A topological completion of refined hedge algebras and a model of fuzziness of linguistic terms and hedges

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Abstract

The main aim of the investigation of hedge algebras is to discover an algebraic structure in the category of universal algebras of terms-domains of linguistic variables or to establish an algebraic approach to the semantic structure of terms-domains. The meaning of linguistic terms in general is vague, however vagueness is a concept which is difficult to be defined formally and reasonably. Motivated by introducing a mathematical foundation of qualitative and quantitative fuzziness of linguistic terms, this paper is aimed to introduce and investigate a completion of refined hedge algebras and obtain a class of complete ones. They will be defined by adding “limit” elements to refined hedge algebras in an axiomatization way. It is shown that a formal model of fuzziness of linguistic terms and hedges can be defined reasonably in this semantic structure and there is a closed relation between fuzziness of terms and semantic-based topology of these algebras.

Keywords: Algebra; Hedge algebra; Fuzziness of terms; Linguistic domain; Semantically ordering relation; Semantic-based topology

1. Introduction

Hedge algebras establish an algebraic approach to the semantic of terms-domains of linguistic variables, which is completely different from the fuzzy sets approach. For easily reading the paper, we shall first give a short overview of this approach.

Approximate reasoning methods in general are based on fuzzy sets theory and fuzzy logics (see [23,24]), in which the meaning of linguistic values is interpreted as a fuzzy set, whose membership function is defined from a set $U$ into the unit-interval $[0, 1]$. From an algebraic point of view, this means that the set of all linguistic values, that has no mathematical structure, are embedded in the set of all fuzzy sets on $U$, denoted by $F(U, [0, 1])$. Since the set of fuzzy sets representing the meaning of linguistic values has no mathematical structure, one has to borrow the analytic structure of the set of all fuzzy sets on $U$. However, from a mathematical viewpoint, it means that the set $F(U, [0, 1])$ cannot model the set of all linguistic values in a right way. On the other hand, the fundamental mathematical structure of a classical logic or non-classical logic is normally a universal algebra (see [1,5,20,21]) of truth values, e.g. it is
On an abstract level, we can consider there exists a partially ordering relation on distributive lattices. Variables become a description of an abstract data type. In words, a function and operation symbols in a signature. It may provide a symbolic representation of a specification or, in other words, about the algebraic structure of terms-domains. It motivates us to find an algebraic approach to this problem.

An algebraic approach to modeling the meaning of terms is to interpret each terms-domain as algebra, in which qualitative meaning of terms is expressed in terms of a semantics-based ordering relation [7,11–18]. Similarly as in classical logics, the semantics of “true” and “false” is interpreted not only by just the elements 1 and 0, but also by the algebraic structure of a Boolean algebra. So, on each terms-set X of a linguistic variable A, we can always assume that there exists a partially ordering relation \( \preceq \) defined by the meaning of terms, called semantically ordering relation. On an abstract level, we can consider X as an algebra \( AX = (X, G, C, H, \preceq) \), where \( G = \{ c^-, c^+ \} \) is the set of primary generators, in which \( c^- \) and \( c^+ \) are the negative and positive primary terms of \( X \) and \( C = \{ W, 0, I \} \), the set of constants interpreted, respectively, as the neutral, the least and the greatest elements in \( X \); \( H = H^+ \cup H^- \), where \( H^+ \cup \{ I \} \) is a modular lattice of hedges, which is considered as the set of unary operations, and \( LH(G \cup C) = X \); \( \preceq \) is a semantically ordering relation on \( X \). Note that elements of \( X \) are expressed in the form \( x = h_n \ldots h_1 c \), where \( h_1, \ldots, h_n \in H \) and \( c \in G \cup C \). For example, a terms-set T of the truth variable TRUTH can be represented as an algebra \( AX = (T, G, C, H, \preceq) \), in which \( G = \{ \text{false}, \text{true} \} \), \( H^+ = \{ M, V \} \) and \( H^- = \{ P, A \} \), where \( M, V, P \) and \( A \) are abbreviations respectively, of more, very, possibly and approximately, and \( \preceq \) is the ordering relation induced by the natural meaning of terms of TRUTH. These algebras are called hedge algebras, since their operations model immediately linguistic hedges. An axioms system was established based on natural properties of terms-meaning formulated in terms of the semantically ordering relation and, it was proved that, the extended hedge algebras become lattices [17,18]. It implies that a composed term \( t = (P_{\text{true}} \text{ OR } A_{\text{true}}) \), for example, which is not in \( X \), can be represented by the join of terms \( P_{\text{true}} \) and \( A_{\text{true}} \) in these algebras. Especially, in the symmetrical hedge algebras a complement operation can be defined and many interesting algebraic–logic properties can be established [18].

However, the order-based structure of these algebras in general is rather rough and so the join of two elements has its meaning too far from that of its operands. Hence, hedge algebras were extended by adding new artificial hedges which are generated from either \( H^+ \) or \( H^- \) in the category of lattices. So, the authors of [18] introduced and examined algebras of the form \( AX = (X, G, C, LH, \preceq) \), in which the set of unary operations is \( LH = LH^+ \cup LH^- \) instead of \( H \), where \( LH^+ \cup \{ I \} \), \( c \in \{ +, - \} \), is a distributive lattice of hedges generated freely from the original hedges in \( H^\prime \). Now, the term \( t \) mentioned above can be expressed by a term \( (P \land A)_{\text{true}} \) in \( X \), where \( \land \) is the meet operation of \( LH^- \) and \( (P \land A) \in LH^- \). By this reason, the new algebras are called refined hedge algebras.

So, hedge algebras are universal algebras and aim to model the meaning of linguistic terms in natural languages or they are semantic models of terms-domains of linguistic variables. It is worth noticing that they are different from term algebras which are abstract ones, whose elements are terms themselves formed syntactically from constants, variables, functions and operation symbols in a signature. It may provide a symbolic representation of a specification or, in other words, a mathematical model of algebraic specification (see [1]). That is, that term algebras provide a mathematical description of an abstract data type.

It has been shown [10,11,13,15] that term-domains regarded as refined hedge algebras of almost all linguistic variables become distributive lattices and, for the linguistic variables having exactly two primary terms, they become symmetrical hedge algebras, in which a complement operation for modeling a logical negation can be defined. So, refined hedge algebras are rich enough to study linguistic logics, i.e. those logics whose truth values are linguistic terms, and to develop a method in linguistic reasoning [8]. Moreover, basic properties of hedge algebras were applied to the study of a new model of linguistic modifiers [3] and for building a system of truth labels that are defined as fuzzy sets on [0,1] (see [4]).
In order to expand possible applicability of hedge algebras, we shall discover additional structural properties of hedge algebras, a topological one. Similarly as in the investigation of extended hedge algebras [18], the study of the completion of refined hedge algebras comes from the observation that the intuitive meaning of the linguistic terms (Very)$^n$ Possibly True will converge to the meaning of True, when $n$ runs to infinity. So, a natural requirement arises to examine a concept of “limit elements” in the category of refined hedge algebras or, in other words, to examine their semantic-based topology. But, these algebras are not complete and it requires a study of the completion of refined hedge algebras and the obtained algebras will be called complete hedge algebras. The final purpose of the research we follow is to show that complete hedge algebras will be a basic mathematical structure for developing approximate reasoning methods and hence it requires at least that they are distributive lattices. However, this objective is not easy to reach. Therefore, in this, we shall try to make their order-based structure as clear as possible for reaching this objective.

Another reason which motivates us to carry out the present research follows from the following observation. It is known that hedge algebras model qualitative semantic characteristics of linguistic terms only, while in several applications of fuzzy disciplines, e.g. of fuzzy control, one needs quantitative semantic ones. A question that arises is how to quantify hedge algebras. In nature, each method of quantifying vague concepts in a terms-set defines a mapping from this terms-set into a real interval, called in the paper a semantically quantifying mapping denoted by $\phi$. For hedge algebras, it will be defined by a formula with parameters to be fuzziness measure of primary concepts and linguistic hedges [9,10,14,16]. There will be a closed relation between the form of formula of semantically quantifying mappings and fuzziness measure of linguistic terms and hedges. Naturally, we need to impose some constraints on fuzziness measure of terms, e.g. we should have the equality $f_m(c^-) + f_m(c^+) = 1$, intuitively, where $f_m(c^-)$ and $f_m(c^+)$ denote the fuzziness measures of the primary terms $c^-$ and $c^+$, that are defined by the “size” of $\phi(LH(c^-))$ and $\phi(LH(c^+))$, where $LH(u)$ is the set which is generated algebraically from $u$ by using hedges in $LH$. It can be observed intuitively from [9,10] that the above equality originated from the density of the set $\phi(LH(c^-)) \cup \phi(LH(c^+))$ in the unit interval $[0, 1]$. That is it relates to a topology of hedge algebras defined by the semantics of its terms-domain, which has still not been investigated.

These reasons suggest the completion of $X$ with new elements which are “limits” of sequences of elements in $X$. However, to obtain a complete set $X \supset X$ we do not add to $X$ all such possible elements, but only those elements so that each subset of $X$ with the form $LH(x)$ will have a greatest upper bound and a least lower bound in $X$, i.e. $X$ becomes $\Omega$-complete, where $\Omega = \{LH(x) : x \in X\}$ (the notion of $\Omega$-completeness of a lattice $X$, for any family $\Omega$ of subsets of $X$, can be found in [22]).

