Five-Coloring Graphs on the Torus

CARSTEN THOMASSEN

Mathematical Institute, Technical University of Denmark,
DK-2800 Lyngby, Denmark

Received March 22, 1993

We prove that a graph on the torus is 5-colorable, unless it contains either $K_6$, the complete graph on six vertices, or $C_1 + C_5$, the join of two cycles of lengths three and five, respectively, or $K_2 + H_7$, the join of $K_2$ and the graph $H_7$ on seven vertices obtained by applying Hajos' construction to two copies of $K_4$, or a triangulation $T_{11}$ with 11 vertices of the torus. This answers questions of Albertson, Hutchinson, Stromquist, and Straight. © 1994 Academic Press, Inc.

1. INTRODUCTION

Heawood [10] proved that a graph on $S_g$, the sphere with $g$ handles added, can be colored in at most $h(g) = \left\lfloor \frac{5}{2}(7 + \sqrt{48g + 1}) \right\rfloor$ colors (for $g \geq 1$). Ringel and Youngs (see [11]) proved that this is best possible. However, Dirac [5] proved that a graph on $S_g$ can be colored in fewer than $h(g)$ colors unless it contains a complete graph on $h(g)$ vertices. In Dirac's terminology, $K_{h(g)}$ is the only $h(g)$-color-critical graph on $S_g$. More generally, Dirac [6] proved that, for each fixed nonnegative integer $g$ and each fixed natural number $k \geq 8$, there are only finitely many $k$-color-critical graphs on $S_g$.

Gallai [8] proved that the vertices of degree $k - 1$ in a $k$-critical graph induce a subgraph whose blocks are either odd cycles or complete graphs. This implies that a 7-critical graph $G$ on $S_g$ has at most $96(g - 1)$ vertices. For if $G$ has more than $96(g - 1)$ vertices, then Euler's formula, combined with the fact that all vertices in a $k$-color-critical graph have degree $\geq k - 1$, implies that $G$ has a vertex $v$ of degree 6, such that all neighbors of $v$ have degree 6 and such that all faces incident with $v$ are bounded by triangles. Hence $v$ and its neighbors belong to a block in the subgraph induced by the vertices of degree 6. As that block is not an odd cycle, it is complete. Hence $G \not\supset K_7$, a contradiction because a 7-critical graph does not contain a 7-critical graph as a proper subgraph.

By similar arguments it follows that for each natural number $k \geq 7$ and for each natural number $m \geq 3$ there are only finitely many $k$-color-critical
graphs on $N_m$, the sphere with $m$ crosscaps added. Fisk (see [3]) gave a construction which implies that there are infinitely many 5-critical graphs on the torus and the projective plane and hence on each $S_g, N_m$, where $g \geq 1, m \geq 1$. We point out that there are infinitely many 5-vertex-critical 6-regular graphs on the torus. This raises the following general question.

**Question 1.** Let $S$ be a fixed surface, $S = S_g \ (g \geq 1)$ or $N_m \ (m \geq 2)$. Do there exist infinitely many 6-critical graphs on $S$?

A negative answer would imply the result in [16] that a graph on $S$ with no short noncontractible cycles is 5-colorable. It would also imply a positive answer to the following question for $k = 5$.

**Question 2.** Let $S$ be a fixed surface and let $k$ be a fixed natural number. Does there exist a polynomially bounded algorithm for deciding if a given graph on $S$ has a $k$-coloring?

For $k = 3$ this problem is NP-complete even for the sphere (see [9]). For $k \geq 6$ it is answered in the affirmative by the above results of Dirac and Gallai. So the cases $k = 4, 5$ remain.

In this paper we answer Question 1 in the negative for the torus. This special case of Question 1 was raised by Albertson and Hutchinson [2] who proved (using the 4-color theorem) that there exists precisely one 6-chromatic 6-regular graph $T_{11}$ on the torus. The problem is also mentioned in [3, Question 3]. Our result (which is independent of the 4-color theorem) has several consequences as mentioned below.

It implies the conjecture ([2, Conjecture 1; 3, Question 1]) that every 6-chromatic graph on the torus must contain at least one noncontractible triangle. For, if $G$ is one of the graphs in the abstract, then the number of triangles in $G$ exceeds the number of face boundaries in any embedding of $G$ on the torus. So $G$ has a triangle, which is not a face boundary. Since $G$ has no separating triangle, the above triangle is noncontractible.

This result can be applied to the conjecture of Straight [13] that every toroidal graph is 5-cocolorable. (In a $k$-cocoloring of a graph each color class is either a complete graph or a set of pairwise nonadjacent vertices). Straight's conjecture clearly implies the 4-color theorem. For, if $G_0$ is a counterexample to the latter, then the union of $K_6$ and five pairwise disjoint copies of $G_0$ is a counterexample to the former. On the other hand, the 4-color theorem combined with the result described in the abstract implies that every toroidal graph $G$ has a 5-cocoloring. For, if $G$ has no noncontractible 3-cycle, then $G$ is 5-colorable by the main result of this paper. On the other hand, if $G$ has a noncontractible 3-cycle, then we color its vertices by color 5 and color the remaining planar graph in colors 1, 2, 3, 4.
The result described in the abstract gives a positive answer to Question 2 for the torus and for \( k = 5 \). Our proof also gives a polynomial time algorithm for actually describing the 5-coloring if it exists.

Finally, our result implies Tutte’s 5-flow conjecture restricted to toroidal graphs. (For definition of 5-flows, see, e.g., [12], where Tutte’s 5-flow conjecture is verified for graphs in the projective plane.) We sketch the proof in the toroidal case. Let \( G \) be a connected, 2-edge-connected (i.e., bridgeless) graph on the torus. If the (geometric) dual graph \( G^* \) of \( G \) is 5-colorable, then \( G \) has a 5-flow. If \( G^* \) contains one of the graphs in the abstract, then \( G \) is contractible to a graph \( H \), which is a (geometric) dual graph of one of the graphs in the abstract. By inspection, \( H \) has a 5-flow. That 5-flow can be extended to a 5-flow of \( G \). Finally, \( G^* \) may have a loop. Then \( G \) has an edge \( e \) such that \( G - e \) is planar. But, it is an easy exercise to extend the dualized proof of the 5-color theorem to show that \( G \) has a 5-flow.

As mentioned earlier, examples of Fisk show that there are 5-chromatic graphs on the torus, which have no short noncontractible cycles. Stromquist asked if any such graph must contain one of Fisk’s examples, which are all triangulations of the torus with precisely two vertices of odd degree. The answer is negative. For, if \( v_0 \) is a vertex of degree 4 in one of Fisk’s examples and \( v_1v_2v_3v_4 \) is the cycle where \( v_1, v_2, v_3, v_4 \) are the neighbors of \( v_0 \), then we delete \( v_0 \) and add instead new vertices \( u_0, u_1, u_2, u_3, u_4 \) and the cycles \( u_1u_2u_3u_4u_1 \) and \( v_1u_1u_2u_3v_3v_4u_4v_1 \) and the four edges from \( u_0 \) to \( \{u_1, u_2, u_3, u_4\} \). The new graph is 5-chromatic and toroidal and does not contain one of Fisk’s examples. So, the class of 5-critical graphs on the torus may be complicated.

