Fuzziness measure on complete hedge algebras and quantifying semantics of terms in linear hedge algebras

Nguyen Cat Ho\textsuperscript{a,}\textsuperscript{*}, Nguyen Van Long\textsuperscript{b}

\textsuperscript{a}Institute of Information Technology, Vietnam Academy of Sciences and Technologies, 18 Hoang Quoc Viet Str. Cau Giay Dist. Hanoi 10 000, Vietnam
\textsuperscript{b}Information Technology Section, University of Transport and Communications, Lang Thuong, Dong Da Dist. Hanoi, Vietnam

Received 30 August 2004; received in revised form 28 September 2006; accepted 18 October 2006

Abstract

In the paper, we shall examine the fuzziness measure (FM) of terms or of complete and linear hedge algebras of a linguistic variable. The notion of semantically quantifying mappings (SQMs) previously examined by the first author will be redefined more generally and a closed relation between the FM of linguistic terms and a family of SQMs with the parameters to be the FM of primary terms and linguistic hedges will be established. A semantics-based topology of hedge algebras and a closed and interesting relation between this topology, the FM and the above family of SQMs will be discovered and examined. An applicability of the FM and SQMs will be shown by an examination of some application examples.

© 2006 Elsevier B.V. All rights reserved.

Keywords: Linear hedge algebras; Fuzziness measure of linguistic terms; Semantically quantifying mappings; Interpolative reasoning method; Membership functions; Fuzziness of linguistic terms

1. Introduction

There are some reasons pushing us to investigate a class of complete and linear hedge algebras, a proper class of the complete hedge algebras investigated in [6] that are considered as an abstract algebra \(A_{\mathcal{X}} = (X, G, C, LH, \Sigma, \Phi, \leq)\), where \(G = \{c^-, c^+\}\) is the set of primary generators, in which \(c^-\) and \(c^+\) are, respectively, the negative primary term and the positive one of a linguistic variable \(\mathcal{X}\) and \(C = \{0, W, 1\}\), a set of constants, which are distinguished elements in \(X\); \(LH = LH^+ \cup LH^-\), where \(LH^\varepsilon\) with \(\varepsilon \in \{+, -\}\) is the distributive lattice of hedges generated from the set of original hedges \(H^\varepsilon\) and is considered as the set of unary operations; \(\leq\) is a semantically ordering relation on \(X\) [5,7–9]. The first reason is that in fuzzy control, real domains of physical variables are usually linear and hence linguistic domains of the corresponding variables are necessarily assumed linearly ordered. In other words, the order relation on the former domains must induce a linear order relation on the latter ones. The second one, as it has been discussed in [6], is that a mathematical formalism is essential to investigate the fuzziness structure of linguistic domains and to

\* This work is supported in part by the National Program on Basic Research of Natural Sciences of Vietnam.

* Corresponding author. Fax: +84 4 7715529.

E-mail addresses: ncatho@hn.vnn.vn (N.C. Ho), utc@hn.vnn.vn (N.V. Long).

0165-0114/$ - see front matter © 2006 Elsevier B.V. All rights reserved.

doi:10.1016/j.fss.2006.10.023
introduce a method of quantifying these domains regarded as hedge algebras. It is worth emphasizing that the fuzziness measure (FM) of uncertainty data is a concept that in general is not easy to define, because of its high abstraction level. In the fuzzy sets framework, there are several approaches to define a FM of fuzzy sets, since it depends on which kind of uncertainty one deals with and which idea one takes to interpret it \([1,3,16,17]\). For example, one can interpret the FM of a fuzzy set as the sum of the fuzziness degree of membership values or as a distance between the fuzzy set and its complement. Although fuzzy sets represent a meaning of linguistic terms, there are no strict constraints on the dependency between fuzzy sets and linguistic terms regarded as their labels. Therefore, terms are mere labels of fuzzy sets and since fuzzy sets and their labels are completely distinct mathematical objects, the FM which is defined based only on fuzzy sets independently from their labels may reflect the fuzziness of terms not properly. That is, that such definitions of the FMs do not originate in the meaning of terms themselves, but in the fuzzy sets only. This may cause some difficulty, because each linguistic term may be expressed, for example, by several different fuzzy sets, depending on the user’s viewpoint, while the notion of fuzziness of terms in nature stems only from the semantics of vague terms.

Hedge algebras provide another viewpoint and formalism to define the concept of fuzziness. Since hedge algebras model directly the relative meaning of terms based on a semantically ordering relation on terms-domains, the structure of hedge algebras, as the title of \([4]\) says, involves already certain fuzziness features of terms of linguistic variables. For example, it was shown in \([6]\) that a semantics-based topology induced by the meaning of terms in hedge algebras can be used to model the fuzziness of linguistic terms. More concretely, we may use the family \(\vartheta^* = \vartheta \cup \{LH(x): x \in Lim(X)\} \cup \{1, W, 0\}\), where \(\vartheta = \{LH(x): x \in LH(G)\}\), to define this concept since it satisfies some natural requirements on the fuzziness of terms. This concept is suggested from the fact that a term \(x\) is fuzzy if a hedge applying to \(x\) still intensifies its meaning properly. It leads also to a suggestive method for defining the FM of vague terms.

In order to quantify linear hedge algebras, a notion of semantically quantifying mappings (SQMs) will be defined under natural and light conditions. It will be shown that a wide class of SQMs, which are defined in an elegant mathematical way using parameters to be the FM of primary terms and hedges, can also be characterized by light and natural conditions.

To show an applicability of the FM and SQMs, we shall re-examine a fuzzy multiple conditional problem and show that an interpolative reasoning method based on suitably determined SQMs can produce better results in comparison with the method based on fuzzy sets examined in \([2]\). It can be seen that the FM of primary terms and hedges which are considered as the parameters of SQMs will make the method also flexible to adapt a given application. Note that the classical fuzzy multiple conditional reasoning (FMCR) methods depend in general on many sophisticated factors, which make these methods becoming a black box, i.e. it is very difficult to recognize their behaviour (see, e.g. \([2]\)) and, therefore, one may lose intuition. The “fuzzy” interpolative reasoning methods considered, for example, in \([12,13]\) seem to be suggestive of its behaviour, but they are restricted by some conditions, e.g. the rule-based systems must be assumed sparse and, in addition, their computation is still sophisticated.

The construction of membership functions of fuzzy sets for a given application is a sophisticated problem, since it depends on several objective as well as subjective factors. And in the fuzzy sets framework, according to our knowledge, there are no constraints imposing on this construction. In the recent paper, an example for solving this problem will be given to show that the FM of terms will also be applicable to certain applications.

The paper is organized as follows: in Section 2, a notion of linear and complete hedge algebras (ComHA) will be introduced and investigated. These algebras provide a basic mathematical structure to investigate a quantitative semantics of terms of a linguistic variable. A notion of fuzziness of the elements of hedge algebras has been introduced in \([6]\). Based on this, in Section 3, we shall introduce and examine a notion of FM of the elements in ComHA. Restricted to the class of linear and ComHA, in Section 4, SQMs will be examined and it is proved that a family of SQMs can be characterized by an analytic expression of the FM of hedges and primary terms. An applicability of the FM and SQMs will be presented in Section 5.

2. Linear and complete hedge algebras

The purpose of an algebraic approach to model the semantics of terms is to interpret each terms-set of a linguistic variable as an algebra whose ordering relation is induced by the natural qualitative meaning of linguistic terms and, therefore, is called a semantically ordering relation. Because real domains of physical variables in general are linear, it is necessary to assume that terms-sets of the corresponding linguistic variables are linear as well. On the other hand,
it is well known that in several application areas of fuzzy control, one needs to use a quantitative meaning of terms rather than a qualitative one. It requires an investigation of a quantifying method for hedge algebras, which will be more easily studied when we restrict ourselves to an examination of the linear hedge algebras, since their structure is much simpler than that of the ComHA. So, the main aim of this section is to introduce and study linear ComHA.

First of all, we recall some notions, notations and some related results. For more details, the readers can refer to [6]. In the algebraic approach to the structure of terms-domain of linguistic variables, each terms-domain can be interpreted as an algebra \( A(X) = (X, G, C, LH, \leq) \), where \( X \) is a term-domain of a linguistic variable \( X \); \( G = [c^-, c^+] \) is a set of generators called the negative primary term and the positive one of \( X \); \( C = \{\emptyset, 0, 1\} \) is a set of constants, which are interpreted, respectively, as the neutral, the least and the greatest element in \( X \); \( LH = LH^+ \cup LH^- \) is the set of one-argument operations, where \( LH^\varepsilon, \varepsilon \in \{+, -\} \), is the distributive lattice of hedges generated freely from the corresponding set of original hedges \( H^\varepsilon \), and \( \leq \) is a semantically ordering relation on \( X \), which is induced from the natural qualitative meaning of terms [4,8–11]. Note that the greatest elements in \( LH^+ \) and \( LH^- \) are denoted, respectively, by \( V \) and \( L \) and are called unit operations. In [6] a notion of ComHA was introduced and examined, by this each ComHA written formally as \( A(X) = (X, G, C, LH_e, \leq) \), where \( LH_e = LH \cup \{\Sigma, \Phi\} \) and \( \Sigma \) and \( \Phi \) are two additional operations, can be considered as an extension of a given hedge algebra \( A(X) \). An axiomatization for these algebras has been established to capture the intended semantics of two additional operations which says that \( \Sigma \) and \( \Phi \) are, respectively, the supremum and the infimum of the set \( LH(x) \) generated from \( x \) by using hedges in \( LH \). In particular, we have \( \Sigma = 1, \Phi = 0, \Sigma^- = \Phi^+ = W \). In the nature, a ComHA can be understood as obtained by completing an original hedge algebra \( A(X) \) with certain elements so that the intended additional operations \( \Sigma \) and \( \Phi \) will be defined on the whole set \( X \).

Put \( Lim(X) = X \setminus LH(G) \).

Given a poset (partially ordered set) \( X \), for \( U, V \subseteq X \), we shall write \( U \leq V \), if \( (\forall x \in U)(\forall y \in V)\{x \leq y\} \). Let \( x \in X \) have a representation of the form \( x = h_n \ldots h_1 u \), where \( u \in X \). Then, for short we denote by \( x(i, u) \), \( 1 \leq i \leq n \), the suffix \( h_{i-1} \ldots h_1 u \) of \( x \) with a convention that for \( i = 1 \), \( h_0 \) is the identity \( I \) and so \( x(1, u) = u \). Recall a formal convention, only used for reducing some formulations, which says that when \( i \) occurs explicitly in a representation of \( x \), any string of hedges in \( LH_e \) applying to \( Iu \) has no effect, i.e. \( h_n \ldots h_1 u = Iu \). A ComHA \( A(X) = (X, G, C, LH_e, \leq) \) is said to be free (or freely generated) provided for all \( x \in LH(G) \), we always have \( hx \neq x \).

Note that for the set \( LH^e, LH^0_e \) denotes the lattice generated freely from \( i \)-th graded semantic level \( H^e_i \) of \( H^e \), where \( \varepsilon \in \{−, +\} \). Recall that the elements in \( H^e_i \) are incomparable.

For convenience, we recall here some results in [6] for the reference in the sequel:

**Axiom (L5) (Ho [6].)** Let \( A(X) = (X, G, C, LH, \Sigma, \Phi, \leq) \) be a ComHA. Then, for all \( h \in LH^e_i \) and \( k \in LH^e_{i-1} \), if \( \Phi x, \Sigma x \in Lim(X) \) (or \( \Phi x, \Sigma x \notin LH(G) \)), then
\[
hx \leq kx \text{ implies that } \Sigma hx = \Phi kx \text{ and } hx \geq kx \text{ implies that } \Phi hx = \Sigma kx.
\]

This axiom says that if \( h \) and \( k \) are consecutive then there are no terms in between \( \Phi hx \) and \( \Sigma kx \). Note that Axiom (L5) is still valid for \( i = 1 \) and in this case, according to the above convention, the statement of Axiom (L5) becomes “\( \ldots \) then \( hx \leq x \) implies that \( \Sigma hx = \Phi lx \) \( x = x \), \( \Sigma x \geq x \) implies that \( \Phi hx = \Sigma lx \) \( x = x \)”.