An algebraic way to solve this completion problem is to examine an algebra $\mathcal{AX}$, which enlarges the given refined hedge algebra $\mathcal{AX}$ so that it contains $\mathcal{AX}$ as its subalgebra. The idea is as follows: we shall equip $\mathcal{AX}$ with two new unary operations $\Sigma$ and $\Phi$ and establish a suitable system of axioms so that for every $x$, the elements $\Sigma x$, $\Phi x$ will be a greatest upper bound and a least lower bound of the set $LH(x)$ in $X$. It will be shown that in this algebraic structure we can introduce a mathematical formalism of the fuzziness of linguistic terms. It is interesting that this model of fuzziness is related closely with some topological properties of this structure. It should be noticed that the concept of fuzziness of vague terms is a nature of inexactness of information that the fuzzy sets deal with and therefore it becomes a more and more important role in studying several areas of artificial intelligence (see [2,6,19]). In general, it is very difficult to define this concept in the fuzzy sets framework and then that why there are many ways to define it. In addition, the concept of fuzziness measure of linguistic hedges is still not defined up to now, according to the author’s knowledge.

The paper is organized as follows. In Section 2, we introduce a system of axioms of complete refined hedge algebras, trying to interpret $\Sigma$ and $\Phi$ as specific hedges, and examine elementary properties of the new algebras. As new hedges, order-based semantics of $\Sigma$ and $\Phi$ will be investigated in Section 3, i.e. some ordering relationships related to elements, in representation of which $\Sigma$ and $\Phi$ occur, will be discovered. A topology of these algebras and a model of fuzziness of vague concepts are introduced and investigated in Section 4.

2. An axiomatization of complete hedge algebras and elementary properties

The aim of the axiomatization of hedge algebras, in general, and of complete refined hedge algebras, in particular, is to find a reasonable system of axioms so that the order-based structure of these algebras will model the relative and qualitative meaning of linguistic terms in a natural way. So, we shall concentrate our effort in establishing additional axioms imposed on hedge algebras, ensuring that they are $\Omega$-complete and order-based relationships between the elements of these algebras will reflect the order-based semantics of linguistic terms properly.
2.1. Elementary notions and some preparation

Let us consider a refined hedge algebra \( A\mathcal{X} = (X, G, C, LH, \leq) \) of a linguistic variable \( \mathcal{X} \). It is known that \( LH \) is the set of one-argument operations (called also hedges) and so \( LH(x) = \{\delta x : \delta \in LH^+\} \), where \( LH^+ \) is the set of all strings of hedges including the empty one and \( LH(G) \cup C = X \). For easily understanding the paper, it is convenient to recall some basic notions related to hedges examined in [7,17,18] (the reader can find them also in e.g. [14,15]). It is necessary to emphasize that all notions and properties of terms-domains can be formulated in terms of the \textit{semantically ordering relation}. Any hedges \( h, k \in LH \) are said to be \textit{converses} (or \( h \) is said to be converse to \( k \) and vice-versa) if the statement \((\forall x \in X)(x \leq h x \iff x \geq k x)\) holds. And if the statement \((\forall x \in X)(x \leq h x \iff x \leq k x)\) holds then \( h \) and \( k \) are said to be \textit{compatible}. For any \( h, k \in LH \), \( h \) is said to be \textit{positive} w.r.t. \( k \) if the statement \((\forall x \in X)((kx \geq x) \implies h k x \leq k x)\) or \((kx \leq x) \implies h k x \leq k x)\) holds, and \( h \) is said to be \textit{negative} w.r.t. \( k \) if the statement \((\forall x \in X)((kx \geq x) \implies h k x \leq k x)\) or \((kx \leq x) \implies h k x \leq k x)\) holds. For example, consider linguistic hedges \( V \) (Very), \( M \) (More), \( L \) (Little), \( P \) (Possibly), \( A \) (Approximately) and \( ML \) (More or Less) of the linguistic variable \( X \). It can be checked that \( V \) is positive w.r.t. \( M, V \) and \( P \), while negative w.r.t. \( P, A \) and \( ML \). And \( L \) is negative w.r.t. \( M, V \) and \( L \), while positive w.r.t. \( P, A \) and \( ML \). Hence, each hedge is either positive or negative w.r.t. any other one and we say that each hedge has a \( PN \)-property w.r.t. any other. So, given a terms-domain, we can always build a table of the \( PN \)-property of hedges, which read as the positiveness or negativeness of a hedge appearing in the first column w.r.t. a hedge appearing in the first row is indicated by the sign in the corresponding cell of the Table 1.

Consider a refined hedge algebra \( A\mathcal{X} = (X, G, C, LH, \leq) \) [17]. The set \( LH \) is constructed as follows (see [12,15]). It is known that \( H = H^+ \cup H^- \), where \( H^+ \) is a set of positive hedges and \( H^- \) is a set of negative ones, and \( H^\varepsilon \cup I \), where \( \varepsilon \in \{+, -\} \) and \( I \) is the identity on \( X \), are assumed to be a \textit{modular lattices} and hence it can be partitioned into graded classes \( H^\varepsilon \) of incomparable hedges, \( i = 1, \ldots, N^\varepsilon \), with a property that if \(|H^\varepsilon| > 1\), where the notation \(|.\|\) denotes the cardinal of the set \( \varepsilon \), then \(|H^\varepsilon_{i-1}| = |H^\varepsilon_{i+1}| = 1 \). Let \( LH^\varepsilon_i \) be a distributive lattice generated freely from the set \( H^\varepsilon_i \) and put \( LH^\varepsilon = \bigcup_{i=1}^{N^\varepsilon} LH^\varepsilon_i \). Then, \( LH \) is defined by \( LH = LH^+ \cup LH^- \), where \( LH^+ \cup I \) and \( LH^- \cup I \) are distributive lattices with unit-operations \( V \) (very) and \( L \) (little), respectively, and a zero-element \( I \). Put \( UOS = \{V, L\} \), the set of the unit-operations.

Based on semantic properties of linguistic hedges in natural languages, we can always assume in the paper that \( V \) is positive w.r.t. the hedges in \( H^+ \) and \( L \), while negative w.r.t. the hedges in \( H^- \setminus \{L\} \); and, since \( L \) is converse to \( V \), it follows that \( L \) is negative w.r.t. \( H^+ \) and \( L \), while positive w.r.t. the hedges in \( H^- \setminus \{L\} \).

Semantically, since \( H^\varepsilon_i \) is the set of incomparable hedges, they have an equivalent semantic effect degree and, hence, we can consider the hedges in \( LH^\varepsilon_i \) as having the same graded semantic level and \( LH^\varepsilon_i \) is called also the \( i \)-th graded semantic class. For example, in the above example we have \( H^-_1 = \{A, P, ML\} \) and the hedge \((A \lor P) \land ML \) \( \in LH^-_1 \) and \( A, P, ML \) have the same graded semantic level. By this interpretation, many semantic properties of terms formulated in terms of the semantically ordering relation, will easily be explained intuitively. For example, for any \( h \in LH^\varepsilon_i, k \in LH^\varepsilon_j \), with \( i \neq j \), the fact that \( hu \leq ku \) (**) implies that \( \partial hu \leq \gamma ku \), for any \( \partial, \gamma \in LH^\varepsilon \), can be understood as follows: since \( h, k \) belong to different graded semantic levels and hedges can change the meaning of terms a bit only, applying any hedges consecutively to \( hu \) and \( ku \) they cannot change the original meaning of \( hu \) and \( ku \), that are expressed by inequality (**). That is that the semantically ordering relationship in (**) inherits the one in (*). However, this property does not hold when \( h, k \in LH^\varepsilon_i \), i.e. when they belong to the same graded semantic class of \( LH^\varepsilon \) (see Proposition 2.2 below).

Recall that an expression \( h_n \ldots h_1 u \) is said to be a \textit{canonical representation} of an element \( x \) w.r.t. \( u \) if \( x = h_n \ldots h_1 u \) and \( h_i \ldots h_1 u \neq h_{i-1} \ldots h_1 u \), for \( \forall i \in N \) such that \( i \leq n \).

Table 1

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For convenience, we use the following notation: for \( x = h_0 \ldots h_1 u \), by \( x(j, u) \) we denote the suffix \( h_{j-1} \ldots h_1 u \) of \( x \) w.r.t. \( u \), i.e. \( x(j, u) \) contains exactly a string of length \( j - 1 \) of hedges applying to \( u \).

Now, we shall recall some results of the refined hedge algebras for reference in the sequel.

**Proposition 2.1.** Let \( AX = (X, G, C, LH, \leq) \) be a refined hedge algebra and \( y = h_n \ldots h_1 h x \) be a canonical representation of \( y \) w.r.t. to \( x \). Then,

(i) For \( \forall h \in LH \), there are two unit-operations \( h^- \) and \( h^+ \) such that \( h^- \) is negative w.r.t. \( h \), whereas \( h^+ \) is positive w.r.t. \( h \) such that, for \( \forall h_1, \ldots, h_n \in LH \), \( V^n h^- h x \leq X \), \( V^n h^+ h x \), for \( h \geq x \) and \( V^n h^+ h x \leq h_n \ldots, h_1 h x \leq V^n h^- h x \), for \( h \leq x \);

(ii) For \( \forall h \in LH^* \), there exist \( o, o' \in \{V, L\} \) such that, for \( \forall h_1, \ldots, h_n \in LH \), \( \forall x \in X, h_n \ldots h_1 h x \leq V^n o x \), with \( x \preceq o x \) and \( h_n \ldots h_1 h x \preceq V^n o' x \), with \( x \succeq o' x \).

**Proposition 2.2.** (i) Property of advancing equally: For \( \forall x \in X \), if \( h x \leq k x \) and \( h, k \) belong together to the same set \( LH_i \), then for \( \forall \delta \in LH^* \), we have \( \partial h x \leq \partial k x \), and for \( \forall h \in LH(hx) \) such that \( u \leq \partial h x \), it is always incomparable with \( \partial k x \), and for \( \forall u \in LH(kx) \) such that \( v \leq \partial k x \), \( v \) is always incomparable with \( \partial h x \).

(ii) For \( \forall x \in X \), if \( h x \leq k x \) and \( h, k \in LH^i \), where \( i \neq j \), then for all \( \delta, \delta' \in LH^* \), we have \( \partial h x \leq \delta' k x \).