2. Terminology

A graph \( G \) consists of a finite vertex set \( V(G) \) and a set \( E(G) \) of unordered pairs of vertices called edges. If the edge \( xy \) is present we say that \( x \) is adjacent to \( y \) or is joined to \( y \) and that \( x \) and \( y \) are neighbors. The degree of a vertex \( x \) is the number of neighbors. It is denoted \( d_G(x) \) or just \( d(x) \). If all vertices have degree \( r \), then \( G \) is \( r \)-regular or just regular. If \( H \) is a subgraph of \( G \), then \( G(H) \), the subgraph of \( G \) induced by \( H \), consists of \( H \) and all edges in \( G \) joining two vertices of \( H \). We define \( G - H = G(V(G) \setminus V(H)) \). If \( v \) is a vertex in a graph \( G \), then \( N(v, G) \) or just \( N(v) \) is the subgraph of \( G \) induced by the neighbors of \( v \). If \( G \) and \( H \) are graphs, then \( G \cup H \) is the disjoint union of \( G \) and \( H \), unless it is clear from the context that they are both subgraphs of a given graph. The join \( G + H \) of \( G \) and \( H \) is obtained from \( G \cup H \) by adding all edges from \( G \) to \( H \).

A walk in a graph \( G \) is a sequence \( x_1x_2 \cdots x_n \) of vertices such that \( x_i \) and \( x_{i+1} \) are neighbors for \( i = 1, 2, \ldots, n - 1 \). If all vertices are distinct, the walk
is called a path or an \( n \)-path and is denoted \( P_n \). The \( n \)-cycle \( C_n \) is obtained from \( P_n \) by adding \( x_n x_1 \). If \( C \) is a cycle in a graph and \( e \) is an edge joining two nonconsecutive vertices of \( C \), then \( e \) is a chord of \( C \). A cycle \( C \) in \( G \) is a Hamiltonian cycle if \( V(C) = V(G) \). \( K_n \) is the complete graph with \( n \) vertices; i.e., all vertices of \( K_n \) have degree \( n - 1 \). A complete subgraph in a graph \( G \) is also called a clique in \( G \). The clique number of \( G \) is the maximum number of vertices of a clique in \( G \).

The graph \( G' \) obtained from a graph \( G \) by identifying two vertices \( x, y \) into a vertex \( v \) is obtained by first deleting \( x, y \), and then adding a new vertex \( v \) and all edges \( vz \), where either \( zx \) or \( zy \) or both are present in \( G \). The multigraph \( G'' \) obtained from \( G \) by identifying \( x \) and \( y \) is obtained from \( G' \) by adding an additional edge \( vz \) for every vertex \( z \) which in \( G \) is joined to both \( x \) and \( y \). We refer to such an edge \( vz \) as a double edge. The operation of going from \( G' \) to \( G \) is called splitting \( v \) into \( x \) and \( y \). If \( x \) and \( y \) are neighbors in \( G \), then \( G' \) (respectively \( G'' \)) is the graph (respectively multigraph) obtained by contracting the edge \( xy \). We shall sometimes write \( v = y \) and say that \( G' \) or \( G'' \) is obtained from \( G \) by contracting \( xy \) into \( y \).

A \( k \)-coloring of a subgraph \( H \) of \( G \) is a map \( c: V(H) \to \{ 1, 2, \ldots, k \} \) (the color set) such that any two neighboring vertices are mapped to distinct colors. If a graph \( G \) has a \( k \)-coloring, it is called \( k \)-colorable. \( G \) is called \( k \)-critical (respectively \( k \)-vertex-critical) if \( G \) is not \((k - 1)\)-colorable, but every proper subgraph (respectively every subgraph with fewer vertices) is \((k - 1)\)-colorable. \( G \) is \( k \)-chromatic if \( G \) is \( k \)-colorable but not \((k - 1)\)-colorable. Gallai [8] proved the following.

**Theorem 2.1.** If \( G \) is a \( k \)-critical graph with at most \( 2k - 2 \) vertices, then \( G \) is of the form \( G = G_1 + G_2 \), where \( G_i \) is \( k_i \)-critical for \( i = 1, 2 \) and \( k_1 + k_2 = k \).

It is easy to see that Theorem 2.1 remains true if "\( k \)-critical" is replaced by "\( k \)-vertex-critical."

Let \( H_7 \) be obtained by applying Hajos' construction to \( K_4 \). That is, \( H_7 \) is obtained from \( K_4 \cup K_4 \) by deleting an edge \( xy \) in one of the \( K_4 \)'s, an edge \( uv \) in the other \( K_4 \). Then we add \( yv \) and identify \( x \) and \( u \). Now \( H_7 \) is 4-critical. There is precisely one other 4-critical graph on seven vertices, namely \( M_7 \), which is obtained from a 6-cycle \( x_1 x_2 \cdots x_6 x_1 \) by adding a new vertex \( v \) and the edges \( x_1 x_3, x_3 x_5, x_5 x_1, vx_2, vx_4, vx_6 \); see [8, 17].

\( S_g \) is the sphere with \( g \) handles added. We shall here only consider the sphere \( S_0 \) and the torus \( S_1 \). \( S_1 \) is homeomorphic to a surface obtained by pasting triangles in the plane together. A graph embedded on a surface \( S \) is a graph on \( S \) such that the edges are polygonal arcs which do not intersect except at a common vertex. We shall speak of the clockwise ordering of the edges incident with a vertex \( v \). A face of an embedded graph is an
arcwise connected component of the surface minus the graph. We shall assume that each face is homeomorphic to a disc. A facial walk is obtained by turning sharp left at every vertex. If this walk has length $k$ we speak of a $k$-face. A facial cycle is a cycle which is also a facial walk. If $G$ is embedded in $S_1$ and $H$ is a subgraph of $G$, then $H$ is also embedded in $S_1$. We speak of the induced embedding. If $C$ is a facial walk of $H$ bounding a face which is homeomorphic to a disc, then that face is the interior of $C$ and is denoted int$(C, H)$ or just int$(C)$. The exterior ext$(C, H)$ is defined similarly. A vertex of $G$ in int$(C, H)$ is said to be inside $C$. If $C$ is a cycle of $G$ separating $S_1$ into components, one of which is homeomorphic to a disc, then $C$ is contractible. Otherwise, $C$ is noncontractible. A triangulation is an embedded graph such that every face is bounded by a triangle (3-cycle). A near-triangulation of the plane is a graph in the plane such that each face, except possibly one, which we call the outer face, is bounded by a 3-cycle. The outer face should be bounded by a cycle, the outer cycle.

The excess of a facial walk is the length of the walk minus 3. The facial excess of the embedding is the excess sum taken over all facial walks.

A graph is planar, respectively toroidal, if it can be embedded on $S_0$ or $S_1$, respectively.

Euler’s formula implies:

**Lemma 2.2.** (a) If $G$ is a multigraph with $n$ vertices and $e$ edges embedded in $S_1$ such that no face is bounded by a 2-cycle, then the facial excess equals $3n - e$.