**Theorem 2.1 (See Ho [6]).** Let \( A(X) = (X, G, C, LH, \Sigma, \Phi, \leq) \) be a ComHA. Then, for all \( y \in LH(x) \) and \( x \in X \), we have
(i) \( \Sigma y \leq \Sigma x \) and \( \Phi y \geq \Phi x \),
(ii) \( \Phi x \leq LH(x) \) \( x \leq \Sigma x \).

**Lemma 2.1 (see Ho [6]).** For all \( x \in LH(G) \), the following hold, where \( UOS = \{V, L\} \) is the set of unit operations:
\[
\Sigma x = \sup\{V^n o^+ x : o^+ \in UOS, o^+ x \geq x, n = 1, 2, \ldots\} = \Sigma V^n o^+ x = \Sigma o^+ x
\]
\[
\Phi x = \inf\{V^n o^- x : o^- \in UOS, o^- x \leq x, n = 1, 2, \ldots\} = \Phi V^n o^- x = \Phi o^- x.
\]

As a consequence, we have \( \Phi x = \inf LH(x) \) and \( \Sigma x = \sup LH(x) \).

Note that \( V \) is positive w.r.t. both unit operations \( V \) and \( L \) and the sequence \( \{V^n o^+ x : o^+ \in UOS, o^+ x \geq x, n = 1, 2, \ldots\} \) is monotonically increasing, while \( \{V^n o^- x : o^- \in UOS, o^- x \leq x, n = 1, 2, \ldots\} \) is monotonically decreasing.
Now, we introduce a definition of the complete and linear hedge algebras.

**Definition 2.1.** A ComHA \( \mathcal{A}X = (X, G, LH, C, \Sigma, \Phi, \leq) \) is said to be linear if the sets \( H^- = \{h_{-1}, \ldots, h_{-q}\} \) and \( H^+ = \{h_1, \ldots, h_p\} \) are linear. Remember that \( H = H^- \cup H^+, h_0 = I \) and it is assumed always in this case that \( h_{-1} < h_{-2} < \cdots < h_{-q}; h_1 < \cdots < h_p. \)

So, in a linear hedge algebra, for every \( x \in X, \) we have either \( h_{-q}x \leq h_{-q+1}x \leq \cdots \leq h_{-1}x \leq \cdots \leq h_px \) or \( h_{-q}x \leq \cdots \leq h_{-1}x \leq h_{-2}x \leq \cdots \leq h_{-q}x. \)

Note that, from the linearity of \( H^- \) and \( H^+, \) it follows that \( LH^+ = H^+ \) and \( LH^- = H^-. \)

We have the following,

**Lemma 2.2 (Ho and Wechler [10]).** Let \( \mathcal{A}X = (X, G, C, H, \leq) \) be a refined, but not ComHA. Then, if \( G \) and \( H^- \), \( H^+ \) are linear, then so is the set \( X = H(G). \)

For the linear ComHAs, a similar statement also holds:

**Theorem 2.2.** For any linear ComHA \( \mathcal{A}X = (X, G, C, LH, \Sigma, \Phi, \leq), \) the underlying set \( X \) is linearly ordered.

**Proof.** It is known in [6] that \( \mathcal{X} = H(G) \cup \text{Lim}(X). \) Since, by Lemma 2.2, \( H(G) \) is a linearly ordered set, it is sufficient to show that any two elements \( x \) and \( y, \) at least one of which is in \( \text{Lim}(X), \) are comparable.

(1) Suppose that exactly one of \( x \) and \( y \) is in \( \text{Lim}(X), \) say \( y, \) and they have the following representation: \( x = h_n \ldots h_1 u, \ y = oy', \) where \( y' = k_m \ldots k_1 u' \) with \( u, u' \in X \) and \( o \in \{\Phi, \Sigma\}. \) By Lemma 2.2, \( H(G) \) is linear and hence \( x \) and \( y' \) must be comparable. Without loss of generality we assume that \( x < y' \) (in the opposite case, the proof is obtained by “duality”).

In the case that \( x \) and \( y' \) are generated from distinct primary terms, then we must have \( x \in H(c^-) \) and \( y' \in H(c^+). \) It follows that \( y = oy' \geq \Phi c^+ \geq \Sigma c^- \geq x, \) i.e. \( x, y \) are comparable. Now, assume that they are generated from the same primary term and let \( u \) be their longest common suffix. Then, there are three cases:

Case 1: \( x = h_n \ldots h_1 u, \ y' = k_m \ldots k_1 u, \) where \( h_1 \) and \( k_1 \) are different from the identity \( I \) and \( h_1 \neq k_1. \) Since \( x < y', \) it follows from the comparison criteria of the refined hedge algebras that \( h_1 u < k_1 u, \) which implies by Axiom (L5) that \( y = oy' \geq \Phi k_1 u \geq \Sigma h_1 u \geq H(h_1 u). \) This yields \( y > x. \)

Case 2: \( u = y' \) and \( x = h_n \ldots h_1 y'. \) The hypothesis \( y < y' \) leads to the fact that \( h_1 y' < y'. \) Hence, it follows from Axiom (L5) that \( \Sigma y' \geq \Phi \geq h(h_1 y') \geq y'. \) Since \( x \in H(h_1 y'), \) the latter imply that the elements \( y = oy' \) and \( x \) are always comparable, where \( o \) independently is either \( \Phi \) or \( \Sigma. \)

Case 3: \( u = x \) and \( y = k_m \ldots k_1 x. \) From \( x < y' \) it follows that \( k_1 x > x \) and, similarly as above, we get \( H(k_1 x) > x. \)

Since \( H(k_1 x) \supset H(y'), \) it implies that \( H(y') > x \) and, hence, \( y = oy' \geq x. \)

(2) Suppose that both \( x \) and \( y \) belong to \( \text{Lim}(X). \) Hence it can be assumed that \( x = ox', \ y = oy', \) where \( o, o' \in \{\Phi, \Sigma\}, \) \( x' = h_n \ldots h_1 u \) and \( y' = k_m \ldots k_1 u'. \) By Lemma 2.2, \( x' \) and \( y' \) are comparable, say \( x' < y'. \) Similarly as above, if \( u \) and \( u' \) are generated from different primary terms by using hedges, then it follows from \( x < y' \) that \( x' \in H(c^-) \) and \( y' \in H(c^+). \) So, we infer that \( y = oy' \geq \Phi c^+ \geq \Sigma c^- \geq ox' = x, \) i.e. \( x \) and \( y \) are comparable. In the case that both \( u, u' \in H(c), \) for some \( c \in \{c^-, c^+\}, \) we can suppose that \( u = u', x' = h_n \ldots h_1 u, \ y' = k_m \ldots k_1 u \) and \( h_1 u \neq k_1 u. \) Then, as above there are three cases for \( x' \) and \( y' \) and, by a similar argument, it can be verified that \( x \) and \( y \) are comparable. The theorem is completely proved. \( \square \)

In the case of the linear ComHAs, we have a version of Theorem 4.2 in [6] as follows, where remember that a subset \( U \) of \( X \) is said to be **everywhere occupying** in \( X \) if for every interval \( (x, y) \) of \( X \) (i.e. \( x < y \) and \( (x, y) = \{u \in X : x < u < y\} \)), there exists an open set \( H(z) \) of the topological space \( (X, \mathcal{A}) \) such that \( H(z) \subseteq U \) and \( x < H(z) < y. \)

**Theorem 2.3.** Let \( \mathcal{A}X = (X, G, C, H, \Phi, \Sigma, \leq) \) be a free ComHA. Then, the set \( H(G) \) is everywhere occupying in \( X. \) Moreover, we have for all \( x, y \in X, \ x < y \Rightarrow (\exists u \in H(G)) \{x < H(u) < y\} \) and \( H(u) \) satisfies one of the following conditions:

(i) \( H(u) \cap H(x') = \emptyset, \) (ii) \( H(u) \cap H(y') = \emptyset, \)

(iii) \( H(u) \subseteq H(x'), \) (iv) \( H(u) \subseteq H(y').\)
where $x'$ and $y'$ are defined so that, for $x, y \in H(G)$ we have $x' = x$ and $y' = y$, and for $x, y \in \text{Lim}(X)$ we have $x = \alpha x'$ and $y = \alpha' y'$.

In [11], a notion of symmetrical extended hedge algebras was introduced. It says that every element $x$ in $X$ has a unique contradictory element in $X$, denoted by $x^-$, which is defined as follows: for $x = h_n \ldots h_1 c, x^- = h_n \ldots h_1 c'$, where $c, c' \in G$ and $c \neq c'$. We define a complement operation and an implication operation as follows: $\neg x = x^-$ and $x \Rightarrow y = \neg x \cup y$, for any $x$ and $y$ of $AX$. In the case that ComHAs are linear, they are extended hedge algebras studied in [11], because they are linearly ordered sets and their corresponding additional operations have the same semantics, since they are, respectively, the supremum and the minimum of the set $H(x)$. So, in the case of the linear ComHAs, it follows from [11] that

Theorem 2.4. Let $AX = (X, G, C, H, \Phi, \Sigma, \leq)$ be a symmetrical linear ComHA. Then,

(i) $\neg (hx) = h \neg x$, for every $h \in LH$ and $x \in X$.

(ii) $\neg (\neg x) = x$, for all $x \in X$.

(iii) $\neg (x \cup y) = \neg x \cap \neg y$ and $\neg (x \cap y) = \neg x \cup \neg y$, for all $x, y \in X$.

(iv) $x \cap \neg x \leq y \cup \neg y$, for all $x, y \in X$.

(v) $x \cap \neg x \leq W \leq x \cup \neg x$, for all $x \in X$.

(vi) $\neg 0 = 0, \neg 1 = 1$ and $\neg W = W$.

(vii) $x > y$ iff $\neg x < \neg y$, for all $x, y \in X$.

(viii) $x \Rightarrow y = \neg y \Rightarrow \neg x$.

(ix) $x \Rightarrow (y \Rightarrow z) = y \Rightarrow (x \Rightarrow z)$.

(x) $x \Rightarrow y \geq x' \Rightarrow y'$ if $x \leq x'$ and/or $y \geq y'$.

(xi) $x \Rightarrow y = \neg 1$ iff either $x = 0$ or $y = 1$.

(xii) $1 \Rightarrow x = x$ and $x \Rightarrow 1 = 1; 0 \Rightarrow x = 1$ and $x \Rightarrow 0 = \neg x$.

(xiii) $x \Rightarrow y \geq W$ iff either $x \leq W$ or $y \geq W$, and $x \Rightarrow y \leq W$ if $x \geq W$ and $y \leq W$.

Theorem 2.4 says that the symmetrical linear ComHAs have rich enough algebraic-logical structure that can be taken as an algebraic foundation for certain fuzzy logics, called linguistic-valued logics, since their elements can be regarded as linguistic terms.

3. A measure of fuzziness of linear ComHAs

Let us consider a linear ComHA $AX = (X, G, C, H_e, \leq)$, where $H_e = H \cup \{\Phi, \Sigma\}$. Assuming that $H(G) = X$, we have $H_e(G) = X = X \cup \text{Lim}(X)$. As investigated in [6], the family $\vartheta = \{H(x) : x \in H(G)\}$ can be considered as a model of the fuzziness of linguistic terms, since the family $\vartheta^* = \vartheta \cup \{H(x) : x \in \text{Lim}(X)\}$ satisfies the following properties:

1. For $x \in G^* = \text{Lim}(X) \cup \{1, W, 0\}$, we have $H(x) = \{x\}$.
2. $H(hx) \subseteq H(x)$, for all hedge $h$ and all $x \in X$.
3. $H(hx) \cap H(kx) = \emptyset$, for $h, k \in H$ and $h \neq k$.
4. $H(x) = \bigcup \{H(hx) : h \in H \cup I\}$.

We recall the intuitive idea mentioned in [6] on the notion of fuzziness of terms represented by the family $\vartheta$. Intuitively, a term $x$ is non-vague if any hedges applying to $x$ cannot generate a proper new meaning and therefore, the vagueness of a term $x$ can be understood that the meaning of $x$ is still changed properly by using hedges. Consequently, the set of terms generated from $x$ by using hedges models a fuzziness of $x$, and so $H(x)$ is a fuzziness image of $x$. So, the condition (1) means that the hedges cannot change the crisp concepts, i.e. the set $H(x)$ consists only of the element $x$. Condition (2) models the following fact: when a hedge $h$ applies to a term $x$ it makes this terms less uncertain or more exact. Then, this condition says that the more exact a term is, the less vague it is. This seems to correspond to our intuition. For condition (3), we note that two terms mentioned in (3) are independent, i.e. the one element cannot be generated from the other by means of hedges. Then, (3) says that the fuzziness of these terms are independent events. Condition (4) can be viewed in term of the fuzziness as follows: the fuzziness of a term $x$ is defined exactly by the fuzziness of those terms, which are more exact than $x$, i.e. the fuzziness of $x$ can be characterized by the hedges only.
This observation suggests us to use the “size” of the set \( H(x) \) to measure the fuzziness degree of \( x \). We shall realize this idea by introducing a system of axioms of the FM.