(iii) In particular, if one of \( h \) and \( k \) is the identity \( I \) then we have \( \partial h x \leq x \), for \( k = I \) (and hence \( LH(hx) \leq x \)) and \( x \leq \delta' k x, for h = I \) (and hence \( LH(kx) \geq x \)).

**Proposition 2.3.** Suppose that \( h, k \) belong together to the same set \( LH_i \). Then, for all \( x \in X \),

(i) \([15, Proposition 3.2] \) For all \( \delta \in LH^* \), \( \partial h x \) is a fixed point (i.e. \( h' \partial h x = \partial h x \), for all \( h' \in LH \)) if \( \partial k x \) is a fixed point.

(ii) \([15, Proposition 3.4] \) \( LH(hu) \) and \( LH(ku) \) are isomorphic in the category of the partially ordered sets with the isomorphism \( f: \partial h u \rightarrow \partial k u \).

Let \( AX = (X, G, C, LH, \leq) \) be a refined hedge algebra. We shall enlarge it as follows: Let \( LH_e = LH \cup \{\Phi, \Sigma\} \) be the set of unary operations with two new ones \( \Sigma \) and \( \Phi \). Consider any abstract algebra \( AX_e = (X, G, C, LH_e, \leq) \) such that \( LH_e(G \cup C) = X \) and the given algebra \( AX = (X, G, C, LH, \leq) \) is its substructure, that is \( LH(G \cup C) = X \). Since, the elements in \( C \) are fixed points of \( AX_e \), i.e. for \( c \in C \) and \( h \in LH \), \( hc = c \), we have \( LH(G \cup C) = LH(G \cup C) = X \). Put \( Lim(X) = LH_e(G \cup C) \setminus LH(G \cup C) \). It is called the set of limit elements of \( AX_e \), since as it will be seen later, these elements can be represented either in the form \( x = \inf LH(u) \) or \( x = \sup LH(u), u \in LH(G) \). By definition, we have \( Lim(X) \cap LH(G \cup C) = \emptyset \).

### 2.2. An axiomatization of complete hedge algebras and elementary properties

Our end-purpose of constructing complete hedge algebras is to prove that the obtained algebras will be distributive lattices. More exactly, we aim to prove that the distributive lattice property of the new algebras can be deduced from an intended system of axioms, which are merely order-based semantic properties of linguistic terms. However, it is not easy to reach this aim in the present paper and, therefore, we shall try to establish a basis to solve it as follows:

We know that the refined hedge algebras are distributive lattices already. So, we shall complete these algebras by adding to them “limit elements” in an algebraic way so that the new algebras will be a minimal (with respect to a semantic-based topology) extension of refined hedge algebras. In such a way we hope that the obtained algebras will also be distributive lattices. Therefore, we shall first try to define new algebras and, then, to make their order-based structure as clear as possible, assuming that it will be a sufficient basis for reaching our main purpose.

Now, we are ready to introduce an axiomatization of completion of hedge algebras.

**Definition 2.1.** An abstract algebra \( AX_e = (X, G, C, LH_e, \leq) \) is said to be a complete hedge algebra (ComHA) if \( substructure (X, G, C, LH, \leq) \) is a refined hedge algebra and \( AX_e \) satisfies the following axioms:

(L1) For \( \forall x \in X \) and \( \forall h \in LH \), we always have \( \Phi x \leq h x \leq \Sigma x \).

(L2) For \( \forall x \in X \) and \( \forall o \in \{V, L\} \), the inequalities \( \Phi x \leq \Phi o x \) and \( \Sigma o x \leq \Sigma x \) hold. Moreover, for \( \forall h, h' \in LH \), we have \( h \Sigma o x \leq h' \Sigma x \) and \( h \Phi x \leq h' \Phi o x \).
(L3) For every \( z \in X \), if for \( \forall x' \in LH(x) \) we have \( x' \leq z \), then \( \Sigma x \leq z \). And, if for \( \forall x' \in LH(x) \) we have \( x' \geq z \), then \( \Phi x \geq z \).

(L4) Let \( h \) be an atom in \( H^E + I \) (that is that it is a minimal element in the lattice \( H^E \)). Then, if \( \Sigma hx \notin LH(x) \) then \( hx \leq x \) implies that \( \Sigma hx = x \) and if \( \Phi hx \notin LH(x) \) then \( hx \geq x \) implies that \( \Phi hx = x \). For any \( y \in X \), \( \Sigma hx < y \) implies that \( x \leq y \) and \( \Phi hx > y \) implies that \( x > y \).

(L5) For any \( h \in LH[^E]_i \) and \( k \in LH[^E]_{i+1} \), i.e. they belong to consecutive graded semantic levels of \( LH[^E] \), if \( \Phi x, \Sigma x \in Lim(X) \) or, equivalently, \( \Phi x, \Sigma x \notin LH(G \cup C) \), then \( hx \leq kx \) implies that \( \Sigma hx = \Phi kx \) and \( hx \geq kx \) implies that \( \Phi hx = 2kx \).

Note that, if we adopt a convention that \( hIx = x \), for any \( h \), and \( \Phi Ix = \Sigma Ix = x \), which mean that any operations applied to each explicit occurrence of \( I \) has no effect, then the first part of (L4) can be considered as a particular case of (L5), in which \( k = I \). With such a convention, some presentations in the sequel can be abbreviated.

The intuitive meaning of these axioms is the following: (L1) means that the acting effect of the new operations \( \Sigma \) and \( \Phi \) is stronger than that of any other one in \( LH \); and since \( \Phi x \leq ox \) and \( ox \leq x \), which are understood as semantically ordering relationships between these terms, the inequalities in the first part of (L2) say \( \Sigma \) and \( \Phi \) cannot change the meaning of these terms, which is expressed by semantics-based ordering relationships, or we can say that \( \Sigma \) and \( \Phi \) should inherit their meaning. The second part of (L2) describes the fact that since the elements \( ox, x, \Phi x \) and \( \Phi ox \) are most specific, they are crisp concepts and, hence, \( h \) and \( h' \) cannot change their relative meaning expressed by their ordering relationships. (L3) ensures that \( \Sigma x \) is the least upper bound and \( \Phi x \) is the greatest lower bound of \( LH(x) \). To figure out the meaning of (L4) and (L5), we observe the following situation: people use linguistic terms like, for example, fast, very fast, low, rather low, . . . to describe linguistically the speed of a vehicle which runs from 0.0 to 200.0 km/h, and all of us understand that any speed of the vehicle in this interval can be considered as being described by at least one of such terms. Therefore, we need to assume that the set of all linguistic terms can approximate to any real value of the speed and so they should be dense in itself or the set of all such linguistic terms has no gaps. That is it requires that \( \Sigma hx = x \) and \( \Phi hx = x \) in (L4) and that \( \Sigma hx = \Phi hx \) and \( \Phi hx = \Sigma kx \) in (L5).

Now, we shall step-by-step clarify the order-based structure of complete hedge algebras and, especially, ordering relationships between elements of these algebras.

First of all, we shall show that the operations \( \Sigma \) and \( \Phi \) will have desired properties.

**Theorem 2.1.** For each ComHA \( AX = (X, G, C, LH, \leq) \) and for every \( x \in X \), we have:

(i) \( \Phi x \leq x \leq \Sigma x \)

(ii) \( \Sigma x = \supremum LH(x) \) and \( \Phi x = \infimum LH(x) \), i.e. the set \( LH(x) \) always has a least upper bound \( \Sigma x \) and a greatest lower bound \( \Phi x \) in \( X \).

**Proof.** (i) By axiom (L1), where \( h \) is arbitrary, and by the fact that the hedges in \( LH^+ \) are converse to the ones in \( LH^- \), we can choose \( h \) such that \( x \leq hx \). Hence, we obtain \( x \leq hx \leq \Sigma x \). By a similar argument, we have \( \Phi x \leq x \), i.e. (i) is valid.

(ii) We first show that \( LH(x) \subseteq \Sigma x \), that is for all \( y \in LH(x) \), \( y \leq \Sigma x \). Indeed, in the case that \( y = x \), by (i) above, the desired inequality holds. In the case that \( y \neq x \), \( y \) should be represented in the form \( y = \delta hx \), \( \delta \in LH^+ \). There are two cases:

- **Case 1:** \( hx \leq x \). According to Proposition 2.2(iii), we always have \( LH(hx) \leq x \), and hence, combining this with the statement (i) we obtain, for all \( y \in LH(hx) \), \( y \leq x \leq \Sigma x \).

- **Case 2:** \( hx \geq x \). Let \( y \) be of the form \( y = h_n \ldots h_1 hx \). By (i) of Proposition 2.1, we have \( y = h_n \ldots h_1 hx \leq V^n h^+ x \). On the other hand, using the first part of (L2) with \( o = h^+ \) we obtain \( \Sigma x \geq \Sigma h^+ x \) and repeatedly applying the first part of (L2) \( n \) times, with \( o = V \), to the obtained results we get \( \Sigma h^+ x \geq \Sigma V h^+ x \geq \cdots \geq \Sigma V^n h^+ x \). By (i) above, it follows that \( \Sigma V^n h^+ x \geq V^n h^+ x \) and hence, combining these results we have \( \Sigma x \geq V^n h^+ x \geq y \), for \( y \in LH(hx) \). This shows that \( LH(x) \leq \Sigma x \) and by Axiom (L3) we obtain \( \Sigma x = \supremum LH(x) \).