(b) A necessary condition for a graph $G$ to be toroidal is that $e \leq 3n$. If equality holds then each $N(v)$ has a Hamiltonian cycle. Moreover, it is possible to select a Hamiltonian cycle in each $N(v)$ such that every edge of $G$ is covered twice by such Hamiltonian cycles.

It is well known that $K_7$ is toroidal. Duke and Haggard [7] proved the following.

**Proposition 2.3.** The minimal nontoroidal graphs on eight vertices are obtained from $K_8$ by deleting the edges of one of the following three graphs: $K_3$, $K_{2,3}$, $K_2 \cup K_2 \cup P_3$.

We use the fact that any graph which contains a subgraph which can be contracted to any of the three graphs above is nontoroidal.

**Lemma 2.4.** If $K_6$ is embedded on $S_1$, then every facial walk is a cycle.

**Proof.** Suppose (reductio ad absurdum) that $W: x_1, x_2, \ldots, x_m, x_1$ is a facial walk, which is not a cycle. Then we can assume that $x_i = x_i$ for some $i$, $1 < i < m - 2$. As $K_6$ has no double edges, $4 < i < m - 2$. If we cut $S_1$ along a simple polygonal curve from $x_1$ to $x_i$ inside the face bounded by $W$, then the
resulting topological space is connected (because \( x_2 \) and \( x_m \) are joined by an edge) and is therefore a cylinder. This cylinder and hence \( K_5 \) can be embedded in \( S_6 \), a contradiction.

We use a special case of the genus additivity theorem [4].

**Lemma 2.5.** If \( G_1 \) and \( G_2 \) are nonplanar graphs having at most one vertex in common, then \( G_1 \cup G_2 \) is nontoroidal.

We use the observation that, if a connected nonplanar graph is embedded on the torus, then each face is (homeomorphic to) a disc. Finally, we discuss the toroidal character of the graphs in the abstract. It is well known that \( K_7 \) triangulates the torus.

![Figure 1](image)

Figure 1 shows a graph in the plane. When the two faces bounded by pentagons are deleted and the pentagons are identified as indicated by the labeling, then the graph becomes \( K_2 + H_7 \) on the torus \( S_1 \). Let \( H_7 \) be obtained from \( H_7 \) by adding an edge between two nonadjacent vertices, which are both joined to the vertex of degree 4 in \( H_7 \). Then Fig. 1 can be extended to a toroidal drawing of \( K_2 + H_7 \). If we add an edge to \( K_2 + H_7 \), such that the resulting graph \( G \) is not isomorphic to \( K_2 + H_7 \), then \( G \) is nontoroidal. For if we contract the new edge, we obtain a nontoroidal
graph by Proposition 2.3. If we contract the new edge in $K_2 + H^*$, we obtain a toroidal embedding of $C_3 + C_5$, which is maximally toroidal. It is easy to see that $K_2 + H^*$ is 5-vertex-critical. As $T_{11}$ defined below is maximally toroidal, we obtain the curious corollary of Theorem 6.1: $K_2 + H^*$ is the only graph on the torus, which is 5-vertex-critical, but not 5-critical. This answers [3, Question 2].

Figure 2 shows an infinite graph obtained from a spiral in the plane. This map can be face-colored in 4-colors but only in one way (except for a permutation of the colors). Now we fix a natural number $m \geq 4$. We call the faces in the spiral in Fig. 2, $\ldots, F_{-1}, F_0, F_1, \ldots$. Then we identify $F_i$ with $F_{i+m}$ for $i \in \{\ldots, -1, 0, 1, \ldots\}$. This transforms the plane into the torus. We define $T_m$ as the dual graph of the resulting toroidal graph (i.e., the vertices of $T_m$ are the $m$ faces of the toroidal graphs of Fig. 2, and two vertices of $T_m$ are adjacent iff the corresponding faces are). Then $T_m$ is 4-colorable if and only if $m$ is divisible by four. For each $k \geq 1$, $T_{4k+1}$ is 5-vertex-critical. $T_{11}$ is 6-critical. $T_{11}$ is called $H_{11,6,e}$ in [15] and $J$ in [1, Fig. 1].

3. 5-COLORING 6-REGULAR GRAPHS ON THE TORUS

**Lemma 3.1.** Let $G$ be a planar graph with outer cycle $S$ such that all vertices inside $S$ have degree at least 6. Let $q$ be the number of vertices inside $S$. If $q \geq 5$, then $|V(S)| \geq 11.$
Proof. Put $|V(S)| = k$. By Euler’s formula, $G$ has at most $2k + 3q - 3$ edges. Hence $G - E(S)$ has at most $k + 3q - 3$ edges. On the other hand, the number of edges of $G - E(S)$ is at least $6q$ minus the number of edges in $G - V(S)$. Now $G - V(S)$ has $q$ vertices and less than $3q - 7$ edges. For if $G - V(S)$ has $3q - 7$ or more edges, then $G - V(S)$ is either a triangulation or a triangulation with one edge missing and with outer cycle of length 3 or 4. But it is not possible that all vertices inside that cycle have degree at least 6. Hence $k + 3q - 3 \geq 6q - (3q - 8)$ which implies that $k \geq 11$.

**Proposition 3.2.** Let $G$ be a 6-regular graph on the torus $S_1$. If $G$ contains a vertex $v$, such that $\{v\} \cup N(v)$ induces a nonplanar graph, then $G = K_7$ or $G$ is obtained from $K_8$ or $K_9$ by deleting the edges of a 1-regular or 2-regular subgraph. In particular, $G$ is 5-colorable unless $G = K_7$.

Proof. By Euler’s formula, $G$ triangulates $S_1$. Let $S: v_1 v_2 \cdots v_k v_1$ be the facial cycle of $G - v$ having $v$ in its interior. As $\{v\} \cup N(v)$ induces a nonplanar graph, we can choose the notation such that $S$ has two chords $v_j v_i$, $v_j v_k$, where $1 < j < i < k \leq 6$. The two 3-cycles $v v_j v_i$ and $v v_j v_k$ are noncontractible because $G$ is 6-regular. Hence none of these triangles separate $S_1$. We consider the induced embedding of the subgraph $G'$ consisting of $v$, $S$, the edges incident with $v$, and $v_1 v_i$, $v_j v_k$. Then $G'$ has six 3-faces and one 10-face. By Lemma 3.1, $G$ has at most four vertices not in $G'$. However, it is easy to see that we cannot add chords to $S$ and then add four or three vertices to $G'$ inside the 10-face and obtain a 6-regular graph. So $|V(G)| = 7, 8$, or 9.

**Theorem 3.3.** Let $G$ be a 6-regular graph on the torus $S_1$. Then $G$ is 5-colorable unless $G = K_7$ or $G = T_{11}$.

Proof. By Proposition 3.2, we can assume that, for each vertex $v$, $\{v\} \cup N(v)$ induces a planar graph. The structure of $G$ is now completely described in [15]. By [15, Theorem 3.2], $G$ is a dual graph of one of the graphs described in [15, Theorem 3.1]. It is now easy to face-color the graphs of [15, Theorem 3.1] in five colors such that neighboring faces obtain different colors unless $G = T_{11}$.

