points under the projection map the pre-image of an open set in \( I \) is a disconnected union of two open sets on the circle, which again is an open set (as the open sets form a topology). It follows, hence, that the projection map is continuous. However, a judgement has to be made, depending upon the value \( \theta \) had assumed during the previous iteration, as to which of the disjoint open sets the pre-image in the present iteration actually belongs. Moreover, every projection map is already an open map (images of open sets under the projection map are open; see Munkres (2000)). Figure 1 epitomizes the argument.

We claim that the projection map is actually a topological conjugate from the unit circle on to the interval \( I \). To see this we simply note from the above argument that the projection map, \( p \), is a homeomorphism and, from figure 2, that the following composition rule holds: \( f \circ p = p \circ g \). As \( g \) is chaotic and \( p \) is a topological conjugate, it follows that \( f: I \to I \) is chaotic on \( I \). This completes the proof for \( f(x) = \cos(\lambda \cos^{-1}(x)) \). For future reference we shall record this conclusion as a proposition.

![Diagrammatic representation of the argument.](image1)

**Figure 1.** Diagrammatic representation of the argument.

![The transcendental chaotic function plotted for various lambda values.](image2)

**Figure 2.** The transcendental chaotic function plotted for various lambda values. The function is chaotic for all lambda restricted to real numbers that belong in the set \((-\infty, -1] \cup [1, \infty)\).
**PROPOSITION**  The iterates of \( \cos(\lambda \cos^{-1}(x)) \) are chaotic on \( I \) whenever \( \lambda \in S_2 \).

Clearly, the iterates of \( f \) tend to a fixed point if \( \lambda \in R \setminus S_2 \), in which case \( g \) performs a contraction map. As the function is defined through a transcendental cosine function, we shall refer to it as a transcendental chaotic function (TCF); see figure 2.

3. Experiment 1

A GCM scaling upward from 300 nodes to 10,000 nodes was considered. The connectivity matrix was assumed to be symmetric, with self-connections among nodes absent. The coupling constant was allowed to vary in the interval \([0.1, 0.4]\), while the parameter of the TCF, lambda, was set in the interval \([1.4, 4.2]\). Iterates were computed according to the formula displayed in equation (1). The parameters set during the runs, which were kept uniform throughout the experiment, are displayed in table 1.

Each initial condition was swept over the network 50 times while each combination of \( \lambda \) and \( \epsilon \) was presented with 1000 such initial vectors. Clusters were identified by sorting the end states the network assumes and setting an appropriate windowing scheme (depending upon the magnitude of the end states) to identify closely spaced real numbers from among the elements of the vector. Clusters in the sorted values can unambiguously be discriminated by setting a small threshold of 0.007. A phase was called *turbulent* if the number of clusters was large (in the order of the number of nodes), or *coherent* if it was small. A colour code was assigned to represent the phase: red indicates a turbulent phase while blue indicates a coherent phase, with intermediate colours for intermediate phases. The resulting phase plots are displayed in figure 3.

It may be observed from the sequence of plots that the turbulent phases (shaded red) shrink and diminish as the network is scaled. At the lower end of the scale (when the nodes are few), partially ordered (yellow) and coherent phases (blue) coexist with a prominent red turbulent phase. The coherent phase has encroached significantly into where the turbulent phases prevailed by the time the network is scaled to 1500 (figure 3(g)). During the initial few plots (figures 3(a–f)), the turbulent phases have shrunk gradually, exhibiting a tendency to migrate within a restricted domain. Figure 3(g) represents a singular phase plot where the coherent phases have severed the diminishing turbulent phases into two disconnected regions, from which point the red regions begin to collapse and localize into smaller regions (figures 3(h–l)). Figure 3(n) portrays the scenario in which the nodes were 10,000, at which point the coherent phases have engulfed all other phases.

We observe that the dynamics of our model exhibit such phase transformations as in figure 3 when the network is scaled up in steps as indicated in the figure until 10,000 nodes. It is unknown at this time how the behaviour might extrapolate when the network is scaled higher up. Of interest of course is how the phases might be in a GCM, scaled to the number of neurons of the human cortex.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value set</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of nodes</td>
<td>300–10,000</td>
</tr>
<tr>
<td>Sweeps for each initial condition</td>
<td>50</td>
</tr>
<tr>
<td>Coupling constant instances, ( \epsilon )</td>
<td>20 in ([0.1, 0.4])</td>
</tr>
<tr>
<td>Chaotic parameter, ( \lambda ); number of instances</td>
<td>20 in ([1.4, 4.2])</td>
</tr>
<tr>
<td>Number of initial conditions (for each ( \epsilon \– \lambda ) pair)</td>
<td>1000</td>
</tr>
</tbody>
</table>
4. Experiment 2

In order to study whether phase plots show a continuous transition with phases exhibiting a gradation as connection density is lesioned from its value of unity (for a GCM), a random lattice of connection density 0.9 was subject to the experiment. Unlike the previous experiment, clusters cannot be distinguished unambiguously by setting one single threshold. As in the present case the range of entries of the sorted vector of end states varied greatly for each instance of initial condition, the sorting criterion had to be adaptive in order to overcome the fuzziness induced by the lesioning. Therefore, an adaptive windowing scheme was implemented to
identify formations of clusters. The phase plot of the random lattice of high connection density (0.9) shown in figure 4 clearly indicates that phase plots do not exhibit a graceful degradation on to networks of a null lesion (the GCM), contrary to what one might expect.

5. Discussion and conclusion

We have examined effects of scale in coupled map dynamics. Our results indicate that, whereas turbulent and partially ordered phases occur in smaller networks, they diminish with increase of scale. As the network is scaled up, the phase plot tends to become coherent; partially ordered phases become increasingly rare and coherent phases engulf the control parameter space.

Absence of invariance of behaviour with respect to scale is typically what we should expect from complex systems. Non-linear systems such as neural networks in general will show compositionality only because, like Hopfield networks, they are explicitly designed to do so. Yet, the particular way in which compositionality is violated by our current system may be considered surprising: when we join two systems of 100 units each into a 200-unit system, this system will behave more coherently than the two it is composed of. To join the two systems, we need extra connections. In general, linear increases in number of nodes imply quadratic increases in the number of connections. We may speculate that the higher dimensionality of the net input to each node has, overall, a stabilizing effect on its activity function, which leads, in turn, to increased coherence at the network level.

When a minor lesion was induced into the 300-node lattice, it turned out that the phases displayed by such system differed strongly from the one displayed by a GCM operating with 300 nodes. Contrary to what one might expect, our effects are not robust with respect to lesion and depend strongly on the symmetry of the network. These results impose limitations on the validity of coupled maps as models of brain circuitry.

The wider significance of our results may be to alert us to the possibility that similar effects of scale may occur in oscillator models of brain activity. For instance, human electroencephalogram has been modelled in terms of coupled relaxation oscillators (Nunez 2000).
Our results suggest that such models should take into consideration network scale and connectivity distribution.

References