The equality for the operation \( \Phi \) in (ii) can be proved similarly. \( \square \)

Now, we examine again the meaning of the concepts which are fixed points. It was proved for the refined hedge algebras that if \( x \) is a fixed point of a hedge \( h \in LH \) i.e. \( hx = x \), then \( x \) is a fixed point of any other one in \( LH \). It explains a simple but interesting fact saying that if the meaning of a term cannot be changed by a hedge then it cannot
be changed by any others. We shall show that it is still valid for the hedges \( \Sigma \) and \( \Phi \) and, moreover, that all elements in \( \text{Lim}(X) \) are fixed points, i.e. we cannot generate new meaning from limit elements, using hedges. This means that the limit elements are most specific or they are crisp concepts.

**Theorem 2.2.** Let us consider a ComHA \( \mathcal{AX} = (X, G, C, \text{LH}_e, \leq) \). Then,

(i) If \( x \) is a fixed point of an \( h \in \text{LH} \), then it is also a fixed point of \( \Sigma \) and \( \Phi \) and so we can use the terminology “fixed point” instead of “fixed point of an operation”. Conversely, if \( x \) is a fixed point of \( \Sigma \) or \( \Phi \), then it is a fixed point of any other one.

(ii) Every \( x \in \text{Lim}(X) \) is a fixed point and it can be represented in the form \( x = \Sigma u \) or \( x = \Phi u \), with \( u \in \text{LH}(G) \).

That is that the set \( X \setminus \text{LH}(G \cup C) \) consists of only elements which are limits of elements of \( \mathcal{AX} \). Note that \( G = \{c^-, c^+\} \).

**Proof.** (i) Suppose that \( x \) is a fixed point of \( h \in \text{LH} \). In the refined hedge algebras, it is known that \( x \) is also a fixed point of any other \( k \in \text{LH} \) and, therefore, \( \text{LH}(x) = \{x\} \). By Theorem 2.1, we get \( \Phi x = \text{infimum} \text{LH}(x) = x = \text{supremum} \text{LH}(x) = \Sigma x \). Now, the remaining statement of (i) follows directly from Theorem 2.1(ii). So, (i) of the theorem is completely proved.

(ii) Let us consider the case \( x \in \text{Lim}(X) \), which is of the form \( x = \Sigma u \) or \( x = \Phi u \), with \( u \in \text{LH}(G) \). Since for \( x = \Phi u \), the proof is similar, it is sufficient to prove the statement for the case \( x = \Sigma u \). Choose an \( h \in \text{LH} \) such that \( x \geq hx = h\Sigma u \). Let \( h' \) be chosen so that \( h'^{-1}ou \geq ou \). By the second part of (L2), we have \( x \geq h^uo \geq h'^{-1}ou \), and by using the first part of (L2), for \( x = ou \) and \( o' = V \), we get \( ou \geq \text{supremum} \text{LH}(ou) \). Applying it again with \( o' = V \) repeatedly to the obtained results, starting from \( ou \), we infer that \( \text{supremum} \text{LH}(ou) \geq \text{supremum} \text{LH}(ou) \), for every \( \forall n \in \mathbb{N} \) and for any unit operation \( o \).

Take any element \( y = h_m \ldots h_1 u \in \text{LH}(u) \). By Proposition 2.1(ii), \( y \) satisfies the inequality \( y = h_m \ldots h_1 u \leq V^{m-1}ou \), for a certain unit operation \( o^+ \). So, from the last inequalities for \( o = o_i^+ \), it follows that \( y \leq \text{supremum} \text{LH}(ou) \), for \( \forall y \in \text{LH}(u) \). Hence, by Axiom (L3) and the assumption made on \( x \) and \( h \), it follows that \( x = \Sigma u \leq h\Sigma u \leq x \), which implies that \( x \) is a fixed point.

Now, assume that \( x \in \text{Lim}(X) \) and \( x \) is of the form \( x = k_m \ldots k_1 a \), with \( a \in G \) (since \( \text{LH}_e(G \cup C) = \mathcal{AX} \) and constants in \( C \) are fixed points). As at least one of these \( k_j \) should be either \( \Sigma \) or \( \Phi \), we may assume that there exists a least index \( j \) such that \( k_j \in \{\Sigma, \Phi\} \), say \( k_j = \Phi \). Then, we have \( x(j + 1, a) = \Phi k_{j-1} \ldots k_1 a = \Phi u \), where \( u = k_{j-1} \ldots k_1 a \in \text{LH}(G) \). By the case proved above, \( \Phi u \) is a fixed point and, hence, by part (i) of the theorem, it follows that \( x = \Phi u \), i.e. it is also a fixed point. Since the proof for the case for \( \Sigma \) is similar, the proof is completed.

To formulate a property below, we need to recall a notion: a ComHA is said to be free (or freely generated from \( G = \{c^-, c^+\} \)) if for every \( x \in \text{LH}(G) \), every \( h \in \text{LH} \), we have \( hx \neq x \).

**Corollary 2.1.** (i) We always have \( \text{Lim}(X) = \text{LH}_e(G) \setminus \text{LH}(G) \).

(ii) If a ComHA \( \mathcal{AX} \) is free, then we have \( \Sigma c^+ = 1 \), \( \Phi c^+ = \Sigma c^- = W \) and \( \Phi c^- = 0 \), which imply that \( \text{LH}_e(G) = X \), i.e. \( G \) is the set of free generators of \( \mathcal{AX} \).

**Proof.** (i) Directly follows from the fact that elements in \( C \) are fixed points.

(ii) For \( x \in \text{LH}(c^+) \), we can write it in the form \( x = h_n \ldots h_1 hc^+ \). By (ii) Proposition 2.1, we have \( h_n \ldots h_1 hc^+ \leq V^n oc^+ \), where \( o = V \). Note that \( \text{LH}(G) = \text{LH}(c^-) \cup \text{LH}(c^+) \) and \( \text{LH}(c^-) \leq \text{LH}(c^+) \). Since \( \mathcal{AX} \) is free, it implies that there are no greatest elements in \( \text{LH}(c^+) \) and hence \( \Sigma c^+ = 1 \).

Since the remaining equalities can be proved similarly, it completes the proof.

Now, we are going to examine the following semantic properties of the operations \( \Sigma \) and \( \Phi \) in order to clarify the order-based structure of the set \( X \) more clearly.

**Proposition 2.4.** For all \( y \in \text{LH}(x) \), where \( x \in X \), we have \( \Sigma y \leq \Sigma x \) and \( \Phi y \geq \Phi x \).

**Proof.** Since it is obvious that \( \text{LH}(y) \subseteq \text{LH}(x) \), the validity of the proposition follows directly from Theorem 2.1.
Theorem 3.1. For any inequality.

Proof. Assume that \( hu \leq ku \), where \( h \in LH_i \), \( k \in LH_j \), with \( i \neq j \). Then,

\[
\begin{align*}
\Sigma x = \Sigma \delta hu & \leq \Phi \gamma ku = \Phi y. 
\end{align*}
\]

Proposition 2.5. Suppose that \( x, y \) can be written in the form \( x = \delta hu, y = \gamma ku \), where \( u \in X \), \( \delta, \gamma \in LH^* \) and \( h \in LH_i, k \in LH_j \), with \( i \neq j \). Then,

\[
\begin{align*}
\Sigma x & = \Sigma \delta hu \leq \Phi \gamma ku = \Phi y. 
\end{align*}
\]

Proof. Since for the case \( hu \leq ku \), we have \( \delta \, hu \leq \gamma \, ku \), for any \( \delta \, \gamma \in LH^* \). Consider any two elements \( x' \in LH(x), y' \in LH(y) \). Obviously, they can be written as \( x' = \delta \, hu \) and \( y' = \gamma \, ku \) and hence from the last inequality, it follows that \( x' \leq y' \), for all \( x' \in LH(x) \). It is implied that \( \Sigma x \leq y' \), for all \( x \in LH(y) \). Now, from (ii) of Theorem 2.1 we infer that \( \Sigma x \leq \text{infimum} LH(y) = \Phi y \), which is the desired inequality.

The above proposition says that any chains of hedges of the forms \( \Sigma \delta \) or \( \Phi \gamma \) cannot change the relative meaning of terms \( hu \) and \( ku \) expressed by the first inequality of the proposition.

Proposition 2.6. For \( x \in LH(G \cup C) \), if the set \( LH(x) \) is finite, then \( \Sigma x, \Phi x \in LH(x) \) and hence \( \Sigma x, \Phi x \notin \text{Lim}(X) \) or there are a least and a greatest element in the set \( LH(x) \).

Proof. It is known that each element \( z \in LH(x) \) has a unique canonical representation with respect to \( x, i.e. \), we have \( z = h_p h_{p-1} \ldots h_1 x \) and \( h_i h_{i-1} \ldots h_1 x \neq h_{i-1} \ldots h_1 x \), for all \( i : 1 \leq i \leq p \). Since \( LH(x) \) is finite, the number of canonical representations of the elements in \( LH(x) \) is also finite. Therefore, among the elements of \( LH(x) \) there exists an element \( z' \) such that the number of hedges of its canonical representation, say \( z' = h_p \ldots h_1 x \), is greatest.

Consider any \( z \in LH(x) \) and let its canonical representation be \( z = h_p \ldots h_1 x \). According to Proposition 2.1(i) and note that \( p' \geq p \) and since \( V \) is positive with respect to \( x \), we have

\[
\begin{align*}
V^{\prime} h'_1 h_1 x & \leq V^{p'-1} h'_1 h_1 x \leq z = h_p \ldots h_1 x \leq V^{p'-1} h'_1 h_1 x \leq V^{p'} h_1 x \quad \text{for} \quad h_1 x \geq x, \\
V^{\prime} h'_1 h_1 x & \leq V^{p'-1} h'_1 h_1 x \leq z = h_p \ldots h_1 x \leq V^{p'-1} h'_1 h_1 x \leq V^{p'} h_1 x \quad \text{for} \quad h_1 x \leq x.
\end{align*}
\]

We can find unit operations \( o \) and \( o' \) such that \( o' \leq x \leq o x \) and, since the unit operations are the strongest, it implies that \( o' \leq x \leq o x \). As \( V \) is positive with respect to \( o \) and \( o' \), it follows that \( V^{p'} o' x \leq \{ V^{p'} h'_1 h_1 x, V^{p'} h_1 h_1 x \} \leq V^{p'} o x \). Therefore, we obtain \( V^{p'} o' x \leq V^{p'} o x \). Because \( z \) is arbitrary, it shows that \( \text{Supremum} LH(x) = V^{p'} ox \) and \( \text{Infimum} LH(x) = V^{p'} o' x \) and, hence, by Theorem 2.1, \( \Sigma x, \Phi x \in LH(x) \).