The triangulation $T_{11}$ was found by Albertson and Hutchinson [2], who used the 4-color theorem to prove that it is the only 6-regular, 6-chromatic graph on the torus. Our proof does not depend on the 4-color theorem. Also, it easily generalizes to an analogous result for the Klein bottle, and it can also be extended to characterize the 6-regular graphs on the torus, which are not 4-colorable. These are all of the form $H_{k,m,e}$ in [15].
4. 5-COLORING PLANAR GRAPHS WITH PRECOLORED OUTER CYCLE

Proposition 4.1. Let $G$ be a planar graph with outer cycle $S: x_1 x_2 \cdots x_k x_1$, $k \leq 6$. Let $c$ be a 5-coloring of $G(S)$. Then $c$ can be extended to a 5-coloring of $G$ if and only if none of (i), (ii), (iii) below hold:

(i) $S$ has five colors. $G$ has a vertex joined to all five colors of $S$.

(ii) $k = 6$, and $S$ has precisely four colors. $G - S$ contains two adjacent vertices each joined to all four colors of $S$.

(iii) $k = 6$, and $S$ has precisely three colors. $G - S$ contains three pairwise adjacent vertices each of which is joined to all three colors of $S$.

Proof. If either (i), (ii), or (iii) holds, then clearly $G$ is not 5-colorable. Assume now that none of (i), (ii), (iii) holds. We prove by induction on $|V(G)|$ that $c$ can be extended to a 5-coloring of $G$. For $|V(G)| \leq 3$, there is nothing to prove so assume that $|V(G)| \geq 4$.

Let $v$ be a vertex joined to say $m$ vertices of $S$, where $m$ is largest possible. Since (i) does not hold, the coloring $c$ can be extended to $v$.

If $m \geq 4$, we apply the induction hypothesis to each face of $G(S \cup \{v\})$. So assume that $m \leq 3$.

If $m = 3$, then we easily complete the proof by induction, unless $v$ is joined to three consecutive vertices of $S$, say $x_1, x_2, x_3$. We apply the induction hypothesis to $S' = x_1vx_3x_4 \cdots x_1$ and its interior. We cannot have (i) as $m \leq 3$.

Assume that $G$ has two adjacent vertices $v_1, v_2$ joined to the same four colors of $S'$. As $m = 3$ both $v_1$ and $v_2$ are joined to $v$. Uncoloring $v$, there are two choices for each of $c(v), c(v_1), c(v_2)$. It is possible to obtain the 5-coloring unless each of $v, v_1, v_2$ are joined to the same three colors of $S$. That is, (iii) holds, a contradiction.

Assume next that (iii) holds for $G - x_2$; i.e., $G - (S \cup \{v\})$ has three pairwise adjacent vertices $v_1, v_2, v_3$ such that each of them is joined to the same three colors of $S'$. If $v$ is joined to only one of $v_1, v_2, v_3$, say $v_3$, then we can color $v_1, v_2, v_3, v$ in that order. So we can assume that $v$ is joined to $v_1, v_2$ and that $G$ has the edges $v_2x_3, v_2x_4, v_3x_4, v_3x_5, v_3x_6, v_1x_6, v_1x_1$. By (iii) we can assume that $c(x_4) = 1, c(x_5) = 2, c(x_6) = 3$. Since also $v_1$ is joined to all three colors of $S'$, $c(v) \in \{1, 2\}$. Since $v_2$ is joined to all three colors of $S'$, $c(v) \in \{2, 3\}$. Hence $c(v) = 2$. But when we colored $v$ we had two choices. Therefore we can assume that $c(v) \neq 2$. This contradiction completes the case $m = 3$.

Assume finally that $m \leq 2$. By Lemma 3.1, $G - S$ has a vertex $v$ of degree $\leq 5$ (in $G$). We can assume that $G$ has no separating triangle, since otherwise we complete the proof by induction. Hence $v$ has two neighbors $u_1, u_2$, which are not adjacent and which are not on $S$. Now delete
and identify \( u_1, u_2 \). If the resulting graph \( G' \) is 5-colorable, then so is \( G \). So we can assume that \( G' \) satisfies either (i), (ii), or (iii). But then \( m \geq 3 \). This contradiction completes the proof.

**Corollary 4.2.** Let \( G \) be a 6-vertex-critical graph on some surface. Let \( H \) be a connected induced subgraph of \( G \) (i.e., \( H = G(H) \)). If each facial walk of \( H \) has length 3 or 4, then \( H = G \).

**Proof.** If \( H \neq G \), then \( H \) is 5-colorable. Any 5-coloring of \( H \) can be extended to a 5-coloring of \( G \) by Proposition 4.1. But \( G \) is not 5-colorable, so \( H = G \).

### 5. Chromatic Properties of \( C_3 + C_5 \) and \( K_2 + H_7 \)

**Lemma 5.1.** Let \( G \) be a copy of \( C_3 + C_5 \) or \( K_2 + H_7 \). Let \( z_0 z_1 z_2 z_3 z_0 \) be a 4-cycle in \( G \) such that \( z_0 \) and \( z_2 \) are neighbors. Let \( G_1 \) be obtained from \( G \) by splitting \( z_0 \) into two nonadjacent vertices \( x, y \) such that \( G_1 \) contains the edges \( z_1 y, z_3 x, z_2 y, z_3 x \) and such that \( z_2 \) is the only vertex joined to both \( x \) and \( y \). Then \( G_1 \) has 5-colorings \( c, c', c'' \) such that

1. \( \{ c(x), c(z_3) \} \neq \{ c(y), c(z_1) \} \).
2. \( c'(y) = c'(z_3) \) or \( c'(x) = c'(z_1) \).
3. Either \( c''(y), c''(z_1), c''(z_2), c''(z_3), c''(x) \) are all distinct or \( c''(y), c(z_1), c''(z_2), c''(z_3) \) are not distinct.

**Proof of (a).** \( G_1 - x \) is a proper subgraph of \( G \) and is therefore 5-colorable. If \( x \) is joined to only 3-colors, then we have at least two choices for \( c(x) \) and we can obtain the desired coloring. So assume that \( x \) is joined to at least four colors. In particular \( x \) (and similarly \( y \)) has degree \( \geq 4 \) in \( G_1 \). Then \( z_0 \) has degree \( \geq 7 \) in \( G \), and so \( z_0 \) is joined to all other vertices of \( G \). If \( G = C_3 + C_5 \), then we can assume that some \( z_i \) \( (i = 1, 2, 3) \) is in the \( C_5 \). Then we 5-color \( G - z_0 \) such that at least one vertex distinct from \( z_i \) has the same color as \( z_i \). This 5-coloring can be extended to the desired 5-coloring of \( G \). (Note that, if \( c(z_1) = c(z_3) \), then \( x \) and \( y \) cannot be joined to the same four colors because that would result in a 5-coloring of \( G \).