In the framework of fuzzy sets, each defuzzification method defines a mapping from \( X \) into a real interval \([a, b]\) or \([0,1]\), for normalization. This mapping can be considered as an SQM, which will be defined and investigated in details in the next section.

Now, we take these mappings in mind to define a notion of the FM of terms as follows. Let us consider a mapping \( f \) from the terms-set \( X \) into the unit interval \([0,1]\), for normalization. This observation suggests us to use the “size” of the set \( H(x) \), for \( x \in X \), can be characterized by the diameter of the image \( f(H(x)) \) of \( H(x) \) contained in the interval \([0,1]\). We will interpret this diameter as the FM of term \( x \). Based on this interpretation, we may adopt the following definition which is considered as an axiomatization of the FM of linguistic terms (see Fig. 1).

**Definition 3.1.** Let \( \mathcal{X} = (X, G, C, H_e, \preceq) \) be a linear ComHA. An \( f_m : X \rightarrow [0,1] \) is said to be an FM of terms in \( X \) provided:

1. (fm1) \( f_m \) is complete, i.e. \( f_m(c^-) + f_m(c^+) = 1 \) and \( \sum_{h \in H} f_m(hu) = f_m(u) \), for all \( u \in X \);
2. (fm2) \( f_m(x) = 0 \), for all \( x \) such that \( H(x) = \{ \} \). Especially, \( f_m(\emptyset) = f_m(W) = f_m(\text{True}) = 0 \);
3. (fm3) \( \forall x, y \in X, \forall h \in H, f_m(hx) = f_m(hy) \), that is the proportion does not depend on particular elements and, hence, it is called the FM of the hedge \( h \) and is denoted by \( \mu(h) \).

Condition (fm1) means that the primary terms and hedges under consideration are complete for modeling the semantics of real domains of a physical variable. In other words, the meaning of the terms under consideration is sufficient to cover the whole real domain of reference. That is, there are no more other primary terms and hedges, except the primary terms and hedges under consideration. (fm2) is intuitively evident. (fm3) also seems to be acceptable: applying a hedge \( h \) to particular vague concepts, the relative modification degree of \( h \) is the same, i.e. this proportion does not depend on particular terms it applies to.

For the sake of reference, we summarize some properties of the FM of linguistic terms and hedges in the following proposition:

**Proposition 3.1.** Let \( f_m \) and \( \mu \) be defined as in Definition 3.1. Then,

1. \( f_m(c^-) + f_m(c^+) = 1 \);
2. \( \sum_{h \in H} \mu(h) = 1 \);
3. \( \sum_{x \in X_k} f_m(x) = 1 \), where \( X_k \) is the set of all terms in \( X = H(G) \) of length \( k \);
4. \( f_m(hx) = \mu(h) f_m(x) \) and, for all \( x \in \text{Lim}(X) \), \( f_m(x) = 0 \);
5. Given \( f_m(c^-), f_m(c^+) \) and \( \mu(h) \), \( \forall h \in H \), then for \( x = h_n \ldots h_1 c^e \), \( e \in \{-, +\} \), one can easily compute \( f_m(x) \) as

\[
f_m(h_n \ldots h_1 c^e) = \mu(h_n) \mu(h_{n-1}) \ldots \mu(h_1) f_m(c^e).
\]

**Proof.** The proposition follows directly from Definition 3.1 and by a proof based on the induction on the length of terms. □
It is worth emphasizing that the notion of fuzziness of vague concepts and, especially, of hedges are very difficult to define, in general. However, in the framework of the theory of hedge algebras, as seen above, the problem seems to become more obvious and it allows to model the fuzziness of terms immediately and to define the FM of terms in a mathematically elegant way based only on the meaning of terms. To illustrate this, we consider some examples.

**Example 3.1.** Let us consider a linear ComHA $\mathcal{AX} = (X,G,H,\Sigma,\Phi,\leq)$ of the linguistic variable $AGE$ with \( H = \{V,M,A,P,L\} \), where $V$, $M$, $A$ and $L$ stand for Very, More, Approximately and Little, respectively, and $G = \{\text{young}, \text{old}\}$. Now, we want to construct an FM of elements of $AX$ of the length not greater than $p$, i.e. the number of alphabets in the string $x$ is not greater than $p$. By Proposition 3.1, we can define $f_m$ and $\mu$, based on the following steps:

*Step 1:* To determine these parameters using the information (expert knowledge or experiment data) in the context of an application.

For example, in general we may consider, based on experts opinion or investigation data, a person of age from 0 to 35 as being young and of age from 36 to 80 as being old. Then, these parameters are defined as follows:

$$f_m(\text{young}) = 35 : 80 = 0.4375 \quad \text{and} \quad f_m(\text{old}) = 1 - f_m(\text{young}) = 0.5625$$

The FMs of hedges are given by $\mu(V) = 0.15$, $\mu(M) = 0.35$, $\mu(A) = 0.26$ and $\mu(L) = 0.24$, where we can put a constraint on hedges formulated that $V$ is less vague than $M$, i.e. $\mu(V) < \mu(M)$ and, analogously, that $\mu(L) < \mu(A)$.

It can be checked that these parameters satisfy (1) and (2) of Proposition 3.1.

*Step 2:* (Loop step) For $i = 1, \ldots, p - 1$, every $x$ of the length $i$ and every $h$, $f_m(hx)$ is computed by the formula $f_m(hx) = \mu(h) f_m(x)$.

For example, with $p = 2$, the FMs of terms of length $\leq 2$, based on Proposition 3.1, are given in the following table:

<table>
<thead>
<tr>
<th>Vyoung</th>
<th>Myoung</th>
<th>young</th>
<th>Ayoung</th>
<th>Lyoung</th>
<th>W</th>
<th>Lold</th>
<th>Aold</th>
<th>old</th>
<th>Mold</th>
<th>Vold</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.065625</td>
<td>0.153125</td>
<td>0.4375</td>
<td>0.11375</td>
<td>0.105</td>
<td>0.00</td>
<td>0.135</td>
<td>0.14625</td>
<td>0.5625</td>
<td>0.196875</td>
<td>0.084375</td>
</tr>
</tbody>
</table>

**Example 3.2.** Let us consider again the ComHA of the linguistic variable of $AGE$ given in Example 3.1. However, we assume here that the variable $AGE$ describes the age of the group of persons who have the academic title Professor or Associate Professor. Then, we can consider such a person of age from 0 to 40 as young and of age from 41 to 80 as old. In this case, we have $f_m(\text{old}) = 0.5$, $f_m(\text{young}) = 0.5$, and we assume that the other parameters are the same as those given above. Then, for $p = 2$, the FMs of terms of length $\leq 2$ are now given in the following table:

<table>
<thead>
<tr>
<th>Vyoung</th>
<th>Myoung</th>
<th>young</th>
<th>Ayoung</th>
<th>Lyoung</th>
<th>W</th>
<th>Lold</th>
<th>Aold</th>
<th>old</th>
<th>Mold</th>
<th>Vold</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.075</td>
<td>0.175</td>
<td>0.5</td>
<td>0.13</td>
<td>0.12</td>
<td>0.0</td>
<td>0.12</td>
<td>0.13</td>
<td>0.5</td>
<td>0.175</td>
<td>0.075</td>
</tr>
</tbody>
</table>

In the framework of the fuzzy sets theory, it is known that there are almost no constraints made on the determination of membership functions. Therefore, we are completely free to construct fuzzy sets to adapt an application. However, the meaning of terms represented by fuzzy sets must be related closely to the FM of terms. So, the conditions made on FM of terms, which models a semantic relation between vague concepts, can be taken as constraints on a construction of fuzzy sets for a given application. This limits the possible constructions. It is worth emphasizing that usually in the literatures, this kind of constraints are not taken into account and hence the authors suppose that these constraints may contribute to a simplification of some methods in the fuzzy sets approach.

### 4. Semantically quantifying mappings of linear ComHAs

Normally, the meaning of linguistic terms is qualitative rather than quantitative. However, in many applications, for example, of the control theory, one needs to use quantitative meaning of terms of physical variables. In fuzzy control, it is well known that the inputs of a fuzzy control method are interpreted as fuzzy sets and, therefore, so are its outputs. Then, it is necessary to transform these fuzzy output data into real data by a certain defuzzification method. In general, we are almost completely free to determine such methods, as well.
To deal with a hedge algebras-based approach to the examination of transforming the terms of a linguistic variable into real values, we introduce a notion of SQMs which assign a real value to each vague term, where the word “semantically” means that they will be defined based on the semantic structure of terms-domains of the linguistic variable or, more concretely, on the FM of terms and hedges. There are a number of parameters used in this method and, therefore, we also have a free domain to define these mappings to adapt applications, but there are also a number of constraints imposed on this domain to limit such possible mappings.

In this section, we restrict our research to the free linear and ComHA: an $A\mathcal{X} = (X, G, C, H, \Sigma, \Phi, \leq)$ is said to be free if for $\forall x \in H(G), \forall h \in H$, we always have $hx \neq x$ (recall that $H(G) \cup \text{Lim}(X) = X$). Note that the equality $LH = H$ always holds. Suppose that the elements of the set $H = H^- \cup H^+$ are naming as follows: $H^- = \{h_1, \ldots, h_q\}$, with $h_1 < \cdots < h_q$, and $H^+ = \{h_1, \ldots, h_p\}$, with $h_1 < \cdots < h_p$. For convenience, it is assumed that $h_0 = I$, where $I$ is the identity on $X$, which can be considered as an artificial hedge. So, for every $x$, we always have either $h_{-1}x < h_{-2}x < \cdots < h_qx < I_0x = x < h_1x < \cdots < h_px$ or $h_{-1}x > h_{-2}x > \cdots > h_qx > I_0x = x > h_1x > \cdots > h_px$.

Now, we introduce a general notion of SQMs in such a way that the conditions made on it are natural and as light as possible.

**Definition 4.1.** Let $A\mathcal{X} = (X, G, C, H, \Sigma, \Phi, \leq)$ be a linear ComHA. A mapping $\varphi : X \rightarrow [0, 1]$ is said to be an SQM of $A\mathcal{X}$, provided that the following conditions hold:

1. (Q1) $\varphi$ is a one-to-one mapping from $X$ into $[0, 1]$ and preserves the order on $X$, i.e. for all $x, y \in X, x < y \Rightarrow \varphi(x) < \varphi(y)$ and $\varphi(0) = 0, \varphi(1) = 1$, where $0, 1 \in C$;

2. (Q2) Continuity: $\forall x \in X, \varphi(\Phi x) = \inf \text{imum} \varphi(H(x))$ and $\varphi(\Sigma x) = \sup \text{imum} \varphi(H(x))$.

(Q1) is a minimum requirement for any quantification methods. The formal requirement given in (Q2) is quite natural, based on the topology of $X$ which ensures that the set $H(G)$ is dense in $X$. Practically, it reflects the fact that there does not exist any term lying in between $\Sigma x$ and $\sup \text{imum} H(x)$ or, semantically, the linguistic terms under the consideration are complete for describing the reference domain of a linguistic variable.

A question arises: how can we define an SQM based on a given FM on $X$?

Firstly, we give a model of the meaning of terms based on the FM of the free linear ComHAs. Let a FM $f m : X \rightarrow [0, 1]$ of a linear ComHA $A\mathcal{X} = (X, G, C, H, \Sigma, \Phi, \leq)$ be given. We shall assign a subinterval $\exists(x)$ of $[0,1]$ to the FM of every element $x \in X = H(G)$, i.e. we shall define a fuzziness semantics assignment (FSA) $\exists : X \rightarrow \text{Inv}([0, 1])$ by an induction on the length $l(x)$ of $x$ as follows, where $\text{Inv}([0, 1])$ is the family of all closed subintervals of $[0,1]$:

1. For $x \in [c^-, c^+]$, define two closed subintervals $\exists(c^-), \exists(c^+)$ of $[0,1]$, such that $|\exists(c^-)| = f m(c^-), |\exists(c^+)| = f m(c^+)$ and $\exists(c^-) \subseteq \exists(c^+)$. Note that, $c^- < c^+$.