Proposition 2.7. Suppose that \( x = \gamma hu \), where \( u \in X, h \in LH \) and \( \gamma \in LH^* \). Then, the following statements hold:

\[
\begin{align*}
hu \geq u & \implies \Phi x \geq u, \quad \text{and} \quad hu \leq u \implies \Sigma x \leq u.
\end{align*}
\]

Proof. Since for the case \( hu \leq u \) the proof is similar, we shall prove the proposition for the case \( hu \geq u \). By Proposition 2.2(ii), we have \( LH(hu) \geq u \). So, it follows, for all \( \gamma \in LH^* \), that \( \gamma hu \geq u \). Now, consider any \( x' \in LH(\gamma hu) \). Because \( x' \) belongs also to \( LH(hu) \), we have \( x' \geq u \). It shows that \( \text{infimum} LH(\gamma hu) \geq u \) or \( \Phi x \geq u \), by Theorem 2.1.

3. Some ordering relationships related to the operations \( \Sigma \) and \( \Phi \)

In order to discover more details of the order-based structure of ComHAs, in this section we shall study the following question: given the ordering relationship between \( hx \) and \( kx \), which ordering relationships between two of those elements can be deduced, at least one of which can be represented in one of the forms \( ohx, oihu, oix, or oikx \), where \( o \in \{ \Sigma, \Phi \} \)?

Firstly, we shall show that if \( h, k \) belong to the same graded semantic level \( LH_i \) and there are no greatest elements in the sets \( LH(hx) \) and \( LH(kx) \), then these sets have the same least upper bound.

Theorem 3.1. For any \( x \in X = LH(G \cup C) \) and \( h, k \in LH_i \) with \( h \neq k \), the condition \( \Sigma hx \notin LH(hx) \) and \( \Sigma kx \notin LH(kx) \) implies that \( \Sigma hx = \Sigma kx \).

A dual statement for \( \Phi \) is also valid and we have \( \Phi hx = \Phi kx \).
Proof. For any \( x \in X \), we have either \( x \leq hx \) or \( x \geq hx \). Since the proof for the case \( x \geq hx \) can be obtained by duality, we consider the case \( x \leq hx \). From the compatibility of \( h \) and \( k, h, k \in LH^\leq \), it follows that \( x \leq hx \). It is known that the lattice \( LH^\leq \) is modular and \( h \neq k \), we infer \( LH^\leq_{i+1} \) consists of a single element denoted by \( l \). Thus, we have \( k < l \) and \( h < l \), which imply that \( hx \leq lx, kx \leq lx \) (see Fig. 1). According to axiom (L4) and (L5), we get \( \Sigma hx = \Phi lx \) and \( \Sigma kx = \Phi lx \), which proves the statement for \( \Sigma \) (note that, in the case that \( l \) is the identity \( l \) the argument is still valid, since in this case (L4) can be considered as a version of (L5), according to our convention).

Since the proof of the statement for \( \Phi \) is similar, the theorem is completely proved. \( \square \)

Lemma 3.1. For \( \forall x \in LH(G) \), we have

\[
\Sigma x = \text{supremum}\{V^n ox : o \in UOS, ox \geq x, n = 1, 2, \ldots\}
\]

and

\[
\Phi x = \text{infimum}\{V^n ox : o \in UOS, ox \leq x, n = 1, 2, \ldots\},
\]

noticing that \( V \) is positive w.r.t. all unit operations in UOS.

Proof. The lemma can easily be deduced from Proposition 2.1(ii) and Theorem 2.1. \( \square \)

Similarly as in the case of the refined hedge algebras, the following lemma shows homogeneity of the hedges in the same graded semantic level \( LH^\leq \).

Lemma 3.2. Let us consider \( x = \delta hu \) and \( y = \delta ku \), where \( h, k \in LH^\leq_i \). Then, \( LH(x) \) is finite iff \( LH(y) \) is finite or, equivalently, \( \Sigma x \notin LH(x) \) iff \( \Sigma y \notin LH(y) \).

Proof. According to Proposition 2.3(ii), \( LH(hu) \) and \( LH(ku) \) are isomorphic in the category of the partially ordered sets under the isomorphism \( f : \delta hu \rightarrow \delta ku \). Thus, the validity of the lemma follows from Proposition 2.6. \( \square \)

Lemma 3.3. Let \( x = \delta h_1 u, y = \delta ku \) and \( h_1 u < ku \), where \( h_1, k \in LH^\leq_i \). If \( \Sigma x \neq \Sigma y \) then there exists a \( z \in LH(h_1 u) \) such that \( \Sigma x \leq z \). Moreover, a \( z \) among such elements can be found so that either \( \Sigma x = z \) or \( \Sigma x = \Phi z \).

Proof. Let us suppose that \( \delta = h_p h_{p-1} \ldots h_2 \). There are two possibilities:

1. There exists \( j \geq 2 \) such that \( h_j \neq V \) and assume that \( j \) is the greatest among such indexes. So, \( h_p = h_{p-1} = \ldots = h_{j+1} = V \).
   (a) Consider the case \( h_j x(j, u) \geq VH_j x(j, u) \), where \( x(j, u) = h_{j-1} \ldots h_{j+1} u \) and \( j \leq p \). Since \( V \in LH^\leq N \) is positive w.r.t. itself, it follows that \( h_j x(j, u) \geq VH_j x(j, u) \geq h_{p-1} \ldots h_1 u = x(p, u) \geq Vx(p, u) = x \). Clearly, there exists
k′ ∈ LH_{n+1}^+ such that V < k′ < I and, hence, Vx(p, u) ≤ k′x(p, u). By virtue of axiom (L5) we have \( \Sigma Vx(p, u) = \Phi k′x(p, u) \). Therefore, \( z = k′x(p, u) \) satisfies the lemma, since \( \Sigma x = \Sigma Vx(p, u) \), by Lemma 3.1 (see Fig. 2(a)).

(b) Consider the case that \( VH_jx(j, u) > hjx(j, u) \). Since \( V \) is positive w.r.t. \( V \), it follows from Lemma 3.1 that \( \Sigma x = \Sigma VH_jx(j, u) = \Sigma hjx(j, u) = \Sigma x \). Thus, the element \( z = k_jx(j, u) \) satisfies the statement of the lemma (Fig. 2(a)).

First, suppose that \( h_j \in LH_i^+ \). So, we get \( h_j < V \in LH^+ \). Then, there should exist \( k_j \in LH_{i+1}^+ \), from which it follows that \( hjx(j, u) ≤ k_jx(j, u) \). By (L5), we obtain \( \Phi k_jx(j, u) = \Sigma hjx(j, u) = \Sigma x \). Therefore, \( z = k_jx(j, u) \) satisfies the statement of the lemma (Fig. 2(a)).

Secondly, suppose that \( h_j \in LH_i^- \). In the case \( h_j = L \) (i.e., \( j = N^- \)), as \( V \) is positive w.r.t. \( h_j \) and \( hjx(j, u) \leq VH_jx(j, u) \), it follows that \( x(j, u) \leq hjx(j, u) = h_jx(j, u) \). By Lemma 3.1, we also have \( \Sigma x = \Sigma hjx(j, u) \). Now, suppose that \( h_{j−1} \in LH_i^− \). If \( h_j \) is negative w.r.t. \( h_{j−1} \), then \( x(j, u) = h_{j−1}x(j − 1, u) \). Since \( h_{j−1}x(j − 1, u) \leq h_jx(j, u) \), it follows that \( h_{j−1}x(j − 1, u) \leq x(j − 1, u) \). Now, consider any \( k_{j−1} \in LH_{j−1}^− \). If \( i' > 1 \), then \( h_{j−1}x(j − 1, u) \leq k_{j−1}x(j − 1, u) \). By Lemma 3.1 and axiom (L5), it follows that \( \Sigma h_{j−1}x(j − 1, u) = \Sigma Lh_{j−1}x(j − 1, u) = \Sigma x = \Phi k_{j−1}x(j − 1, u) \). Since \( h_{j−1}x(j − 1, u) \) is the desired element (see Fig. 2(a)). If \( i' \neq 0 \), then \( k_{j−1} = I \) and we have \( \Sigma h_{j−1}x(j − 1, u) = \Sigma x = x(j − 1, u) \) and then \( z = x(j − 1, u) \) is the desired element.

If \( h_j \) is positive w.r.t. \( h_{j−1} \), then \( x(j − 1, u) \leq h_{j−1}x(j − 1, u) \). Since \( L \) is negative w.r.t. \( L \) and all hedges in \( LH^+ \), it follows that \( h_{j−1} \in LH^− \) and \( h_{j−1} ≠ L \). Hence, in \( LH_{j−1}^− \) there should exist a \( k_{j−1} \) and we have \( h_{j−1}x(j − 1, u) \leq k_{j−1}x(j − 1, u) \). Thus, \( \Sigma h_{j−1}x(j − 1, u) = \Sigma Lh_{j−1}x(j − 1, u) = \Sigma x = \Phi k_{j−1}x(j − 1, u) \). and then \( z = k_{j−1}x(j − 1, u) \) is the element we want to find.