We now assume that \( G = K_2 + H_7 \). Again both \( x \) and \( y \) have degree \( \geq 4 \) in \( G_1 \) (and \( z_0 \) has degree 8 in \( G \)). So we can assume that \( N(y) \) and \( N(x) \) have vertex sets \( \{ z_1, z_2, w_1, w_2, w_3 \} \) and \( \{ z_2, z_3, u_1, u_2 \} \), respectively. As \( G - \{ z_0, z_3 \} \) is 4-colorable, there exists a 5-coloring \( c \) of \( G_1 - x \) such that \( y \) and \( z_3 \) are the only vertices of color 1. Also we can assume that \( x \) is joined to four colors; i.e., \( u_1, u_2, z_2 \) have colors 2, 3, 4, respectively. If \( c(z_1) \neq 5 \), we color \( x \) by 5 and complete the proof. So assume that \( c(z_1) = 5 \).
If \( y \) is not joined to colors 2, 3, we change \( c(y) \) to 2 or 3 and complete the proof. So we can assume that \( c(w_1) = 2, \ c(w_2) = 3 \). We now interchange colors of \( w_1 \) and \( y \) (and change \( c(w_3) \) to 1 if \( c(w_3) = 2 \)). This new coloring can be extended to the desired 5-coloring unless \( z_3 \) is joined to \( w_1 \) (or to \( w_3 \) if \( c(w_3) = 2 \)). Similarly, \( z_3 \) is joined to \( w_2 \) (or to \( w_3 \) if \( c(w_3) = 3 \)). So \( z_3 \) is joined to at least two of \( w_1, w_2, w_3 \), say \( w_1 \) and \( w_2 \). Also, \( z_3 \) is joined to \( u_1 \) since otherwise we change \( c(u_1) \) to 1 and color \( x \) by 2. Similarly, \( z_3 \) is joined to \( u_2 \). So \( d(z_3) \geq 6 \). We have previously proved that in any 4-coloring of \( G - \{z_0, z_3\}, u_1, u_2, z_1, z_2 \) are forced to have distinct colors. Hence one of these vertices has degree 8. (That vertex must be one of \( z_1, z_2 \).) So \( z_3 \) is the unique vertex in \( G \) of degree 6. But if we 4-color \( G - \{z_0, z_3\} \) and then delete the vertex in \( G - \{z_0, z_3\} \) which is joined to all other vertices, then we obtain a 3-coloring of \( H_7 \) minus the vertex \( z_3 \) of degree 4 (in \( H_7 \)). In that 3-coloring the four neighbors of \( z_3 \) which are \( u_1, u_2, w_1, w_2 \) must have three distinct colors. This contradiction proves (a).

**Proof of (b).** At least one of \( x, y \) (say \( x \)) has degree \( \leq 4 \) in \( G_1 \). As in (a) we can 5-color \( G_1 - x \) such that \( y \) and \( z_3 \) are vertices of color 1. This proves (b).

**Proof of (c).** We can assume that \( x \) has degree \( \geq 5 \) in \( G_1 \), since otherwise we use the proof of (b). If \( y \) has degree 2 in \( G_1 \), then we first 5-color \( G_1 \) and then change the color of \( y \) if necessary. So we can assume that \( y \) has degree \( \geq 3 \) in \( G_1 \). Hence \( z_0 \) is joined to all other vertices of \( G \). Let \( w \) be a vertex in \( N(y) - \{z_1, z_2\} \). As \( G - \{w, z_0\} \) is 4-colorable, \( G_1 - y \) has a 5-coloring \( c'' \) such that \( x \) and \( w \) are the only vertices of color 5. Extend \( c'' \) to 5-coloring (which we also call \( c'' \)) of \( G_1 \). Then \( c'' \) has the desired property.

**Lemma 5.2.** Let \( G \) be a copy of \( C_3 + C_5 \). Let \( S: z_0z_1z_2z_0 \) be a 3-cycle in \( G \) and let \( u_1 \) be a vertex in \( G - S \) joined to \( z_0 \). Let \( G' \) be obtained from \( G \) by splitting \( z_0 \) into two nonadjacent vertices \( x \) and \( y \) such that \( u_1 \) and at most one more vertex \( u_0 \) in \( G' \) is joined to both \( x \) and \( y \) and such that \( yz_1z_2x \) is a path in \( G' \). Let \( G'' \) be obtained from \( G' \) by adding a vertex \( v_0 \) and joining \( v_0 \) to \( x, y, u_1, z_1, z_2 \). If \( G'' \) is toroidal and non-5-colorable, then \( G'' \cong C_3 + C_5 \).

**Proof.** Assume that \( G'' \not\cong C_3 + C_5 \). If one of \( x, y \) has the same neighbors in \( G' \) as \( z_0 \) does in \( G \), then clearly \( G' \not\cong C_3 + C_5 \), a contradiction. So assume that \( z_0 \) has two neighbors in \( G \) such that one is a neighbor of \( x \) but not \( y \), and the other is a neighbor of \( y \) but not \( x \). We can assume that each of \( x, y \) has degree at least 5 in \( G'' \), and hence \( z_0 \) has degree \( \geq 6 \) in \( G \). For if \( x, y \), say, has degree \( \leq 4 \) in \( G'' \), then
\( G'' - \{x, v_0\} \) is a proper subgraph of \( C_5 + C_5 \) and has therefore a 5-coloring. Any such 5-coloring can be extended to a 5-coloring of \( G'' \) by first coloring \( v_0 \) and then \( x \).

\( G \) consists of 5-cycle \( p_1, p_2, p_3, p_4, p_5, p_1 \) and a 3-cycle \( q_1, q_2, q_3, q_1 \) and all 15 edges \( p_i q_j (1 \leq i \leq 5, 1 \leq j \leq 3) \). As \( d_G(z_0) \geq 6, z_0 \in \{q_1, q_2, q_3\} \).

Consider first the case where \( z_0, z_1, z_2 \) are \( q_3, q_1, q_2 \), respectively. If both \( u_0 \) and \( u_1 \) are in \( \{p_1, p_2, p_3, p_4, p_5\} \), then we color \( y, z_1, z_2, x \) by 2, 1, 2, 1, respectively. The remaining vertices in \( G'' \) are colored 3, 4, 5. If \( u_1 = p_1 \) and \( u_0 = z_1 \), we color \( y, z_1, z_2, x, u_1 \) by 2, 1, 2, 3, 4, respectively. As \( y \) has degree \( \geq 5 \) in \( G'' \), some vertex in \( \{p_2, p_3, p_4, p_5\} \) can obtain color 3. The remaining vertices are colored by 4, 5.

Consider next the case where \( z_0, z_1, z_2 \) are \( q_1, p_1, p_2 \), respectively. We consider the subcase where \( u_0 \) is not in \( \{z_1, z_2\} \). Then we color \( y, z_1, z_2, x, u_0, u_1 \) by 2, 1, 2, 1, 3, 4, respectively. This coloring can be extended to a 5-coloring of \( G'' \) (coloring \( v_0 \) last), except in the following three cases (or cases which are equivalent to one of them by permuting colors or using automorphisms of \( G'' \) or renaming \( u_0 \) and \( u_1 \)): \( u_0 = q_2 \) and \( u_1 = p_4 \) (in which case we color \( q_3 \) by the same color as \( x \) or \( y \) and recolor either \( z_1 \) or \( z_2 \) by 4 and color the remaining vertices 5) or \( u_0 = p_3 \) and \( u_1 = p_4 \) (in which case we color \( q_3 \) by 1 or 2 and recolor \( z_1 \) or \( z_2 \) by 4 as in the previous case; then we color \( p_5, q_2 \) by 3, 5, respectively). Finally, if \( u_1 = p_3 \) and \( u_0 = p_5 \), then color \( q_3 \) by 1 or 2 and recolor one of \( z_1, z_2 \) by 3 and we recolor \( p_3, p_4, p_5, q_2 \) by 4, 3, 4, 5, respectively.