2. Suppose that for all $x \in X = H(G)$ with $l(x) = n$, the closed subintervals $\exists(x)$ have already been defined so that $|\exists(x)| = f m(x)$ and, whenever $x < y$, we have $\exists(x) \subseteq \exists(y)$. Consider the sequence $\{h_{ix} : i \in [-q^+ p]\}$, where $[-q^+ p] = \{i : -q \leq i \leq p \land i \neq 0\}$. It is known that this sequence is either increasing or decreasing w.r.t. $i$, say $h_0x < \cdots < h_1x < h_{-2}x < \cdots < h_qx$. Then, we partition the interval $\exists(x)$ into $p + q$ closed subintervals, each of which is associated with one and only one label $h_{ix}, i \in [-q^+ p]$, such that $|\exists(h_{ix})| = f m(h_{ix})$ and, whenever $h_{ix} < h_{jx}$, we have $\exists(h_{ix}) \subseteq \exists(h_{jx})$, where $i, j \in [-q^+ p]$.

By Proposition 3.1, it is obvious that the above definition is correct and the above FSA is uniquely defined for a given fuzziness measure of $fm$ on $X$.

From the definition of FSA, we can regard each interval $\exists(x), x \in X$, as representing a meaning of term $x$ together with its fuzziness degree and, hence, it is called the fuzziness interval of $x$. To examine FSAs, we introduce a notion of semi-partition of an interval. A family $\{A_i : i \in I\}$ of closed subintervals of $[a, b]$ is said to be a semi-partition of $[a, b] if \bigcup_{i \in I} A_i = [a, b]$, and the intersection of any two distinct subintervals of this family contains at most one element, which is their common end point.

Let $X_k$ denote the set of all terms $x \in X = H(G)$ of length $k$, i.e. $l(x) = k$. The following lemma can be easily derived from the above definition:

**Lemma 4.1.** Let $fm$ be a fuzziness measure of $A\mathcal{X}$ and $\exists$ be an FSA associated with $fm$. Then,

1. $\exists(c^-), \exists(c^+)$ is a semi-partition of $[0,1]$ and, for $x \in X$, the family $\{\exists(h_{ix}) : i \in [-q^+ p]\}$ is a semi-partition of the interval $\exists(x)$.
The family \( \{ \mathcal{S}(x) : x \in X_n \} \) is a semi-partition of \([0,1]\) and if \( x < y \) and \( l(x) = l(y) = n \) then \( \mathcal{S}(x) \subseteq \mathcal{S}(y) \).

(iii) For \( y = \sigma x \), where \( \sigma \) is a string of hedges, we have \( \mathcal{S}(y) \subseteq \mathcal{S}(x) \).

(iv) For \( x, y \in X \) and \( x < y \), \( H(x) \cap H(y) = \emptyset \) implies that \( \mathcal{S}(x) \subseteq \mathcal{S}(y) \).

**Proof.** (i) follows immediately from the definition of the FSA \( \mathcal{S} \).

The statement (ii) can be proved by induction on \( n \): by the definition of the FSA \( \mathcal{S} \), it is obvious that (ii) is true for \( n = 1 \), since in this case \( \mathcal{S}(c^-), \mathcal{S}(c^+) \) is a semi-partition of \([0,1]\). Assume that (ii) is true for \( n = k - 1 \). For each \( x \in X \), \( l(x) = k - 1 \), by (i) the family \( \{ \mathcal{S}(h_i x) : i \in [-q^-p] \} \) is a semi-partition of the interval \( \mathcal{S}(x) \) and whenever \( h_i x < h_j x \), we have \( \mathcal{S}(h_i x) \subseteq \mathcal{S}(h_j x) \). Therefore, it implies that \( \{ \mathcal{S}(y) : y \in X, l(y) = k \} = \{ \mathcal{S}(h_i x) : i \in [-q^-p] \} \) and \( x \in X_{k-1} \) is a semi-partition of \([0,1]\). Now, it is easy to verify that if \( y < y' \) then \( \mathcal{S}(y) \subseteq \mathcal{S}(y') \), for \( l(y) = l(y') = k \).

Since (iii) can easily be deduced from the definition of \( \mathcal{S} \), we are going to prove (iv). It is obviously true if \( x \in H(c^-) \) and \( y \in H(c^+) \). Now, suppose that \( x, y \in H(c) \) such that \( x < y \) and \( H(x) \cap H(y) = \emptyset \). Assume that the canonical representation of \( x, y \) are \( x = h_k \ldots h_i u \) and \( y = k_m \ldots k_1 u \), where \( u \) is their longest common suffix. Since \( H(x) \cap H(y) = \emptyset \), it follows that \( h_1 \neq 1, k_1 \neq 1 \) and \( h_1 u < k_1 u \). This inequality leads to \( \mathcal{S}(h_1 x) \subseteq \mathcal{S}(k_1 x) \), which together with (iii) imply that \( \mathcal{S}(x) \subseteq \mathcal{S}(y) \). □

To define the following notion, we recall that a hedge is positive (or negative) w.r.t. a primary term \( c \in \{ c^-, c^+ \} \) if it strengthens (or weakens, respectively) its semantic tendency.

**Definition 4.2 (Sign function).** A function \( \text{Sign} : X \to \{-1, 0, 1\} \) is a mapping which is defined recursively as follows, where \( h, h' \in H \) and \( c \in \{ c^-, c^+ \} \):

(a) \( \text{Sign}(c^-) = -1 \), \( \text{Sign}(c^+) = +1 \);

(b) \( \text{Sign}(hc) = -\text{Sign}(c) \) if \( h \) is negative w.r.t. \( c \); \( \text{Sign}(hc) = +\text{Sign}(c) \) if \( h \) is positive w.r.t. \( c \);

(c) \( \text{Sign}(h'hx) = -\text{Sign}(hx) \), if \( h'hx \neq hx \) and \( h' \) is negative w.r.t. \( h \); \( \text{Sign}(h'hx) = +\text{Sign}(hx) \), if \( h'hx \neq hx \) and \( h' \) is positive w.r.t. \( h \).

(d) \( \text{Sign}(h'hx) = 0 \) if \( h'hx = hx \).

Based on \( \text{Sign} \) function, we have a criterion for comparing \( hx \) and \( x \).

**Lemma 4.2 (Ho et al. [9]).** For any \( h \) and \( x \), if \( \text{Sign}(hx) = +1 \) then \( hx > x \), if \( \text{Sign}(hx) = -1 \) then \( hx < x \).

It is known that a fuzziness measure \( fm \) is defined by giving the parameters \( fm(c^-), fm(c^+) \) and \( \mu(h_i) \), where \( i \in [-q^-p] \). We always assume that all these parameters are greater than 0 and \( fm(c^-) + fm(c^+) = 1 \), or, equivalently, \( fm(W) = \emptyset = 0 \). Put \( \sum_{i=-q}^{-p} \mu(h_i) = \alpha, \sum_{i=1}^{p} \mu(h_i) = \beta = 1 - \alpha \).

To understand how a formula for computing an SQM can be established in the definition below, let us consider a fuzziness measure \( fm \) of the variable \emph{TRUTH}, a segment of the FSA \( \mathcal{S} \) of which is given in Fig. 1. We shall interpret the marked points in the interval \([W, 1]\) in this figure as the SQM values of the corresponding terms associated with these points. In this case, it is observed that the point of label “Little True” with \( \text{Sign}(\text{Little True}) = -1 \) divides the interval of the length \( fm(L\text{True}) \) in the proportion \( \beta : \alpha \), while the point of label “Very True” with \( \text{Sign}(\text{Very True}) = +1 \) divides the interval of the length \( fm(\text{Very True}) \) in the proportion \( \alpha : \beta \). Taking this observation in mind, we can establish a formula to compute an SQM \( \varphi \) as follows:

**Definition 4.3.** Let \( \mathcal{L} \) be a free linear ComHA and \( fm \) be a fuzziness measure on \( X \). Then, a mapping \( \varphi : X \to [0, 1] \) is said to be induced by \( fm \), if it is defined recursively as follows:

1. \( \varphi(W) = \emptyset = fm(c^-), \varphi(c^-) = \theta - \alpha fm(c^-) = \beta fm(c^-), \varphi(c^+) = \theta + \alpha fm(c^+) \);
2. \( \varphi(h_j x) = \varphi(x) + \text{Sign}(h_j x) \sum_{i=-\text{sign}(j)}^{\text{sign}(j)} \mu(h_i) fm(x) + \omega(h_j x) \mu(h_j) fm(x) \), for \( j \in [-q^-p] \).

\[
\omega(h_j x) = \frac{1}{2} \left[ (\alpha + \beta) - \text{Sign}(h_j x) \text{Sign}(h_j x) (\beta - \alpha) \right] = \begin{cases} 
\alpha & \text{if } \text{Sign}(h_j x) \text{Sign}(h_j h_j x) = +1, \\
\beta & \text{if } \text{Sign}(h_j x) \text{Sign}(h_j h_j x) = -1.
\end{cases}
\]
(3) $\varphi(\Phi c^-) = 0$, $\varphi(\Sigma c^-) = \theta = \varphi(\Phi c^+)$, $\varphi(\Sigma c^+) = 1$, and for $j \in [-q^\wedge p],$

$$\varphi(\Phi h_j x) = \varphi(x) + \text{Sign}(h_j x) \left\{ \sum_{i=\text{sign}(j)}^{j-\text{sign}(j)} \mu(h_i) f m(x) \right\} - \frac{1}{2}(1 - \text{Sign}(h_j x))\mu(h_j) f m(x),$$

$$\varphi(\Sigma h_j x) = \varphi(x) + \text{Sign}(h_j x) \left\{ \sum_{i=\text{sign}(j)}^{j-\text{sign}(j)} \mu(h_i) f m(x) \right\} + \frac{1}{2}(1 + \text{Sign}(h_j x))\mu(h_j) f m(x).$$

**Example 4.1.** Consider again the hedge algebra of the variable AGE in Example 3.1, where the FM of the primary terms and hedges of AGE are $f m(young) = 0.4375$ and so $f m(old) = 0.5625$, $\mu(V) = 0.15$, $\mu(M) = 0.35$, $\mu(A) = 0.26$ and $\mu(L) = 0.24$. So, $q = p = 2$ and $\alpha = \beta = 0.5$. We can compute by Definition 4.3 the SQM values of some linguistic terms of AGE as follows:

For $x = young$, we have $\varphi(young) = \beta f m(young) = 0.5 \times 0.4375 = 0.21875$. Since, the reference domain of AGE is $[0, 80]$, the real value of $young$ is $80 \times 0.21875 = 17.5$;

$\varphi(old) = f m(young) + \alpha f m(old) = 0.71875$ and the real value of $old$ is 57.5.

For $x = Lc^- = young$, we have $j = -2$ and, then,

$$\varphi(Lc^-) = \varphi(c^-) + \text{Sign}(Lc^-) \left\{ \sum_{i=\text{sign}(j)}^{j-\text{sign}(j)} \mu(h_i) f m(c^-) + \omega(h_j c^-)\mu(h_j) f m(c^-) \right\}$$

$$= \varphi(young) + [\mu(A) f m(young) + \alpha\mu(L) f m(young)],$$

$$\text{Sign}(young)\text{Sign}(VLyoung) = (-1)(-1)(+1)(-1)(-1) = +1,$$ i.e., $\omega(Ly) = \alpha$.

So, $\varphi(young) = 0.21875 + \{0.26 \times 0.4375 + 0.5 \times 0.24 \times 0.4375\} = 0.385$ and the real value of $Lyoung$ is $80 \times 0.385 = 30.8$.

Now, we consider the hedge algebra of the variable AGE of the group of persons mentioned in Example 3.2 with FM of the hedges are the same as above, but the FM of primary terms are $f m(young) = f m(old) = 0.5$. Then, the SQM values of the terms mentioned above are the following:

For $x = young$, we have $\varphi(young) = \beta f m(young) = 0.5 \times 0.5 = 0.25$ and the real value of $young$ in this case is $80 \times 0.25 = 20$. And for $x = old$,

$$\varphi(old) = f m(young) + \alpha f m(old) = 0.75$$ and the real value of $old$ in this case is $80 \times 0.75 = 60$.