Now, consider the case \( h_j ≠ L \). By our convention, no hedges in \( \delta \) are the identity and so there should exist \( h_j \in LH_i^− \) with \( i' ≠ 0 \). Therefore, if \( i' > 1 \), there exists always a \( k_j \in LH_{j−1}^− \) and it implies that \( h_{j−1}x(j, u) ≤ k_{j−1}x(j, u) \). Because we assume that \( VH_jx(j, u) > h_{j−1}x(j, u) \), it follows that \( VH_jx(j, u) = \Sigma x = \Phi k_{j−1}x(j, u) \). If \( i' = 1 \), then \( k_j = I \) (the identity) and so we have \( \Phi k_{j−1}x(j, u) = \Phi Ix(j, u) = x(j, u) \), by our conventional notation. Thus, in both cases, \( z = k_{j−1}x(j − 1, u) \) satisfies the lemma.

In the case that \( h_j = V \), for all \( j ≥ 2 \), we have \( x = h_ph_{p−1}…h_1u = V^{p−1}h_1u \). For \( VH_1u ≥ h_1u \), Lemma 3.1 ensures that \( \Sigma x = \Sigma h_1u \). According to Theorem 3.2, if \( LH(h_1u) \) is infinite then so is \( LH(ku) \) and, hence, Theorem 3.1 implies that \( \Sigma h_1u = \Sigma ku = \Sigma V^{p−1}ku = \Sigma y \), which contradicts the assumption. So, \( LH(h_1u) \) should be finite and, by Lemma 3.1, there is an \( n \) such that \( \Sigma h_1u = V^noh_1u \) and \( z = V^noh_1u \) satisfies the lemma (Fig. 2(b)).

For \( VH_1u ≤ h_1u \), the fact that \( V \) is positive w.r.t. itself implies that \( x = V^{p−1}h_1u = Vx(p, u) ≤ x(p, u) = h_ph_{p−1}…h_1u \). Now, using the same argument as in the case (a) above, we obtain \( \Sigma Vx(p, u) = \Phi kx(p, u) \).

**Theorem 3.2** (Property of advancing equally for \( \Sigma \) and \( \Phi \)). Suppose that \( x = \delta hu, y = \delta ku \) and \( h, k \in LH_i^+ \), that is \( h \) and \( k \) belong to the same graded semantic level. Then,

(i) \( hu ≤ ku \) implies that \( \Sigma hhu ≤ \Sigma kku \) and \( \Phi khu ≤ \Phi khu \);

(ii) \( \Sigma hhu ≠ \Sigma kku \) implies that, for \( \forall v \in LH(ku) \) such that \( v ≤ \Sigma hhu \), \( v \) and \( \Sigma kku \) are incomparable and, for \( \forall v \in LH(hu) \) such that \( v ≥ \Sigma kku \), \( v \) and \( \Sigma hhu \) are incomparable.

A similar formulation for \( \Phi \) also holds, i.e. \( \Phi hhu ≠ \Phi kku \) implies that for \( \forall v \in LH(ku) \) such that \( v ≤ \Phi kku \), \( v \) and \( \Phi hhu \) are incomparable and, for \( \forall v \in LH(hu) \) such that \( v ≥ \Phi hhu \), \( v \) and \( \Phi kku \) are incomparable, as well.
Proof. (i) Since $hu \leq ku$, Proposition 2.2(i) leads to $\delta'hu \leq \delta'ku$, for every string $\delta'$ of hedges. Take any $z \in LH(x)$ and so it can be written in the form $z = \gamma\delta hu$. According to the last inequality, for $\delta' = \gamma\delta$, we have $z = \gamma\delta hu \leq \gamma\delta ku \in LH(y)$ and using Theorem 2.1, it leads to $z \leq \Sigma y$, for $\forall z \in LH(x)$. By this and Axiom (L3) we obtain $\Sigma x \leq \Sigma y$ (see Fig. 3).

The validity of $\Phi x \leq \Phi y$ can be proved by duality.

(ii) Suppose that $\Sigma x \neq \Sigma y$. Then, the part (i) of the theorem implies that $\Sigma x < \Sigma y$. Take any element $v \in LH(ku)$ such that $v \not\in \Sigma y$. Then, $v$ can be written as follows $v = \delta'ku$. It is necessary to show that $v$ and $\Sigma x$ are incomparable. We assume the contradiction that $\Sigma x$ and $v$ are comparable. Suppose that $\Sigma x > v$. According to Lemma 3.3, there exists a $z \in LH(hu)$ such that $z \geq \Sigma x$ and hence $z > v$. Since the inequality $z \leq \delta'hu$ implies a contradiction that $\Sigma x \leq v$, we should have $z \not\subseteq \delta'hu$. By Proposition 2.2(i), $hu < ku$ implies that $\delta'hu \leq \delta'ku$ and hence $z$ should be incomparable with $\delta'ku = v$. It contradicts the fact that $z > v$.

Now, suppose that $\Sigma x \leq v$. There are two cases: (1) $\exists z' \in LH(y), v \not\supseteq z'$. Hence, $z' = \gamma\delta ku$. Again by Proposition 2.2(i), we infer that $\gamma\delta hw \leq \gamma\delta ku$ and $v$ is incomparable with $\gamma\delta hu$. It is contrary to the inequalities that $LH(x) \supseteq \gamma\delta hu \leq \Sigma x \leq v$. (2) $\forall z' \in LH(y), z' \leq v$. By Axiom (L3), we have $\Sigma y \leq v$, which contradicts the assumption made on $v$.

Accordingly, the comparability of $v$ and $\Sigma x$ always leads to a contradiction and, therefore, $\Sigma x$ and $v$ are incomparable.

Now, we consider the case that $v \in LH(hu)$ and $v \not\subseteq \Sigma x$ (see Fig. 3). It is necessary to show that $\Sigma y$ and $v$ are incomparable. It can be seen that $v \not\subseteq \gamma\delta hu$, for any $z' = \gamma\delta hu \in LH(x)$. Based again on Proposition 2.2(i), we infer that $\gamma\delta hu \leq \gamma\delta ku$ and $u$ is incomparable with $\gamma\delta ku$. This implies that $v \not\subseteq \Sigma y > \gamma\delta ku$. Assume the contrary that $v < \Sigma y$ and consider a $z \in LH(hu)$ which satisfies $\Sigma x \leq z$. Since $\Sigma x < \Sigma y$, on account of Lemma 3.3, there exists $z' \in LH(hu)$ such that $\Phi z' = \Sigma x$ or $z' = \Sigma x$. If $v \not\subseteq z$, for all $z \geq \Sigma x$, it follows that $u \not\subseteq \Phi z'$, i.e. $u \not\subseteq \Sigma x$, which contradicts the assumption made on $u$. Thus, there exists a $z = \gamma\delta hu \in LH(hu)$ such that $\Sigma x \leq z$ and $v \not\subseteq z$. By Proposition 2.2(i), we have $\gamma\delta hu \leq \gamma\delta ku$ and $u$ is incomparable with $\gamma\delta ku$. Because the values of $\Sigma, \Phi$ are defined only by ordered structure and, according to Proposition 2.3(ii), $LH(hu)$ and $LH(ku)$ are isomorphic, it can be seen that $\Sigma x \leq \gamma\delta hu$ implies that $\Sigma x \not\subseteq \gamma\delta ku$. From the last inequality we infer $v < \gamma\delta ku = z$, a contradiction.

The remaining statement for $v \not\subseteq \Phi ku = y$ (see Fig. 3) is proved by duality. □

4. Fuzziness of linguistic terms and semantic-based topology of ComHAs

Because the meaning of linguistic terms reflects a semantic approximation of terms or a semantics-based topology and, in our algebraic approach, fuzziness is also a notion of semantic closeness of objects in reality, we are able to show that there is a close connection between fuzziness of terms and a semantics-based topology of $AX'$. Then, as it was required in the introduction of the paper, we shall prove that each ComHA $AX'$ is a minimal extension of a refined hedge algebra $AX$. The results of this section provide a foundation to investigate the fuzziness measure of terms and quantification problem of ComHAs.

4.1. A model of fuzziness of vague terms and semantic-based topology of ComHAs

Fuzziness of linguistic terms is a notion which is in general very difficult to be formalized in a reasonable way. Since hedge algebras model a natural semantic-based structure of terms-domains, we hope that they may give a useful
and proper formalism to define fuzziness of vague concepts. Our idea comes from a usual view-point as follows: it is well known that one of the main functions of natural languages is to reflect the reality and the meaning of a term has been taking shape by an assignment of appropriate things or phenomena in the real world to this term. Since the things or phenomena are infinite, while the set of terms in a natural language is finite, there are a large number of terms in the language, each of which should reflect many different things in reality. So, a term becomes vague or inexact if it expresses a number of things or phenomena and, hence, the set of things which are expressed by a term should depict an essential feature of its fuzziness. This observation can be taken to formalize the concept of fuzziness of terms.