Consider now the subcase where \( z_0, z_1, z_2 \) are \( q_1, p_1, p_2 \), respectively, and \( u_0 \in \{p_3, p_4, p_5\} \), say \( u_0 = p_2 \). If \( u_1 \in \{p_3, p_4, p_5\} \), then we color \( y, z_1, z_2, x \) by 2, 4, 3, 1, and we color \( u_1 \) by 3 unless \( u_1 = p_3 \), in which case we color it by 4. Then we color one of \( q_2, q_3 \) by 1 or 2. If both \( q_2, q_3 \) can be colored 1, 2, then it is easy to complete the coloring. So we can assume that \( q_2, q_3 \) are colored by 2, 5, respectively, and that both \( q_2, q_3 \) are adjacent to \( x \). (If they are both adjacent to \( y \), the proof is similar.) Since \( y \) has degree at least 4 in \( G' \), at least one vertex in \( \{p_3, p_4, p_5\} \setminus u_1 \) is joined to \( y \) and is colored 1. Now it is easy to complete the 5-coloring (by possibly interchanging the color of \( z_1 \) and \( z_2 \)).

If \( z_0, z_1, z_2 \) are \( q_1, p_1, p_2 \), respectively, and \( u_0 = p_2, u_1 = q_3 \), then we color \( y, z_1, z_2, x, q_2 \) by 2, 1, 2, 3, 1, 4, respectively. If \( q_3 \) can be colored 2, then we color \( p_3, p_4, p_5, v_0 \) by 5, 3, 5, 5, respectively. So assume that \( q_3 \) is adjacent to \( y \). Then we color \( q_3 \) by 5. If we can color \( \{p_3, p_4, p_5\} \) by colors \( \{1, 2, 3\} \), then color \( v_0 \) by 5. So assume that \( \{p_3, p_4, p_5\} \) cannot be colored \( \{1, 2, 3\} \). Then \( p_3, p_4 \) are joined to the same vertex in \( \{x, y\} \). That vertex must be \( x \), since \( x \) has degree at least 4 in \( G' \). Then we can assume that \( p_5 \) is adjacent to \( y \), since otherwise we color \( p_3, p_4, p_5 \) by 2, 3, 2, respectively. (Although we shall not need it, we mention that the graph
we are now considering is not 5-colorable.) We now show that $G''$ is
nontoroidal. For, if we contract $v_0x$ into $x$ and $p_5p_4$ into $p_4$, then the only
missing edges in the resulting graph are $p_1p_3$, $p_3y$, $p_2p_4$, $xq_3$, and, hence
$G''$ is nontoroidal by Proposition 2.3.

Finally we consider the case where $z_0, z_1, z_2$ are $q_1, q_2, p_1$, respectively. If $u_0 \neq \{z_1, z_2\}$, then we color $y, z_1, z_2, x, p_2, p_3, p_4, p_5, q_3$ by
$2, 1, 2, 1, 3, 4, 3, 4, 5$, respectively. If $u_0 = p_1$, then we color $y, z_1, z_2, x$ by
$2, 1, 3, 1$, respectively. If $q_3$ is not joined to $y$, then we color $q_3$ by 2 and
the vertices $p_3, \ldots, p_5$ by 4 and 5. If $q_3$ is adjacent to $y$, then we color
$q_3$ by 5. The assumption that $x$ has degree at least 4 in $G'$ implies that
some vertex in $\{p_2, \ldots, p_5\}$ can be colored 2. The others are colored by
3, 4. So we assume that $u_0 = q_2 = z_1$. We color $y, z_1, z_2, x, u_1$ by
$2, 3, 2, 1, 4$ and we try to extend this coloring. If $q_3$ can be colored 1,
then we color $p_2, p_3, \ldots$ by 4 and 5. So we assume that $q_3$ is joined to $x$.
If $u_1 = p_3$, then we recolor $z_2$ by 4 and color $q_3$ by 2. As $y$ has degree
at least 4 in $G'$, it is joined to at least one of $p_4, p_5$ which we color 1.
The remaining vertices of $\{p_1, \ldots, p_5\}$ are colored 5. If $u_1 = q_3$, then we
color one of $p_2$ or $p_5$ by 1 if possible, and we complete the coloring by
using 5 for two vertices in $\{p_2, p_3, p_4, p_5\}$. So assume that both $p_2$ and
$p_5$ are joined to $x$. Since $y$ has degree at least 4 in $G'$, it is joined to
both of $p_3, p_4$. We claim that $G''$ is nontoroidal. For if $G''$ has an
embedding on the torus, then that embedding is a triangulation (because
$G''$ has 10 vertices and 30 edges). Consider the induced embeddings of
$G'' - p_2$, $G'' - p_5$, and $G'' - v_0$, respectively. The face of $G'' - p_2$ contain-
ing $p_2$ is bounded by a Hamiltonian cycle $R_1$ of $N(p_2)$. Similarly, we let
$R_2, R_3$ denote Hamiltonian cycles of $N(p_3)$ and $N(v_0)$, respectively.
Each of $R_1, R_2, R_3$ contains $xp_1$. So $xp_1$ is in three facial triangles,
a contradiction.

Finally, we consider the subcase, where $z_0, z_1, z_2, u_0, u_1$ are
$q_1, q_2, p_1, q_2, p_2$, respectively. We color $y, z_1, z_2, x, u_1, q_3$ by $2, 3, 2, 1, 4, 5$,
respectively. We can assume that $q_3$ is joined to $x$ since otherwise we
recolor $q_3$ by 1 and complete the coloring. We can assume that $p_3$ is
joined to $x$ since otherwise we color $p_5, p_4$ by 1, 4 and complete the
coloring. Now we color $p_5$ by 4. The coloring can be completed unless
$p_3$ and $p_4$ are both joined to the same vertex in $\{x, y\}$. That vertex must
be $y$ as $y$ has degree at least 4 in $G'$. If we contract $v_0y$ into $y$ and $p_4, p_5$
into $p_4$, then the only missing edges in the resulting graph are $q_3y$, $xp_3,$
$p_3p_1, p_3p_4$. So this graph (and hence also $G''$) is nontoroidal by
Proposition 2.3.

Lemma 5.3. Let $G$, $G'$, and let $G''$ be as in Lemma 5.2, except that now
$G$ is a copy of $K_2 + H_7$. If $G''$ is toroidal and non-5-colorable, then
$G'' \cong K_2 + H_7$. 
Proof. The proof is like that of Lemma 5.2. In the present lemma, the embedding part is easier in that we can replace the condition that \( G'' \) is toroidal by the weaker condition that \( N(x) \) is Hamiltonian for each vertex \( x \) in \( G \). We leave the tedious details for the reader.

6. The 5-Color Theorem

**Theorem 6.1.** Let \( G \) be a graph on the torus. Then \( G \) is 5-colorable if and only if \( G \) does not contain \( K_6 \) or \( C_3 + C_5 \) or \( K_2 + H_7 \) or \( T_{11} \).