For $x = Lc^- = young$, we have $j = -2$ and, then,

$$\varphi(Lc^-) = \varphi(c^-) + \text{Sign}(Lc^-) \left\{ \sum_{i=\text{sign}(j)}^{j-\text{sign}(j)} \mu(h_i) f m(c^-) + \omega(h_j c^-)\mu(h_j) f m(c^-) \right\}$$

$$= \varphi(young) + [\mu(A) f m(young) + \alpha\mu(L) f m(young)].$$

So, $\varphi(young) = 0.25 + \{0.26 \times 0.5 + 0.5 \times 0.24 \times 0.5\} = 0.44$ and the real value of $L young$ in this case is $80 \times 0.44 = 35.2$.

**Remark 4.1.** If $f m(x) \neq 0$, the value $\varphi(h_j x)$ divides internally the interval $\Im(h_j x)$ in the proportion $\beta: \alpha$, if $\text{Sign}(h_j x)\text{Sign}(h_j h_j x) = -1$, and in the proportion $\alpha: \beta$, if $\text{Sign}(h_j x)\text{Sign}(h_j h_j x) = +1$. Hence, $\varphi(x)$ is an internal point of $\Im(x)$ and, more exactly, it is a common end point of the subintervals $\Im(h_j x)$ and $\Im(h_j x)$ of $\Im(x)$.

**Theorem 4.1.** Let $AX$ be a free linear ComHA and $\varphi$ be a mapping induced by a fuzziness measure $f m$. Then, $\varphi$ is an SQM and the image $\varphi(H_e(x))$ is countable and dense in the interval $\Im(x) = [\varphi(\Phi x), \varphi(\Sigma x)]$, $\forall x \in X$. Moreover, we have $f m(x) = \varphi(\Sigma x) - \varphi(\Phi x)$, i.e. it equals the length of $\Im(x)$ and, hence, $f m(x) = d(\varphi(H(x)))$, where $d(.)$ denotes the diameter of the set $\Im(x)$.

**Proof.** It is evident firstly that $H_e(x)$ is countable and so is the set $\varphi(H_e(x))$. Now we shall prove that the set $\varphi(H(x))$ is dense in $\Im(x)$. Take an arbitrary subinterval $[a, b] \subseteq [0, 1]$ of the length $e > 0$. Put $\lambda = \text{Max}\{\mu(h_j) : j \in [-q^\wedge p]\} < 1$. For any $y \in H(x)$, $y = \delta x = h_0 \ldots h_x, h_1, \ldots, h_n \in H$ and $|\delta| \geq 1$, the length of $\Im(y)$ is $f m(y) = \mu(h_n)\mu(h_{n-1}) \ldots \mu(h_1) f m(x) \leq \delta^a f m(x)$. By the definition of the FSA $\Im$, the family $\{\Im(y) : y = \delta x \text{ and } |\delta| = n\}$ is
a semi-partition of \( \mathcal{S}(x) \). Therefore, with \( n \) sufficiently large, \( |\mathcal{S}(y)| < \varepsilon/2 \) and, hence, there exists an \( \mathcal{S}(y) \) such that 
\( \mathcal{S}(y) \subseteq [a, b] \) which implies that \( \phi(y) \in [a, b] \). It shows that \( \phi(H(x)) \) is dense in \( \mathcal{S}(x) \).

From the expression in (3) of Definition 4.3, it follows that \( \phi(\Phi(x)) \) and \( \phi(\Sigma(x)) \) are, respectively, the left and the right end points of the interval \( \mathcal{S}(x) \). Since \( \phi(H(x)) \) is dense in \( \mathcal{S}(x) \), it is obvious that \( \phi(\Phi(x)) = \inf(\mathcal{S}(x)) \), \( \phi(\Sigma(x)) = \sup(\mathcal{S}(x)) \), i.e. \( \phi \) is continuous, and \( d(\phi(H(x))) = \mu(x) = \phi(\Sigma(x)) - \phi(\Phi(x)) \).

Now, we shall prove that \( \phi \) is an isomorphism in the categories of the linearly ordered sets, i.e. it is one-to-one from \( X \) into \([0,1]\) and preserves the order relation of \( X \). Indeed, let us consider any two elements \( x, y \in X \) such that \( x < y \). Suppose firstly that \( x, y \in H(G) \) and their canonical representations are \( x = \delta \varepsilon = k_m \ldots k_1 c, y = \delta' \varepsilon = k'_n \ldots k'_1 c' \), where \( c, c' \in \{c^-, c^+\} \). Assuming \( c \neq c' \), \( x < y \) implies that \( c < c' \) and so \( (c) \subseteq (c') \). From (iv), Lemma 4.1 and Remark 4.1, it follows that \( \phi(x) < \phi(y) \). Now, suppose that \( c = c' \in \{c^-, c^+\} \) and \( u \) is the longest suffix of \( x \) and \( y \), i.e. \( x = k_m \ldots k_i u, y = k'_n \ldots k'_i u \) and \( k_i u \neq k_i u' \). In the case that \( i = \min\{m, n\} + 1, \) e.g. \( i = m + 1 \leq n \), we have by our convention that \( k_i = I \)-the identity on \( \overline{X} \). Hence, \( y = k'_n \ldots k'_1 x \in H(x) \). From \( x < y \), it follows that \( x < k'_j x \). Since, by Remark 4.1, \( \phi(x) \) lies in between \( \mathcal{S}(h_{-j}x) \) and \( \mathcal{S}(h_{+j}x) \), it can be seen that \( \{\phi(x)\} \subseteq \mathcal{S}(k'_j x) \). Hence, the condition \( \mathcal{S}(y) \subseteq \mathcal{S}(k'_j x) \) leads to \( \phi(x) < \phi(y) \). In the case that \( i \leq \min\{m, n\} \), \( k_i u \neq k_i u' \) as previously, is an end point of \( \mathcal{S}(u) \) defined by \( \mu \). Since \( \mathcal{S} \) preserves the order relation, we have \( \mathcal{S}(h_1 u) \subseteq \mathcal{S}(k_i u) \), which leads to \( \phi(x) < \phi(y) \) with a notice that \( \phi(x) \in \mathcal{S}(h_1 u) \) and \( \phi(y) \in \mathcal{S}(k_i u), \) by Remark 4.1.

Now, suppose that exactly one of \( x \) and \( y \) is in \( \overline{\text{Lim}(X)} \). We shall prove for the case \( y \in \overline{\text{Lim}(X)}, y = o'y' \), where \( y' \in H(G), o' \in \{\Phi, \Sigma\} \) (for the case \( y \in H(G) \) and \( x = o'x' \) with \( x' \in H(G) \) the proof is obtained by “duality”).

On account of Theorem 2.3, we have \( \{\exists z \in H(G)\} \{x < H(z) < y\}, \) where \( H(z) \) satisfies one of conditions (i)-(iv). If (ii) holds, i.e. \( H(z) \cap H(y') = \emptyset \), then \( z < y \) implies \( z < y' \) and, hence, (iv) of Lemma 4.1 yields \( \exists z \subseteq \mathcal{S}(z') \). Since \( x, z \in H(G) \), by the case proved above, it follows that \( \phi(x) < \phi(z) \subseteq \phi(\Phi(y')) \subseteq \phi(o'y') = \phi(y) \). If (iv) in Theorem 2.3 holds, i.e. \( H(z) \subseteq H(y') \), then from the condition \( H(z) < y \), we infer that \( o' \) must be \( \Sigma \) and hence \( \phi(x) < \phi(z) \subseteq \phi(\Sigma(y')) = \phi(y) \). (note that, by Remark 4.1, \( \phi(z) \) is always an internal point of \( \mathcal{S}(z) \)). And, if (iii) of Theorem 2.2 holds, i.e. \( H(z) \subseteq H(x) \), then, by a similar argument, we obtain \( \phi(x) < \mathcal{S}(z), \) since \( x < H(z) \). On the other hand, between \( H(z) \) and \( H(y') \) there are only two possibilities that either \( H(z) \cap H(y') = \emptyset \), or \( H(z) \subseteq H(y') \), and \( \phi(o'y') \), as it is proved previously, is an end point of \( \mathcal{S}(y') \), it follows from \( H(z) < y \) that \( \mathcal{S}(z) < \phi(o'y') \). Therefore, we get \( \phi(x) < \phi(y) \).

Consider the case that \( x, y \in \overline{\text{Lim}(X)} \), i.e. \( x = o x' \) and \( y = o'y' \), where \( x', y' \in H(G) \). In this case, we have \( x < H(z) < y = o'y' \). By the same argument as that for the sets \( H(z) \) and \( H(y') \) above, we obtain \( \mathcal{S}(z) < \phi(o'y') \). Interchanging the role of \( y = o'y' \) with the one of \( o x' \) and using a dual argument, we get \( \phi(o x') < \mathcal{S}(z) \). So, we obtain again \( \phi(x) < \phi(y) \). 

This theorem gives us a good model of the FM of terms: given a term \( x \), the FM of \( x \) is the diameter of the image of the set \( H(x) \) under an SQM. In addition, we have \( \frac{d\phi(H(x))}{d\phi(H(y))} = \frac{d\phi(H(hx))}{d\phi(H(hy))} \), which means that the relative degree of modification of a hedge does not depend on particular terms it applies.

As an immediate consequence of this theorem, we have:

**Corollary 4.1.** The image set \( \phi(H(G)) \) of \( H(G) \) is countable and dense in \([0,1]\).

At first glance, we have a feeling that the mapping \( \phi \) defined by Definition 4.3 seems to be rather special, while the conditions imposed on the notion of SQMs in general are light and natural. However, the following theorem shows that the conditions in Definition 4.3 are also light and natural.

**Theorem 4.2.** Let \( AX \equiv (X, G, C, H, \Sigma, \Phi, \leq) \) be a free linear ComHA, where \( H^- = \{h_{-1}, \ldots, h_{-q}\}, \) with \( h_{-1} < h_{-2} < \cdots < h_{-q} \), and \( H^+ = \{h_1, \ldots, h_p\}, \) with \( h_1 < \cdots < h_p \). Suppose that the mapping \( v : X \rightarrow [0, 1] \) satisfying the following conditions:

- \( (v_1) \) \( v \) is a one to one and preserves the order relation on \( X \), i.e. \( x < y \Rightarrow v(x) < v(y) \);
- \( (v_2) \) The image set \( v(H((c^-, c^+))) \) is dense in the unit interval \([0,1] \);
- \( (v_3) \) For all \( x, y \in X \) and all \( h \in H \), the following proportion holds:

\[
\frac{d(v(H(hx)))}{d(v(H(xy)))} = \frac{d(v(H(hy)))}{d(v(H(uy)))}.
\]
Then, \(v\) is an SQM defined by Definition 4.3, where the FM \(\mu\) will be defined by \(\mu(h) = \frac{d(v(H(hx)))}{d(v(H(c^+)))}\), for any \(h \in H\), and \(fm(c^+) = d(v(H(c^+) \cup H(c^-)))\). In addition, we have \(fm(x) = d(v(H(x)))\), \(\forall x \in X\), is an FM on \(X\).

**Proof.** Let us denote the interval \([v(\Phi x), v(\Sigma x)]\) by \(\mathcal{S}(x)\). We shall show by induction on the length \(l(x)\) of \(x\) that \(v(H(x))\) is dense in \(\mathcal{S}(x)\). For \(l(x) = 1\), i.e. \(x \in [c^-, c^+]\), it follows from (v1) and (v2), that \(v(H(c^-)) < v(H(c^+))\) and \(v(H(c^-)) \cup v(H(c^+))\) is dense in \([0, 1]\). Hence, it can be checked that infimum \(v(H(c^-)) = v(\Phi c^-)\), supremum \(v(H(c^+)) = \infimum v(H(c^+)) = v(W)\). Since, by (v1), for \(u \in H(x)\), \(v(u) < v(\Sigma x)\) and \(v(u) > v(\Phi x)\), it implies that \(v(\Sigma c^-) = v(\Phi c^-) = v(W)\), and supremum \(v(H(c^+)) = 1 = v(\Sigma c^+)\). Thus, \(\mathcal{S}(c^-)\) and \(\mathcal{S}(c^+)\) constitute a semi-partition of \([0, 1]\) and \(v(H(c))\) is dense in \(\mathcal{S}(c)\), \(c \in [c^-, c^+]\). Obviously, \(d(v(H(c^-))) + d(v(H(c^+))) = 1\).