Let $\mathcal{AX} = (X, G, LH_{e}, \leq)$ be a free ComHA which model a linguistic domain $Dom(\mathcal{X})$ of a variable $\mathcal{X}$. Semantically, the set $LH(x)$ consists of all terms, each of which still reflects a definite essential meaning of a concept $x$. For example, consider two terms $x = PTrue$ and $y = ATrue$. The term $VPTrue$ reflects a definite meaning of $PTrue$, but not of $ATrue$, while the term $VATrue$ reflects a definite meaning of $ATrue$, but not of $PTrue$. As the set of things mentioned above, the set $LH(x)$ consisting of all terms, which are semantically related to $x$, expresses certain characteristics of fuzziness of the term $x$ and hence we may use this set to qualitatively model its fuzziness. This observation also comes from the fact that a term $x$ is vague if and only if its meaning is still changed by using hedges. It suggests that we should examine the family $\vartheta = \{LH(x): x \in LH(G)\}$. The family $\vartheta^* = \vartheta \cup \{LH(x): x \in Lim(X)\} \cup \{1, W, 0\}$ has the following properties which model main intuitive features of fuzziness of vague concepts:

1. For $x \in G^* = Lim(X) \cup \{1, W, 0\}$, we have $LH(x) = \{x\}$. This means that no any proper meaning can be generated from $x$ by using hedges and, therefore, $x$ is an exact concept.
2. $LH(hx) \subseteq LH(x)$, for all hedge $h$ and all $x \in X$. This property describes the fact that the more specific a term is, the less fuzziness it is. It seems to correspond to our intuition.
3. $LH(hx) \cap LH(kx) = \emptyset$, for $h, k \in LH$ and $hx \neq kx$. It says that $hx$ and $kx$ define different and independent meanings and therefore they define their own meaning. Thus, the fuzziness of the terms $hx$ and $kx$ is determined by just their own meaning.
4. $LH(x) = \bigcup_{h \in LH, i \in I} LH(hx)$. Since, by its nature, a term $x$ is vague if and only if its meaning can be modified by hedges, this equation is inferred from the fact that the fuzziness of a term $x$ can be defined by and only by means of hedges.

So, in our study we may use the set $LH(x)$ to model the fuzziness of a concept $x \in X$.

In nature, fuzziness of a term also reflects semantic similarity of things in the real world indicated by this term. Therefore, we believe that the family $\vartheta$, as a formal tool to model fuzziness of vague terms, will play a special role in studying a topological structure of ComHAs. Since several properties of $\vartheta$ in the category of linear ComHAs will be examined in a future paper, we shall study in this subsection the general mathematical structure of $\vartheta$ in order to show that it defines a semantic-based topology of $\mathcal{AX}$ and the set $Lim(X)$ we add to $X$ is sparse based on this topology.

**Theorem 4.1.** Let $\mathcal{AX} = (X, G, LH_{e}, \leq)$ be a ComHA. Then, there exists a unique topology $\mathcal{I}$ on $X$ such that the family $\vartheta^* = \vartheta \cup \{LH(x): x \in LH(G)\} \cup \{1, W, 0\}$ is a basis of $\mathcal{I}$, i.e. $(X, \mathcal{I})$ becomes a topological space.

**Proof.** Recall that if a family $\Omega$ of subsets of $X$ has the following properties:

1. $X \in \Omega$;
2. For $\forall U_i \in \Omega, i = 1, \ldots, k, \cap_{i=1}^{k} U_i \in \Omega$,

then there exists a unique topology $\mathcal{I}$ on $X$ such that $\Omega$ is a basis of $\mathcal{I}$.

It is clear that $\vartheta^*$ satisfies (o1). Now, consider any two subsets $U_1 = LH(u_1) \in \Omega, i = 1, 2$. If $u_1$ and $u_2$ is independent, then in the theory of refined hedge algebras it is well known that $LH(u_1) \cap LH(u_2) = \emptyset$. In the opposite case, we should have either $u_1 \in LH(u_2)$ or $u_2 \in LH(u_1)$. Assume that $u_1 \in LH(u_2)$, we obtain $LH(u_1) \cap LH(u_2) = LH(u_1)$. Obviously, it implies by induction that $\vartheta$ satisfies (o2). □

Because the sets $LH(x), x \in LH(G)$, model fuzziness of linguistic terms, it can be seen that if the set $X$ is too large in comparison with $X = LH(G \cup C)$, then some expected characteristics of fuzziness which should be defined in the whole $X$ will be lost. So, the extension $X$ of $X$ should be required to be minimal in some sense. In order to clarify it, we recall some notions.

A subset $U$ of $X$ is said to be nowhere dense in $(X, \mathcal{I})$ if the interior of its closure is empty.
The following corollary follows immediately from the above theorem. Note that the set \( X = LH(G) \) is a largest open proper subset in \((X', \mathcal{I})\).

**Corollary 4.1.** The set \( G^\circ = \lim(X) \cup \{I, W, 0\} \) is a nowhere dense closed set in the topological space \((X, \mathcal{I})\) defined in Theorem 4.1.

4.2. Minimal extension property of the ComHAs

Sparseness of the set \( \lim(X) \) in the topological sense examined above is intuitively not enough to clarify that this extension is “minimal”. For example, when in a topology on \( X \) there are only few open sets, a nowhere dense and closed set does not mean that it is sparse. So, we need to show that the topology defined above is sufficiently fine. To solve this question, we shall examine this topology in the context of the semantic order-based structure of ComHAs.

It requires us to introduce first the following notion:

**Definition 4.1.** (i) Let \( AX = (X, G, LH, \preceq) \) be a ComHA. A subset \( U \) of \( X \) is said to be everywhere occupying in \( X \) if for every interval \((x, y) \) of \( X \) (that is we have \( x < y \)) there exists an open subset \( LH(z) \) of \( U \) such that \( x < LH(z) < y \).

The notion everywhere occupying subset \( U \) defined in a connection with the order-based structure of \( X \) means that \( X/U \) is very sparse or, more suggestively, the power of \( X/U \) is almost nothing in comparison with that of the space \( X \). First, we shall prove the following lemma.

**Lemma 4.1.** Let \( AX = (X, G, LH, \preceq) \) be a free ComHA and consider an arbitrary element \( x = h_n \ldots h_1 u \in LH(G) \) and a hedge \( o \in \{\Phi, \Sigma\} \). Then, if \( \Phi h_1 u < o x \) (or \( \Sigma h_1 u > ox \), respectively), then the interval \( (\Phi h_1 u, ox) \) \((ox, \Sigma h_1 u)\), respectively) includes at least one open set. Moreover, there exists an element \( z_i = k_i h_{i-1} \ldots h_1 u = k_i x(i-1, u) \) such that \( k_i x(i-1, u) < h_1 x(i-1, u) (k_i x(i-1, u) > h_1 x(i-1, u) \), respectively), \( LH(z_i) \cap LH(x) = \emptyset \) and \( \Phi h_1 u = \Phi z_i < LH(z_i) = LH(k_i x(i-1, u)) < ox \) (\( \Sigma z_i > LH(z_i) = LH(k_i x(i-1, u)) > ox \), respectively).

**Proof.** First, we consider the case \( \Phi h_1 u < ox \), where \( x = h_n \ldots h_1 u \). According to Lemma 3.1, it implies that \( \Phi h_1 u = \Phi V_n^{-1} o^{-1} h_1 u < o h_n \ldots h_1 u \). Put \( z = V_n^{-1} o^{-1} h_1 u \). Then, there exists a greatest common suffix of \( z \) and \( x \) or, in other words, there exists an index \( i \) such that \( z = z k_i x(i-1, u) \), \( x = \beta h_1 x(i-1, u) \), where \( k_i \neq h_i \), and \( \alpha \) and \( \beta \) are strings of hedges which are prefixes of \( z \) and \( x \), respectively. Note that, \( k_i \) should be a unit operation and hence \( k_i x \) and \( h_1 x \) are comparable. By Proposition 2.1 and Lemma 3.1, we obtain \( \Phi h_1 u = \Phi k_i x(i-1, u) = \Phi z \) (see Fig. 4). From the last inequality, it follows that \( k_i x(i-1, u) < h_1 x(i-1, u) \), because in the opposite case we have \( h_1 x(i-1, u) \preceq k_i x(i-1, u) \), which implies that \( o h_n \ldots h_1 u \preceq \Sigma h_1 x(i-1, u) \preceq \Phi k_i x(i-1, u) = \Phi h_1 u \). This contradicts the hypothesis that \( \Phi h_1 u < ox \). Therefore, we infer that \( LH(k_i x(i-1, u)) < LH(h_1 x(i-1, u)) \) and hence, \( x \in LH(h_1 x(i-1, u)) \) and \( LH(k_i x(i-1, u)) \cap LH(x) = \emptyset \), we have \( \Phi h_1 u = \Phi k_i x(i-1, u) < LH(k_i x(i-1, u)) < \Phi h_1 x(i-1, u) \preceq ox \).

Since the proof for the case \( \Sigma h_1 u > ox \) can be obtained by duality, it completes the proof.

**Theorem 4.2.** Let \( AX = (X, G, LH, \preceq) \) be a free ComHA. Then, the set \( LH(G) \) is everywhere occupying in \( X \). Moreover, we have for all \( x, y \in X, x < y \Rightarrow (\exists u \in LH(G))(x < LH(u) < y) \), and \( LH(u) \) satisfies one of the following conditions:

(i) \( LH(u) \cap LH(x') = \emptyset \),
(ii) \( LH(u) \cap LH(y') = \emptyset \),
(iii) \( LH(u) \subseteq LH(x') \),
(iv) \( LH(u) \subseteq LH(y') \),

where \( x' \) and \( y' \) are defined by \( x' = x, y' = y \), for \( x, y \in LH(G) \), or \( x = ox' \), \( y = o'y' \), for \( x, y \in \lim(X) \) and \( o, o' \in \{\Phi, \Sigma\} \).

**Proof.** Recall that \( X = LH(G \cup C) \cup \lim(X) \) and \( LH(G \cup C) \cap \lim(X) = \emptyset \). Notice that in a free ComHA \( AX \), we always have \( k x \neq x \), for \( \forall x \in LH(G) \) and \( \forall k \in LH \). We shall prove the theorem by cases.

1. \( x, y \in LH(G) \) and suppose that \( x = h_n \ldots h_1 u, y = k_m \ldots k_1 u' \). Then, there is a possibility that \( x \in LH(c^-), y \in LH(c^+) \) and hence we should have \( x = h_n \ldots h_1 c^- \) and \( y = k_m \ldots k_1 c^+ \). Since \( LH \) contains converse hedges, there
always exists an \( h \in LH \) such that \( hx > x \), and so \( LH(hx) > x \). Because \( y \in LH(c^+) > LH(c^-) \supseteq LH(hx) \), it follows that \( y > LH(hx) > x \), which are the desired inequalities and, obviously, \( LH(hx) \) satisfies (iii), i.e. \( LH(hx) \subseteq LH(x) \), where \( u = hx \).