**Proof.** The “only if” part is trivial. We prove the “if” part by contradiction. Assume that \( G_0 \) is a counterexample such that

(i) \(|V(G_0)|\) is minimum.

By Theorem 3.3, \( G_0 \) has a vertex \( v_0 \) of degree \( \leq 5 \). We choose \( G_0 \) and \( v_0 \) such that

(ii) the clique number of \( N(v_0) \) is maximum subject to (i).

(iii) The number of largest complete subgraphs in \( N(v_0) \) is maximum subject to (i) and (ii).

(iv) The number of edges in \( N(v_0) \) is maximum subject to (i), (ii), (iii).

(v) \(|E(G_0)|\) is minimum subject to (i) – (iv).

We now derive a number of properties of \( G_0 \) and finally reach a contradiction.

(1) \( G_0 \) is 6-vertex-critical.

**Proof of (1).** For any vertex \( v \) of \( G_0 \), \( G_0 - v \) is not a counterexample to Theorem 6.1. Hence \( G_0 - v \) is 5-colorable.

(2) Each vertex of \( G_0 \) has degree at least 5.

**Proof of (2).** Every 6-vertex-critical graph has minimum degree at least 5.

By (2), we have

(3) \( v_0 \) has degree 5.

As \( G_0 \) does not contain \( K_6 \) we conclude that

(4) \( N(v_0) \) is not a complete graph.

Let \( x, y \) be any pair of nonadjacent vertices in \( N(v_0) \). Let \( G_{xy} \) denote the graph obtained from \( G_0 - v_0 \) by identifying \( x \) and \( y \). Clearly, \( G_{xy} \) is
toroidal. If $G_{xy}$ has a 5-coloring, then it is easy to modify that 5-coloring to a 5-coloring of $G_0$. Hence

(5) $G_{xy}$ is not 5-colorable.

Then (i) implies that

(6) $G_{xy}$ contains either $K_6$ or $C_3 + C_5$ or $K_2 + H_7$ or $T_{11}$.

Let $G'_{xy}$ be a copy of $K_6$ or $C_3 + C_5$ or $K_2 + H_7$ or $T_{11}$ in $G_{xy}$. Let $G_{xy}'$ be the multigraph in $G_{xy}$ induced by $G_{xy}'$. Then

(7) $G_0$ consists of $v_0$, $N(v_0)$, the five edges from $v_0$ to $N(v_0)$, and the union of all the graphs obtained from $G_{xy}$ by splitting the contracted vertex into $x$ and $y$, where the union is taken over all pairs of nonadjacent vertices $x, y$ in $N(v_0)$.

Proof of (7). The subgraph described in (7) is not 5-colorable. For if it had a 5-coloring, then some two nonadjacent vertices $x, y$ in $N(v_0)$ would have the same color. This would result in a 5-coloring of $G_{xy}$, a contradiction. Now (7) follows by the minimality properties (i), (v) in Theorem 6.1.

(7) implies that $G_0$ has at most 106 vertices and, hence, what remains is a finite problem.

(8) $|V(G_0)| \geq 10$. If equality holds, then $G_0$ has a vertex which is adjacent to all other vertices.

Proof of (8). Assume that $|V(G_0)| \leq 10$. Then Theorem 2.1 implies that $G_0$ is of the form $H_1 + H_2$, where $H_i$ is $k_i$-vertex-critical, $k_1 \leq k_2, k_1 + k_2 = 6$. If $k_1 = k_2 = 3$, then $G_0 = K_6$ or $G_0 = C_3 + C_5$, a contradiction. So $k_1 \leq 2$ and, hence, $G_0$ has a vertex adjacent to all other vertices. Suppose now (reductio ad absurdum) that $|V(G_0)| \leq 9$. If $k_1 = 1$, then $|V(H_2)| \leq 8$ and, hence, $H_2$ is of the form $H_2' + H_2''$, where $H_2' = K_2$ or $K_1$. So we can assume that $k_1 = 2$ and that $H_2$ is 4-vertex-critical. The only 4-vertex-critical graphs with $\leq 7$ vertices are $K_4, K_1 + C_4, H_7$, and $M_7$ (see the remark following Theorem 2.1). By assumption, $G_0 \not\cong C_3 + C_5 = K_2 + (K_1 + C_3)$ and $G_0 \not\cong K_2 + H_7$. Hence $G_0 \cong K_2 + M_7$. Now $G_0$ has nine vertices and at (least) 27 edges. Hence, $G_0 = K_2 + M_7$ triangulates $S_1$. But $K_2 + M_7$ has a vertex $v$ such that $N(v)$ is nonhamiltonian. This contradiction proves (8).

(9) If $x$ and $y$ are two nonadjacent vertices of $N(v_0)$, then $G_{xy}'$ is a $K_6$ or $T_{11}$.

Proof of (9). Suppose (reductio ad absurdum) that $G_{xy}' = C_3 + C_5$ or $G_{xy}' = K_2 + H_7$. Let $z_0$ be the vertex in $G_{xy}'$ corresponding to $\{x, y\}$ in $G_0$. Let $xu_1u_2 \cdots u_kyz_2z_3 \cdots z_mx$ be the facial walk in the subgraph of $G_0 - v_0$ induced by $(V(G_{xy}') \setminus \{z_0\}) \cup \{x, y\}$ bounding the face containing $v_0$. We
can assume that $1 \leq k \leq m$, $\{x, y\} \cap \{u_1, \ldots, u_k\} = \emptyset$ and that $G_0$ is drawn on the torus such that $k + m$ is minimum. We obtain $G''_{xy}$ by deleting one edge in each double edge of $G''_{xy}$. As $G'_{xy}$ is a $C_3 + C_5$ or $K_2 + H_7$, $G''_{xy}$ has precisely one face bounded by a 4-cycle by Lemma 2.2. All other faces are bounded by 3-cycles. As $G''_{xy}$ is obtained from $G'_{xy}$ by adding edges, $G''_{xy}$ has at most one face bounded by a 4-cycle. All other faces are bounded by 3-cycles or 2-cycles. Hence $k \leq 2$ and $m \leq 3$. Also, the vertices $y, z_1, \ldots, z_m, x$ are distinct. ($z_1 \neq y$ because $yz_1$ is an edge in $G_{0}$, and $z_1 \neq x$ because $xy$ is not an edge in $G_{0}$.) Also $z_2 \neq x, y$ because $G_{0}$ does not have multiple edges.) Finally, all vertices of $G''_{xy}$ are either in $G'_{xy}$ or inside one of the cycles $R_1: v_0, xu_1, \ldots, u_k, yv_0$ or $R_2: v_0, yz_1, \ldots, z_m, xv_0$ by the proof of Corollary 4.2. Let $q_i$ be the number of vertices inside $R_i$ ($i = 1, 2$).

Consider first the case $m = 3$. Then $G''_{xy}$ has no 2-cycle except possibly $xu_1, y$ if $k = 1$. (For any other 2-cycle would be of the form $xwxy$ and it would be nonfacial, contradicting Lemma 2.2.) It follows that all vertices $x, u_1, \ldots, u_k, y, z_1, \ldots, z_k$ are distinct except that possibly $z_2$ equals one of $u_1, u_2$.