Assume, as the induction hypothesis, that \(v(H(x))\) is dense in \(\mathcal{S}(x)\). Without loss of generality, we assume that \(h_{-q} x < \cdots < h_{-1} x < x < h_1 x < h_2 x < \cdots h_p x\), which leads to the inequality \(H(h_j x) < H(h_k x)\), for \(j\) and \(k\). Clearly, \(H(x) = \{x\} \cup \bigcup \{H(h_j x) : j \in [−q^−, p]\}\) and \(v(H(x)) = \{v(x)\} \cup \bigcup \{v(H(h_j x)) : j \in [−q^−, p]\}\). If \(j\) and \(k\) are consecutive and \(j < k\), then they have the same (negative or positive) sign, i.e. \(h_j\) and \(h_k\) belong together to either \(H^-\) or \(H^+\). From Axiom (L5) (in Section 2), it follows that \(\mathcal{S}h_j x = \Phi h_k x\). By (v1), we have \(v(H(h_j x)) < v(H(h_k x))\), and hence \(v(H(h_j x)) < v(\Sigma h_j x) = v(\Phi h_q x) < v(H(h_k x))\). Since \(v(H(x))\) is dense in \(\mathcal{S}(x)\), it follows that \(v(\Sigma h_j x) = \supremum v(H(h_1 x)) = \infimum v(H(h_1 x)) = v(\Phi h_1 x)\). Similarly, for \(j = 0\), we can prove that \(\infimum v(H(h_{−q} x)) = \infimum v(H(x)) = v(\Phi x)\) and for \(k = 1\), we have \(v(\mathcal{S}h_{−1} x) = \supremum v(H(h_{−1} x)) = \infimum v(H(h_1 x)) = v(\Phi h_1 x) = v(x)\). Now, it can easily be seen that \(v(H(h_k x))\) is dense in \(\mathcal{S}(h_k x) = \{v(\Phi h_k x), v(\Sigma h_k x)\}\) and \(d(v(H(h_k x))) = d(v(\mathcal{S}(h_k x)))\). Obviously, the family \(\{\mathcal{S}(h_k x) : k \in [−q^−, p]\}\) is a semi-partition of \(\mathcal{S}(x)\). This implies that \(\sum_{k \in [−q^−, p]} d(v(H(h_k x))) = d(v(H(x)))\). Put \(fm(x) = d(v(H(x)))\) and \(\mu(h) = \frac{d(v(H(x)))}{d(v(H(h_k x)))}\) we get \(fm(c^-) + fm(c^+) = 1\) and \(\sum_{k \in [−q^−, p]} \mu(h_k)\). Since \(x = \sum_{k \in [−q^−, p]} \mu(h_k)\), it can be verified that \(v(x)\) divides internally the interval \(\mathcal{S}(x)\) in the proportion \(x: \beta\) and that \(v\) is computed by the formulas given in Definition 4.3. \(\square\)

So, we have shown that there is a closed relation between the SQM \(v\) and the family of fuzziness intervals of \(X\). This suggests us to define an order relation on these fuzziness intervals as follows: for any \(x, y \in X\), \(\mathcal{S}(x) \subseteq \mathcal{S}(y)\) if and only if \(v(x) \leq v(y)\). As a consequence of Theorem 4.2, (iv) of Lemma 4.1 and Theorem 2.4, we have:

**Corollary 4.2.** The family \(\{\mathcal{S}(x) : x \in X\}\) is linearly ordered and isomorphic onto \((X, \leq)\) and, hence, this family of fuzziness intervals satisfies the statements formulated in Theorem 2.4.

5. Some applications

5.1. A problem of construction of memberships functions of linguistic terms

A great deal of research has been carried out into the problem of memberships functions, some of which deals with the construction of membership functions, which are appropriate to a given application [14,15,18]. Certain requirements for such a construction can be observed in accounts of this research:

- The construction should obey some rules imposed by reality, such as equalization rule [18]. That is, ones require that the desired membership functions should model the reality appropriately.
- It should have a suitable mathematical structure [15] which can model a semantic structure of a linguistic variable.
- It should be easily computed [14,15,18].

In [15], a mathematical model of hedges effect, which was used in constructing membership functions of the truth variable, has been introduced and examined. It gives a simple method for computing membership functions, the order relation of which is the ordinary one on the functions and it satisfies some conditions imposed by certain properties of hedge algebras. In [18], an algorithm for building fuzzy sets based on the respective probability density function of available data and fuzzy equalization rule is applied. The main idea of the algorithm is roughly as follows:

Let \(p(u)\) be a given probability density function on an interval \([a,a']\).

(i) Specify the number \(n\) of fuzzy sets in the form of triangles or trapezoids we want to build for the linguistic variable under consideration.
(ii) Determine the internal points $a_i, i = 0, 1, \ldots, n + 1$, of $[a, a']$ in turn so that $P(A_1) = P(A_2) = \cdots = P(A_n) = 1/n$ (called equalization rule), where $P(A) = \int_U \mu_{AI}(u)p(u)\,du$ and $\mu_{AI}(u)$ is a membership function of the fuzzy set $A_i$ defined on $[a_{i-1}, a_{i+1}]$.

So, it can be seen that this method does not deal with the meaning of terms and hedges and hence the relative position of the obtained fuzzy sets, that reflects an aspect of the meaning of terms, depends only on $n$ and $p(u)$, but not on the meaning of terms which may be taken as the linguistic labels of the obtained fuzzy sets. Therefore, the meaning of the linguistic labels cannot be expressed by the corresponding fuzzy sets properly. It seems not applicable to many applications where the designed linguistic labels must be compatible with their fuzzy sets.

In this subsection, we shall introduce a new method which takes the advantage of the FMs of terms in hedge algebras examined in Section 3 to construct membership functions for a fuzzy conditional reasoning problem so that the constructed fuzzy sets will be compatible with the meaning of linguistic terms.

We shall examine two closely related problems:

5.1.1. Construct fuzzy sets appropriate to linguistic terms of a linguistic variable

The problem is formulated as follows: given a term-domain $X$ of a linguistic variable $X$, construct a set of membership functions, which will represent the meaning of the terms in $X$ appropriately. The problem seems to be rather general: construct fuzzy sets based on only the meaning of their linguistic labels. Restricting the fuzzy sets under consideration to the form of triangles or trapezoids, most construction methods that the authors found in the research papers seem to lead to the problem saying that find a set of fuzzy sets that form a soft partition of a given real interval (see, e.g., [2,19]). In general, there are no clear and explicit constraints imposing upon this construction in many research works. Therefore, in several works, the fuzzy sets were built on equal base sides independently of whether some of their linguistic labels are vaguer than others [2,19].

Let $\mathcal{A}X = (X, G, C, H, \Sigma, \Phi, \leq)$ be the free linear ComHA of the linguistic variable $X$. We shall construct fuzzy sets representing the terms of $X$, taking advantage of the meaning of terms represented by the fuzziness intervals associated with the respective terms of $X$.

Since the real domain of $X$ may be the interval $[0, L]$, in order to obtain the fuzziness intervals defined on $[0, L]$, the FM of terms must be multiplied by $L$. As previously, the fuzziness interval of a term $x$ defined on $[0, L]$ is denoted also by $I(x)$ and $X_k$ denotes the set of all terms in $X$ of length $k$, i.e. there are exactly $k - 1$ hedges in their canonical representation. Suppose that the elements of $X_k$ are naming as $\{u_{k,i} : i = 1, 2, \ldots, m\}$ so that $u_{k,i} < u_{k,j}$, and hence $I(u_{k,i}) \leq I(u_{k,j})$, whenever $i < j$.

For every term $x$ of $X_k$, the fuzzy set associated with the fuzziness interval $I(x)$ will be constructed as follows:

For $k = 1$, $X_1 = \{c^-, c^+\}$ and the fuzziness intervals $I(c^-)$ and $I(c^+)$ form a semi-partition of $[0, L]$. For $I(c^-)$, we construct a trapezoid with two base sides of lengths $L(fm(c^-) + \frac{1}{2}fm(c^+))$ and $\frac{1}{2}Lfmc^-)$. Similarly, for $I(c^+)$ we construct a trapezoid of two base sides of lengths $L(fm(c^+) + \frac{1}{2}fm(c^-))$ and $\frac{1}{2}Lfmc^+)$.

For $k > 1$ and $x \in X_k$, the fuzzy set associated with $I(x)$ will be constructed as follows:

- Associated with the first fuzziness interval $I(u_{k,1})$, we build a trapezoid with two base sides of lengths $L(fm(u_{k,1}) + \frac{1}{2}fm(u_{k,2}))$ and $\frac{1}{2}Lfmu_{k,1})$. Similarly, the constructed trapezoid associated with the end fuzziness interval $I(u_{k,m})$ has two base sides of lengths $L(fm(u_{k,m}) + \frac{1}{2}fm(u_{k,m-1}))$ and $\frac{1}{2}Lfmu_{k,m})$ (see Fig. 2).
- Associated with $I(u_{k,i})$, for any $i \neq 1$ and $i \neq m$, we build a triangle with the base side of length $L(fm(u_{k,i}) + \frac{1}{2}fm(u_{k,i-1}) + \frac{1}{2}fm(u_{k,i+1}))$. The membership function of this fuzzy set attains the value 1 at the center of the subinterval $I(u_{k,i})$.

So, for $k = 1, 2, \ldots$, we obtain a family $F$ of fuzzy sets in the form of either triangles or trapezoids associated with the corresponding terms of the given linguistic variable. For convenience, the fuzzy set associated with a term $x$ some times is called the fuzzy set of the term $x$. The method of the construction depends only on the FM of primary terms and hedges considered as the parameters of the method. By adjusting these parameters, we hope that one can construct the fuzzy sets suitably and flexibly to adapt the application.

Remark. According to this construction, there is a one-to-one correspondence between these fuzzy sets and the fuzziness intervals of the terms in $X$. It can be seen that the order relation on the fuzziness intervals examined in the end of Section 4 will induce an order relation $\leq$ on $F$ and hence $(F, \leq)$ is isomorphic onto $((\exists x) : x \in X, \leq)$ or...
(X, ≤), where the use of the same notation ≤ has no confusion, since the respective underlying sets are completely different. So, it is different from other methods whose sets of outputs have no mathematical structure (see, e.g., [15,18]), the set (F, ≤) produced by the new method has a quite rich computation structure established by Corollary 4.2.

5.1.2. Construct a set of fuzzy sets for an application

In applications, normally one needs to use only a small number of fuzzy sets. So, another problem is that, for a given application, how can we define a small set of fuzzy sets, which are sufficient to solve the application problem.

In fuzzy control problems, one often faces with the FMCR problem whose fuzzy model is given as follows:

\[ F(x) = \begin{cases} a_1 & \text{if } x < a_1 \\ a_2 & \text{if } a_1 \leq x \leq a_2 \\ \vdots \\ a_n & \text{if } x > a_n \end{cases} \]

(5.1)

where \( a_i \), \( i = 1, \ldots, n \) are verbal descriptions of physical variables \( X \) and \( Y \), respectively.

The fuzzy model (5.1) models a real curve \( C_r \) defined on a universe of discussion \( U = [0, L] \subseteq R \) — the set of all real numbers. In many applications, this curve \( C_r \) can be assumed given, e.g. \( C_r \) can be formed from a collection of experiment data [2] or it is defined by certain mathematical equations [19].

The problem is how can we define a fuzzy model (5.1), which models the given real curve \( C_r \) suitably?

Step 1: Determine linguistic terms and their fuzzy sets for the linguistic variable \( X \); the main idea of the method can be formulated as: given \( C_r \), find fuzzy sets associated with some terms of the linguistic variable \( X \) that hold necessary information to describe the shape of \( C_r \).

It can be observed that the method errors in solving the FMCR problem depend on the degree of irregular variation of \( C_r \) with respect to the variation of the base variable \( u \). So, on the intervals where the curve changes rather quickly or irregularly, the method may cause a big error and, hence, it is necessary to build fuzzy sets thereon. Practically, we can find out these intervals by recognizing a quick change of the area of the figure defined by the intersection of the domain bounded by \( C_r \) and a triangle or trapezoid when its base side moves from one fuzziness interval to another one. Because we only try to discover on which fuzziness intervals the real curve varies quickly and since the fuzzy sets take values only in the interval \([0,1]\), for the sake of convenience instead of \( C_r \), we consider the curve \( C_r' = (1/\text{Sup } C_r)C_r \), where the notation \( \text{Sup } C_r \) denotes the high of \( C_r \), which is a constant.