The second possibility is that \( x, y \in LH(c^e) \), that is that they have a greatest common suffix and, hence, we may first suppose that \( u = u' \) and \( h_1u \neq k_1u \). By hypothesis, \( x < y \) and it implies that \( h_1u < k_1u \) and so \( LH(h_1u) < LH(k_1u) \).

As above, we can choose an \( h \in LH \) such that \( hx > x \). Therefore, we have \( LH(hx) > x \), \( hx \in LH(h_1u) \), \( y \in LH(k_1u') \) and \( LH(hx) \subseteq LH(h_1u) \), which lead to the required inequalities and the validity of (iii). Secondly, suppose that \( x = h_n \ldots h_1y \). From \( x < y \), it follows that \( h_1y < x \) and \( x \in LH(h_1y) \). Since, there is an \( h \in LH \) such that \( LH(h_1y) \supseteq hx > x \), it can be seen that \( y > LH(hx) > x \) and (iii) holds. By a similar argument, we have a proof for the case \( y = k_m \ldots k_1x \) and the same result will be derived by interchanging the role of \( x \) and \( y \).

(2) \( x \in LH(G) \) and \( y = oy' \in Lim(X) \), where \( o \in \{ \Phi, \Sigma \} \). Suppose that \( x = h_n \ldots h_1u \) and \( y' = k_m \ldots k_1u' \). As above, in the cases \( u = c^- \) and \( u' = c^- \), or \( u = u' \) and \( h_1u \neq k_1u' \), the condition \( x < y \) leads to \( LH(x) < LH(y) \).

Choose an \( h \in LH \) such that \( hx > x \). Since \( LH(hx) \subseteq LH(x) \), it follows that \( y = oy' \geq \Phi y' \geq \Sigma x > LH(hx) > x \) (see Fig. 5(a)). This shows the validity of the theorem with \( u = hx \). In the case \( x = h_n \ldots h_1y' \), we have \( x \in LH(h_1y') \).

From the conditions \( x < y \) and \( LH(h_1y') \subseteq \Sigma y' \), it can be verified that \( o = \Sigma \), i.e. \( y = \Sigma y' \). Therefore, if we choose an \( h \) such that \( hx > x \), we have \( y = \Sigma y' > LH(hx) > x \).

If \( y' = k_{i_j} \ldots k_{i_1}x \), then \( x < y \) implies that \( k_{i_j}x > x \) and \( y = oy' \geq \Phi k_{i_j}x \geq x \). Assuming \( oy' = \Phi k_{i_j}x \), it follows that \( y = \Phi k_{i_j}x > x \) and hence, by (LA) and Theorem 3.1, \( k_{i_j} \) does not belong to the first graded level \( LH^1 \). So, there is a \( k' \in LH^1 \) such that \( LH(k_{i_j}x) > LH(k'x) > x \) and these lead to the inequalities \( x \leq \Phi k_{i_j}x > LH(k'x) > x \) (see Fig. 5b). Moreover, we also have \( LH(u) \subseteq LH(x) \), where \( u = k'x \) and \( x \in LH(G) \). Assuming \( oy' > \Phi k_{i_j}x \), according to Lemma 4.1 there exists an element \( u \) such that \( x \leq \Phi k_{i_j}x = \Phi z < LH(z) < oy' \) and \( LH(z) \cap LH(y') = \emptyset \), which are what is required to prove.

(3) For the case that \( x = ox' \in Lim(X) \) and \( y \in LH(G) \), where \( o \in \{ \Phi, \Sigma \} \), the proof is similar as that in the case (2).

(4) \( x = ax' \in Lim(X) \) and \( y = a'y' \in Lim(X) \).
Suppose that \( x' = h_n \ldots h_1 u \) and \( y' = k_m \ldots k_1 u' \). As in (2), when \( u = c^- \) and \( u' = c^+ \), or \( u = u' \) and \( h_1 u \neq k_1 u' \), the condition \( x < y \) implies that \( LH(h_1 u) < LH(k_1 u') \). Assuming that \( \Sigma h_1 u = \Phi k_1 u \), then there are two possibilities to be considered. Firstly, let \( x = \alpha x' = \Sigma h_1 u \) hold, and so we have \( x = \Phi k_1 u < o'y' \). According to Lemma 4.1 applied to the element \( y \), there exists a \( z_i = k_i'y'(i - 1, u') < k_i'y'(i - 1, u) \) such that \( x = \alpha = \Phi k_1 u = \Phi z_i \leq LH(z_i) = LH(k_i'y'(i - 1, u')) < o'y' = y \) and we have \( LH(z_i) \cap LH(y') = \emptyset \), i.e. the desired results are derived. Secondly, let \( x = \alpha x' < \Sigma h_1 u \) hold. Again by Lemma 4.1 applied to the element \( x \), there exists a \( z_i = h_i x'(i - 1, u) > h_i x'(i - 1, u) \) such that \( x = \alpha x' < LH(z_i) = LH(h_i x'(i - 1, u)) < \Sigma z_i = \Sigma h_1 u = \Phi k_1 u \leq \Phi y' \leq o'y' = y \) and it can be seen also that \( LH(z_i) \cap LH(y') = \emptyset \). These are just what we require.

Now, assume that \( \Sigma h_1 u \neq \Phi k_1 u \). Analogously as above, the condition \( x < y \) leads to \( h_1 u < k_1 u \) and, hence, \( LH(h_1 u) < LH(k_1 u) \). It implies that \( \alpha x' \leq \Sigma h_1 u < \Phi k_1 u \leq o'y' \). In the case that the hedges \( h_1, k_1 \) do not belong to the same \( LH^c \), then they must be converse and, hence, we have \( h_1 u < k_1 u \). If they belong correspondingly to the first graded semantic level of \( LH^+ \) and of \( LH^- \), then, by Theorem 3.1, it implies that \( \Sigma h_1 u = \Phi k_1 u = u \), which contradicts the assumption. Therefore, at least one of the hedges \( h_1, k_1 \) does not belong to the first graded level, say \( h_1 \). Then, there is an \( h' \) in the first graded level such that \( LH(h_1 u) < LH(h' u) < u < LH(k_1 u) \). In the case that \( h_1, k_1 \) belong together to any one of the sets \( LH^+ \) and \( LH^- \), by Theorem 3.1, they cannot belong to the same graded semantic level of \( LH^c \) and moreover they do not belong to consecutive graded levels, since otherwise by Axiom (L5) we have \( \Sigma h_1 u = \Phi k_1 u \), again a contradiction. So, there exists an \( h' \) such that \( LH(h_1 u) < LH(h' u) < LH(k_1 u) \) (Fig. 6a). From this, it can be verified that \( x = \alpha x' \leq \Sigma h_1 u < LH(h' u) < \Phi k_1 u \leq o'y' = y \), which are the required inequalities, and, moreover, we have \( LH(h' u) \cap LH(y') = \emptyset \).

Now, we consider the case \( x' \) and \( y' \) can be written in the form \( x' = h_n \ldots h_1 y' \). Since \( x' \in LH(h_1 y') \), the inequality \( x < y \) implies that \( h_1 y' < y' \) and, hence, \( LH(h_1 y') < y' \). Evidently, \( x' \in LH(y') \) and, so, we infer that \( \Phi y' \leq x' = \alpha i' \leq \Sigma h_1 y' \leq y' \leq \Phi k_1 u \leq \Phi y' \). Since \( x < y \), we should have \( o'y = \Sigma \). Choose an \( h' \) such that \( y' < h' y' \). It can be verified that \( x = \alpha x' < y' < LH(h' y') \leq \Sigma y' = y \) and, in this case, we have \( LH(h' y') \subseteq LH(y') \) (Fig. 6b).

For the remaining case \( y' = h_n \ldots h_1 x' \), by a similar argument we obtain \( o = \Phi \) and \( x = \Phi x' < x' < LH(h' x') \leq \Sigma x' \leq \Phi y' < y' < \Sigma y' \). Therefore, we have \( x < LH(h' x') \) and \( LH(h' x') \subseteq LH(x') \) and the theorem is completely proved.

5. Conclusions

Hedge algebras establish an algebraic approach to the semantics of term-domains of linguistic variables. They were first examined in [7,17] and then extended hedge algebras were studied in [18], which are a completion of hedge algebras, and refined hedge algebras were investigated in [12,14,15]. Now, in this paper a completion of the refined hedge algebras has been examined to establish a basis for studying the lattice property of refined hedge algebras and investigating fuzziness measure and quantification problem of ComHAs.

The algebraic approach is completely different from the fuzzy sets approach, maybe called an analytic one, since the meaning of each vague term in the former approach is represented not by a single fuzzy set but by an order-based substructure related to this term, which is a mathematical object much richer than fuzzy sets themselves. That is
expressing the meaning of terms using hedge algebras can catch more information involved in a term. Because of this reason, many properties of terms’ meanings formulated in terms of semantically ordering relation have been discovered to clarify the inherent algebraic structure of terms-domains. We have shown that there exists a closed relationship between this structure of terms-domains, fuzziness of terms and a topology on the corresponding terms domains. We hope that hedge algebras may provide a good mathematical foundation for studying several problems of fuzzy logics and human reasoning [3,4,8,9].

Now, we close the paper by formulating an open problem: Is a ComHA \( AX = (X, G, C, LH, \leq) \) a distributive lattice?

The problem seems not to be easy and, hence, we encounter some difficulties in solving it. Because of this reason, this open problem may also be formulated as follows: whether the axiom system given in Definition 2.1 is sufficient to prove the distributive lattice property of ComHAs?

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