As an example, consider a real piece-linear curve \( C_r' \) given in Fig. 3. Suppose that \( X \) is a linguistic variable with the reference domain \([0, L]\) and \( \mathcal{AX} = (X, G, H, \Sigma, \Phi, \leq) \) is its free linear ComHA, where \( H^- = \{h_{-1}, \ldots, h_{-q}\} \), with \( h_{-1} < h_{-2} < \ldots < h_{-q} \), and \( H^+ = \{h_1, \ldots, h_p\} \), with \( h_1 < \ldots < h_p \). For simplifying presentation, we suppose that \( p = q = 2 \). Let \( \mu(h_{-1}), \mu(h_{-2}), \mu(h_1) \) and \( \mu(h_2) \) be the parameters of an SQM of \( \mathcal{AX} \) defined by Definition 4.3 and suppose that \( f_m(c^-) = f_m(c^+) \) and \( \mu(h_{-1}) = \mu(h_{-2}) = \mu(h_1) = \mu(h_2) \).

The algorithm consists of the following steps:

1. Specify a number \( n \) (for example, \( n = 7 \)) of fuzzy sets which are required to build; specify a number \( K \), a changing degree of the area of the figure defined by the intersection of the field bounded by \( C_r' \) and a triangle or trapezoid (in this example, we assume that \( K = 1.3 \)).

2. Determine values of the parameters of the SQM mentioned above so that \( f_m(c^-) + f_m(c^+) = 1 \) and \( \mu(h_{-1}) + \mu(h_{-2}) + \mu(h_1) + \mu(h_2) = 1 \). Based on the method described in Section 5.1.1, we construct the triangles (not trapezoids) associated with the fuzziness intervals \( I(c^-) \) and \( I(c^+) \) on the domain \( U = [0, L] \). Put these triangles into the family \( F = F_1 \). So, \( F_1 \) consists of the triangles with the labels \( c^- \) and \( c^+ \).
Suppose that we have constructed the family $F_1$, whose triangles or trapezoids are listed in the order they occur on $U$ from left to right (in the first step, we have $i = 1$ and $F_1$ consists of the triangles No. III and No. VIII in Fig. 3).

(3) Consider the list $F_i$ and the set $X_i$. Construct either triangles or trapezoids associated with the fuzziness intervals of the terms in $X_i$, using the method given in Section 5.1.1. These fuzzy sets are put into a temporal list $A$ in the order they occur from left to right on the interval $[0, L]$.

For instance, in Fig. 4, the output list $A$ of fuzzy sets of this step for $i = 1$ consists of eight triangles named by $I$, $II$, $IV$, $V$, $VI$, $VII$, $IX$ and $X$, which are associate with the fuzziness intervals $I(h_2c^-)$, $I(h_1c^-)$, $I(h_{-1}c^-)$, $I(h_{-2}c^-)$, $I(h_{-1}c^+)$, $I(h_1c^+)$ and $I(h_2c^+)$, respectively.

(4) Put the first triangle of $A$, denoted by $A'$, into the list $F_{i+1}$. Consider $A$ and $A'$, where $A'$ is the consecutive triangle of $A$ in the list $A$.

Loop step: Compute in turn two areas formed by the intersection of the curve $C_r'$ and the triangles $A$ and $A'$. If the condition $Cond = "The changing degree of $C_r'$ defined by the proportion of the larger area to the smaller one is not less than the given threshold $K"$ is satisfied, put both triangles into $F_{i+1}$.

Take the triangle $A'$ which has just been considered to play the role of $A$ and the consecutive triangle of $A'$ in $A$ to plays the role of $A'$, if there are still such a triangle in $A$, and go to the Loop Step.

(5) If $F_{i+1}$ is still not empty and the number of triangles in the family $F = F_1 \cup \cdots \cup F_{i+1}$ is still less than $n$, put the index $i := i + 1$ and go to (3).

(6) Transform the first and the end triangles (No. $(I)$, $(X)$) in the list $F$ into the corresponding trapezoids as shown in Fig. 3. The results are the fuzzy sets listed in $F$.

In Fig. 3 we consider a piece-linear curve connecting the points $(0.0, 1.00)$, $(1.7, 1.00)$, $(3.0, 0.90)$, $(3.8, 0.50)$, $(5.0, 0.45)$, $(8.5, 0.35)$ and $(10.0, 0.10)$. The algorithm generated a list of outputs consisting of the fuzzy sets named by $(I)$, $(II)$, $(IV)$, $(V)$, $(VII)$, $(IX)$ and $(X)$, which correspond to the linguistic terms Very small, small, Possibly small, Less small, large, More large and Very large, where $h_2 = V$ (Very), $h_1 = M$ (More), $h_{-1} = Possibly (P)$ and $h_{-2} = Little (L)$.

Step 2: Determine linguistic terms and fuzzy sets for the variable $Y$: in practice, the allocation of the fuzzy sets on $[0, L]$ of $X$ constructed in Step 1 plays a key role in minimizing the error of an FMCR method. Therefore, we may take the advantage of these fuzzy sets to compute fuzzy data of $Y$ based on the curve $C_r$. Because $C_r$ and $C_r'$ are the same with a difference of a multiplier only, we may use the curve $C_r'$ to determine the values of $Y$ in the following way.
In the fuzzy environment, there are maybe several ways to solve this question. Here, simply we shall benefit by the points that hold much information related to the fuzzy data, i.e. the points at which the values of fuzzy sets are maximal. These points are called the representatives of the respective fuzzy sets. Note that, since $fm(c^-) = fm(c^+) = 0.5$ and $\mu(h_1) = \mu(h_2) = 0.25$ for both the variables $X$ and $Y$, these representatives are just the centers of the fuzziness intervals associated with the respective fuzzy sets constructed by the method given in Section 5.1.1. Then, the values of the real curve $C'_r$ at the representatives of the fuzzy sets produced by Step 1 will be computed. Since the maximal length of the terms of $X$ determined in Step 1 is 2, we may determine the fuzzy sets of $Y$ corresponding to those of $X$ among the fuzzy sets of the terms of $Y$ constructed by the method given in Section 5.1.1 of the length not greater than 2 that satisfy the criterion saying that the representative of the fuzzy set under consideration is most close to at least one of the values of $C'_r$ which have just been computed above.

The representatives of the fuzzy sets represented by the triangles ($I$), ($III$)–($V$) and ($VIII$)–($X$) defined in Step 1, respectively, are determined to be $u_1 = 0.625$, $u_2 = 2.50$, $u_3 = 3.125$, $u_4 = 4.375$, $u_5 = 7.5$, $u_6 = 8.125$ and $u_7 = 9.375$. The values of the curve $C'_r$ at these points are, respectively, the following: 1.0, 0.94, 0.8375, 0.37857, 0.3607, 0.20417 and 0.476.

Now, we consider all fuzziness intervals of terms of the length not greater than 2 of the variable $Y$ and compute their representatives and obtain 0.0625, 0.1875, 0.25, 0.3125, 0.4375, 0.5625, 0.6875, 0.75, 0.8125 and 0.9375. Based on our criterion, the fuzziness intervals of terms of $Y$ are those whose representatives are 0.1875, 0.3125, 0.4375, 0.8125 and 0.9375. The terms associated with these fuzziness intervals are $Mc^-$, $Lc^-$, $Pc^-$, $Mc^+$ and $Ve^+$. The base sides of the fuzzy sets of these terms of $Y$ are subintervals of $[0,1]$. In order to obtain the fuzzy sets defined on the real reference domain of $Y$, we may use a linear transformation of $[0,1]$ into the real domain of $Y$.

Now, a fuzzy model that is appropriate to the real curve $C_r$ will be formulated as follows, where $c^+ = \text{large}$ and $c^- = \text{small}$:

\[
\begin{align*}
\text{If } X &= \text{Very small} & \text{then } Y &= \text{Very large}, \\
\text{If } X &= \text{small} & \text{then } Y &= \text{Very large}, \\
\text{If } X &= \text{Possibly small} & \text{then } Y &= \text{More large}, \\
\text{If } X &= \text{Less small} & \text{then } Y &= \text{Less large}, \\
\text{If } X &= \text{large} & \text{then } Y &= \text{Possibly small}, \\
\text{If } X &= \text{More large} & \text{then } Y &= \text{Possibly small}, \\
\text{If } X &= \text{Very large} & \text{then } Y &= \text{More small}.
\end{align*}
\]

5.2. An interpolation reasoning method based on SQMs of hedge algebras

To illustrate an additional applicability of the quantification of hedge algebras, we take an example studied in [2] to re-examine.

Let us consider the FMCR problem saying that “For a given fuzzy model (5.1), which models a real curve $C$ in the reality, and an input $A$, find an output $B$ corresponding to $A$”. It is known that each FMCR method provides a mechanism for computing an output $B_0$ for a given input $A_0$ of model (5.1). The physical variables of these problems are normally modeled by linguistic variables, whose reference domains usually are linearly ordered sets. So, HAs as models of the domains of these variables must be linearly ordered sets as well. This suggests us to deal with a new fuzzy interpolative method to solve FMCR problem, based on SQMs examined in Section 4.

Using fuzzy sets-based FMCR methods to solve a practical problem modeled by a fuzzy model (5.1), we have to carry out many tasks:

(i) To determine fuzzy sets which represent the meaning of linguistic terms occurring in the given fuzzy model.

(ii) To determine an appropriate FMCR method: normally, to build a suitable FMCR method of reasoning, one must define many factors:

a. Implication operators for representing the semantics of if–then sentences in the fuzzy model (5.1). Then, each if–then sentence will define a fuzzy relation.

b. Aggregation operators for aggregating the obtained fuzzy relations. One can choose a t-norm or an s-norm or even any one of the two dual operators Max and Min for aggregation.

c. Compositional rule: original compositional rule is computed by Max–Min composition. More generally, one can use any t-norm or s-norm operator instead of Min and Max, respectively.
To find one among many defuzzification methods: the output of (ii) in general is a fuzzy set. In many applications one requires that the outputs must be real data and so one needs to define an appropriate method of defuzzification to transform a fuzzy set into a real datum; and so on.

Note that these tasks above are big problems: the membership problem, the implication problem, the aggregation problem and defuzzification problem. To solve these problems, different theories have been intensively developed. So, FMCR method, which depends on many hard factors, is not easy to define appropriately for a particular application. For example, the fuzzy relation as an output of the task b of (ii), which in nature is a mathematical representation of the fuzzy model (5.1) and depends on many hard factors, will make the model to become a black box. Therefore, dealing with this fuzzy relation, the developer is very hard to recognize the behaviour of model (5.1) and his FMCR method.

Now, we introduce another approach based on the quantification of the linear hedge algebras to solve the FMCR problem mentioned above, which may provide a clearer and more informative method than the above one. Suggested by an ordinary interpolative method and by fuzzy interpolative methods investigated in, e.g. [12,13], the idea of our method can be described simply as follows: for a given fuzzy model (5.1), we regard each if–then sentence of this model as defining a point and, hence, this model describes a fuzzy curve \( C_f \) in the Cartesian product \( X \times Y \), where \( X \) and \( Y \) are considered as underlying sets of the linear ComHAS of the linguistic variables \( \mathcal{X} \) and \( \mathcal{Y} \), respectively. Then, the FMCR problem may be understood as an interpolative problem with respect to the fuzzy curve \( C_f \) in \( X \times Y \). It seems to be very similar to the traditional interpolation and this makes the FMCR problems becoming very clear. In particular, one may recognize the behaviour of the fuzzy model (5.1), based on the shape of the fuzzy curve \( C_f \) which in nature is similar to the given real curve \( C \).

So, main steps of the new method are roughly as follows:

1. Construct SQMs \( u_X \) and \( u_Y \), which map \( X \) and \( Y \) into \([0, a]\) and \([0, b]\), respectively. These mappings are computed using Definition 3.4 and determining the respective parameters \( \mu(h), h \in H, \) and \( \text{fm}(c^–) \).

2. Under \( u_X \) and \( u_Y \), the fuzzy curve \( C_f \) is transformed into a real curve \( C_t \) in \([0, a] \times [0, b]\), where \([0, a]\) and \([0, b]\) are the domains of the basic variables of \( \mathcal{X} \) and \( \mathcal{Y} \), respectively.

3. Apply an ordinary linear interpolative method (in general, we may use any other ones) to the obtained curve \( C_t \) to compute outputs corresponding to the given inputs.

In Step 1, one actually needs to determine the parameters \( \mu(h) \) with \( h \in H \) and \( \text{fm}(c^–) \). Although, it is not an easy problem but it is still much simpler than the determination of the factors in building an FMCR method mentioned above. For instance, we may use our intuitive knowledge to establish some conditions made on these parameters, e.g. we require that \( \mu(\text{More}) > \mu(\text{Very}) \) and/or \( \mu(\text{Possibly}) > \mu(\text{Little}) \), since \( \text{Very} \) and \( \text{Little} \) are less vague than \( \text{More} \) and \( \text{Possibly} \), respectively. Note that, it is different from the usual applications of fuzzy sets, in our method we may assume in general that \( \text{fm}(c^–) \not= \frac{1}{r} \) and, since the shape of the fuzzy curve \( C_f \) is similar to the real curve, the parameters \( \mu(h), h \in H, \) and \( \text{fm}(c^–) \) can be determined based on the observation of the shape of the real curve formed by the collected experiment data. In addition, it can be seen that the new method provides a possibility to develop a learning algorithm to adapt applications by regulating the parameters of the chosen SQMs.

To evaluate the new method, we re-examine again the problem investigated by Cao and Kandel in [2]. Their aim was to examine the applicability of 72 implication operators. To do this, they designed seven examples of real curves with the different shapes given in Figs. 5(b)–(h) that express the dependencies of the rotation speed \( N \) of a series of electrical motors on the current intensity \( I \). These curves can be considered as being determined by a collection of real data and their fuzzy models are given in Table 2. Applying traditional FMCR methods they computed approximate values of the real curves based on the fuzzy models and shown that among 72 given operators, five operators named by \( 5^*, 22^*, 8, 25 \) and \( 31 \) are best applicable in the sense that for any curves defined by seven examples and for a small modification of the values of the designed membership functions, they cause the method errors less than the maximal model error. Here, the model error of these fuzzy models has been defined in [2] to be 400. The maximal method errors of these operators for these curves are given in Table 1.

To compare two methods, we shall again compute approximate values of the same real curves, using the proposed method and show that the maximal method errors become much smaller than the above ones.

It is observed that the results of the examination in [2] depend on the following subjective factors: (i) The fuzzy models which are formulated to model the real curves; (ii) The FMCR methods, whose key logical factor is the implication operator one uses to compute the fuzzy relations representing the respective sentences in (5.1). The given real curves and the model errors are criteria to evaluate an applicability of implication operators or, more generally, of approximate
Table 1

<table>
<thead>
<tr>
<th>Methods</th>
<th>EX1</th>
<th>EX2</th>
<th>EX3</th>
<th>EX4</th>
<th>EX5</th>
<th>EX6</th>
<th>EX7</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Max error of Cao–Kandel method with operator 5*</td>
<td>200</td>
<td>300</td>
<td>400</td>
<td>400</td>
<td>80</td>
<td>400</td>
</tr>
<tr>
<td>2</td>
<td>Max error of Cao–Kandel method with operator 22*</td>
<td>200</td>
<td>350</td>
<td>400</td>
<td>400</td>
<td>80</td>
<td>200</td>
</tr>
<tr>
<td>3</td>
<td>Max error of Cao–Kandel method with operator 8</td>
<td>300</td>
<td>350</td>
<td>200</td>
<td>150</td>
<td>200</td>
<td>100</td>
</tr>
<tr>
<td>4</td>
<td>Max error of Cao–Kandel method with operator 25</td>
<td>300</td>
<td>400</td>
<td>200</td>
<td>150</td>
<td>200</td>
<td>100</td>
</tr>
<tr>
<td>5</td>
<td>Max error of Cao–Kandel method with operator 31</td>
<td>300</td>
<td>400</td>
<td>200</td>
<td>150</td>
<td>200</td>
<td>100</td>
</tr>
<tr>
<td>6</td>
<td>Maximal error caused by our method with operator 0, ( \alpha = 0.4, \mu(L) = 0.2, \mu(M) = 0.3, \mu(V) = 0.3 ), for I</td>
<td>353</td>
<td>800</td>
<td>412</td>
<td>229</td>
<td>248</td>
<td>248</td>
</tr>
<tr>
<td>7</td>
<td>Maximal error caused by our method with operator 0.628, ( \alpha = 0.4, \mu(L) = 0.22, \mu(P) = 0.18, \mu(M) = 0.28, \mu(V) = 0.32 ), for I</td>
<td>254</td>
<td>228</td>
<td>104</td>
<td>77</td>
<td>66</td>
<td>104</td>
</tr>
<tr>
<td>8</td>
<td>Maximal error caused by our method with operator 0.628, ( \alpha = 0.4, \mu(L) = 0.22, \mu(P) = 0.18, \mu(M) = 0.28, \mu(V) = 0.32 ), for I</td>
<td>236</td>
<td>197</td>
<td>102</td>
<td>80</td>
<td>77</td>
<td>47</td>
</tr>
</tbody>
</table>

Table 2

Fuzzy models for EX1–EX7 examined in [2]

<table>
<thead>
<tr>
<th>EX1</th>
<th>Values of I</th>
<th>Values of N</th>
<th>EX2</th>
<th>Values of I</th>
<th>Values of N</th>
<th>EX3</th>
<th>Values of I</th>
<th>Values of N</th>
<th>EX4</th>
<th>Values of I</th>
<th>Values of N</th>
</tr>
</thead>
<tbody>
<tr>
<td>Null</td>
<td>Very_Large</td>
<td>Null</td>
<td>Very_Large</td>
<td>Null</td>
<td>Very_Large</td>
<td>Null</td>
<td>Very_Large</td>
<td>Null</td>
<td>Very_Large</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Zero</td>
<td>Large</td>
<td>Zero</td>
<td>Zero</td>
<td>Small</td>
<td>Medium</td>
<td>Small</td>
<td>Zero</td>
<td>Small</td>
<td>Small</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Small</td>
<td>Medium</td>
<td>Small</td>
<td>Large</td>
<td>Medium</td>
<td>Medium</td>
<td>Medium</td>
<td>Medium</td>
<td>Large</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Medium</td>
<td>Small</td>
<td>Medium</td>
<td>Medium</td>
<td>Large</td>
<td>Large</td>
<td>Very_Large</td>
<td>Medium</td>
<td>Very_Large</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Large</td>
<td>Zero</td>
<td>Very_Large</td>
<td>Zero</td>
<td>Very_Large</td>
<td>Very_Large</td>
<td>Very_Large</td>
<td>Zero</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Very_Large</td>
<td>Zero</td>
<td>Very_Large</td>
<td>Very_Large</td>
<td>Very_Large</td>
<td>Very_Large</td>
<td>Very_Large</td>
<td>Zero</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>EX5</th>
<th>Values of I</th>
<th>Values of N</th>
<th>EX6</th>
<th>Values of I</th>
<th>Values of N</th>
<th>EX7</th>
<th>Values of I</th>
<th>Values of N</th>
</tr>
</thead>
<tbody>
<tr>
<td>Null</td>
<td>Very_Large</td>
<td>Null</td>
<td>Zero</td>
<td>Large</td>
<td>Small</td>
<td>Large</td>
<td>Medium</td>
<td>Very_Large</td>
</tr>
<tr>
<td>Zero</td>
<td>Very_Large</td>
<td>Zero</td>
<td>Zero</td>
<td>Small</td>
<td>Medium</td>
<td>Medium</td>
<td>Large</td>
<td></td>
</tr>
<tr>
<td>Small</td>
<td>Large</td>
<td>Small</td>
<td>Large</td>
<td>Large</td>
<td>Medium</td>
<td>Large</td>
<td>Very_Large</td>
<td></td>
</tr>
<tr>
<td>Medium</td>
<td>Large</td>
<td>Medium</td>
<td>Large</td>
<td>Medium</td>
<td>Medium</td>
<td>Medium</td>
<td>Large</td>
<td></td>
</tr>
<tr>
<td>Large</td>
<td>Medium</td>
<td>Large</td>
<td>Large</td>
<td>Medium</td>
<td>Medium</td>
<td>Large</td>
<td>Very_Large</td>
<td></td>
</tr>
<tr>
<td>Very_Large</td>
<td>Zero</td>
<td>Very_Large</td>
<td>Very_Large</td>
<td>Very_Large</td>
<td>Very_Large</td>
<td>Very_Large</td>
<td>Zero</td>
<td></td>
</tr>
</tbody>
</table>

reasoning methods. They are objective and, therefore, will also be used as the criteria for evaluating the applicability of the new method. This means that to determine new method we can change the fuzzy models and the method of reasoning, but must use the same criteria to evaluate the method.

Now, for each fuzzy model and for the given real values of \( I \) as inputs, the corresponding output values of \( N \) will be computed using the new method and they are given in row No. 6 of Table 1. The maximal result errors are two large in EX2 (Error = 800) and rather large (greater than 300) in EX1 and EX3, despite our effort to find several systems of the parameters \( \alpha \) and \( \beta \), \( \beta \in \mathbb{R} \). The reason is that the fuzzy models describing the real curves is not appropriate, since the right-end triangle with the membership function \( \mu_{V_{\text{Large}}} \) of the term ‘VLarge’ of the variable \( I \), defined on the reference domain \([0.0,10.0] \) (see the dash line triangle in Fig. 5(a)), must represent the meaning of another term, which is considerably greater than VLarge. Indeed, since \( \mu_{V_{\text{Large}}} (u) = 1 \) only when \( u = 2000 \), or it attains the greatest value only at the right-end point of the basic variable interval, there is no another triangle with the membership function \( \mu_A \) lying properly on the right side of the triangle \( \mu_{V_{\text{Large}}} \) in the sense that there exists an \( u \in [0.0, 10.0] \) such that \( \mu_{V_{\text{Large}}} (u) = 1 > \mu_A (u) \). And, since linguistic terms are mere labels of fuzzy sets and since the fuzzy set with the membership function \( \mu_{V_{\text{Large}}} \) was shown in [2] to be appropriate to solve EX2, we can assign another term which has the meaning more compatible with this triangle.
Note that the proposed algorithm presented in Section 5.1.1 for building fuzzy sets is aimed to ensure that the output fuzzy sets will be compatible with the meaning of their labels and have a closed relation to the FM of terms. By this algorithm, the triangle associated with the term \textit{VV Large} has the base side of length 0.46875 and it is represented by the right-end triangle in Fig. 5(a). Observing Fig. 5(a), we see that the triangle \textit{\mu_{VV Large}} with dash line examined in [2], whose base side equals 1.0, is more similar to the triangle with label ‘\textit{VV Large}’ than the one with label ‘\textit{VLarge}’ constructed by the method described in Section 5.1.1. A similar comment may be made for the terms \textit{Null} and \textit{Zero} of the variable \(I\) and for the terms of the variable \(N\). Since the membership functions examined in [2] were shown to be
appropriate to the problem considered there, their labels must be understood as unsuitable. Because our method deals with the linguistic labels directly, before performing our method unsuitable verbal descriptions for the both linguistic variables in the fuzzy models must be changed as follows: \( \text{Null} := \text{VV Small} \), \( \text{Zero} := \text{VM Small} \), \( \text{V Large} := \text{VV Large} \). The other verbal descriptions remain unchanged.

The second negative effect of the new method is caused by the parameter \( \mu(c^-) \). Usually, in the fuzzy sets framework one chooses the neutral value to be 0.5 or, in our formalism, \( \mu(c^-) = 0.5 \). However, it causes large result errors for the new method. Analysing the results of our computation process we recognize that \( \mu(c^-) \) must be greater than 0.5. For example, for the examples under consideration, the value \( \mu(c^-) \) is chosen to be 0.628 instead of 0.5, the FM of hedges \( \mu(h) \) of variable \( I \) are given in row No. 6 and the parameters for the variable \( N \) are chosen the same for all examined examples. Then, the results computed by the new method are shown by graphic figures in Fig. 5(b)–(h) and the result errors are given in row No. 7 of Table 1, which are obviously much improved.

To point out that the method also provides a flexible tool, the row No. 8 shows that when we regulate the parameters, the results are still improved better. Comparing the computing results given in row Nos. 7 and 8, it is shows that the FM of hedges also has a meaningful influence on the output results.

References