By Proposition 4.1 and Corollary 4.2, $q_1 \leq 1$ and $q_2 \leq 3$. If $q_2 = 3$, then by Lemma 2.2, $N(v_0)$ contains no $C_3$. But, since $G'_{xy}$ is a $C_3 + C_5$ or a $K_2 + H_7$, it is easy to find a vertex $v_1$ of degree 5 (not only in $G'_{xy}$ but also in $G_0$) such that $N(v_1)$ has a $C_3$ (contracting (ii) in Theorem 6.2) unless $G'_{xy} = C_3 + C_5$, $k = 2$, and $z_1, z_2, z_3, u_1, u_2$ are the vertices of the $C_3$. But in that special case it is easy to 5-color $G_0$: We color $x, y$ by 1, 2, respectively. As $G''_{xy}$ has no double edges when $k = 2$, we can color the other two vertices of the $C_3$ in $C_3 + C_5$ by 1 and 3. Then we color $z_3$ by 2 and the remaining four vertices in the $C_5$ in $C_3 + C_5$ by 4, 5. This coloring can be extended to a 5-coloring of $G_0$. This gives a contradiction when $q_2 = 3$.

If $m = 3$ and $q_2 = 2$, then we obtain a contradiction as in the case $q_2 = 3$, unless the interior of $R_2$ consists of vertices $w_1, w_2$ and the edges $w_1w_2$, $w_1v_0, w_1y, w_1z_1, w_1z_2, w_2z_2, w_2z_3, w_2x, w_2v_0$, and $v_0$ is also joined to $z_2 \in \{u_1, u_2\}$. In this case $q_1 = 0$, by Corollary 4.2, and $G_0$ is 5-colorable, by Lemma 5.1(a). This contradiction shows that $q_2 \leq 2$ when $m = 3$.

If $m = 3$ and $q_2 = 1$ and the vertex inside $R_2$ is called $w_1$, then $w_1$ is joined to at least five vertices of $R_2$. If $w_1$ is joined to both $x$ and $y$, then $k = 2$ and $v_0$ is joined to $x, w_1, y, u_2, u_1$. Then $N(v_0)$ contains at most one $C_3$ (if $z_2 \in \{u_1, u_2\}$). But it is easy to find a vertex $v$ of degree 5 in $G'_{xy}$ and in $G_0$ such that $N(v)$ has at least two $C_3 - s$ because $\{z_1, z_2, z_3, u_1, u_2\}$ has less than five vertices. This contradiction shows that $w_1$ is not joined to both $x$ and $y$. In fact, the above argument implies that the notation can be chosen such that $w_1$ is joined to $v_0, y, z_1, z_2, z_3$, and $v_0$ is joined to $z_3$ and to $z_2$ which is in $\{u_1, u_2\}$. In particular, $q_1 = 0$. Any 5-coloring of $G_0 - \{v_0, w_1\}$ satisfying the conclusion of Lemma 5.1(c) can be extended to a 5-coloring of $G$. This contradiction proves that $q_2 = 0$ if $m = 3$. 

CARSTEN THOMMASEN
If \( m = 3 \) and \( q_2 = 0 \), then \( v_0 \) has at least one neighbor in \( \{ z_1, z_2, z_3 \} \). But then \( v_0 \) is joined to both \( z_1 \) and \( z_3 \). For if \( v_0 \) is not joined to \( z_3 \) but to \( z_i \) (\( i = 2 \) or \( 1 \)) then we can add the edge \( z_i x \), contradicting either the maximality property (iv) in Theorem 6.1 or the minimality of \( k + m \). (Note that if \( v_0 \) is joined to \( z_2 \) and \( z_2 = u_1 \), then \( q_1 = 0 \), since otherwise we add \( z_2 x \) and delete \( x u_1 \) which brings us back to the case \( q_2 = 1 \). But then \( v_0 \) is joined to \( z_1 \) and we can add \( z_1 x \).) Now the 5-coloring of \( G_0 - v_0 \) in Lemma 5.1(b) can be extended to a 5-coloring of \( G_0 \). This contradiction shows that \( 1 \leq k \leq m \leq 2 \). Possibly \( \{ z_1, z_2 \} \cap \{ u_1, \ldots, u_k \} \neq \emptyset \).

Now we 5-color \( G_0 \) minus the interior of the walk \( W = y z_1 z_2 x u_1 \cdots u_k y \). By Proposition 4.1, the interior of \( W \) either contains just one vertex (which must be \( v_0 \)) joined to all five colors, or it contains two vertices \( v_0, v_0' \) and the edges \( v_0 y, v_0 z_1, v_0 z_2, v_0 x, v_0 v_0', v_0 x, v_0 u_1, v_0 u_2, v_0 y \). Assume first that the interior of \( W \) contains both \( v_0 \) and \( v_0' \). Since \( G_{x_1} \) is a \( C_3 + C_5 \) or \( K_2 + H_7 \) with one additional vertex (namely \( v_0' \)), it is easy to find a vertex \( v_1 \) of degree 5 in \( G_0 \) such that \( N(v_1) \) has at least one triangle. Moreover, if \( N(v_0) \) has at least one triangle, then \( v_1 \) can be chosen such that \( N(v_1) \) has at least two triangles. Hence \( N(v_0) \) has at least two triangles by the maximality property (iii) in Theorem 6.1. By Lemma 2.2(a), \( G_0 \) cannot contain both edges \( x z_1, y z_2 \). Hence \( k = 2 \) and \( \{ z_1, z_2 \} \cap \{ u_1, u_2 \} \neq \emptyset \). We cannot have \( \{ z_1, z_2 \} = \{ u_1, u_2 \} \) because then \( G_{x_1}'' \) would have two double edges contradicting Lemma 2.2(a). So we can assume that \( u_1 = z_1 \) and \( u_2 \neq z_2 \). By Proposition 4.1, \( z_2 \) and \( u_2 \) have the same color in any 5-coloring of \( G_0 - \{ v_0, v_0' \} \). So \( G_1 = (G_0 - \{ v_0, v_0' \}) \cup \{ z_2 u_2 \} \) is not 5-colorable. Hence \( G_1 \) contains a subgraph \( G_1' \) which is one of the graphs \( K_6, C_3 + C_5, K_2 + H_7, T_{11} \), by the minimality of \( G_0 \). If \( z_2 u_2 \) is contained in a facial cycle \( z_2 u_2 q_2 z_2 \) of \( G_1' \), then either \( q z_2 x u_1 u_2 q \) or \( q z_2 z_1 y u_2 q \) is a contractible 5-cycle with more than one vertex in its interior, contradicting Proposition 4.1. So \( z_2 u_2 \) is not in a facial 3-cycle of \( G_1' \). Hence \( G_1' = K_6 \).

This \( K_6 \) contains \( z_2 \) and \( u_2 \) but none of \( x, y \) because \( G_{x_1}'' \) has only one double edge; \( x \) or \( y \) has degree \( \geq 6 \) in \( G_0 \), since otherwise \( x, y, v_0, v_0' \) are inside a walk of length 6. So \( G_1' = K_6 \) can be obtained from \( G_{x_1}'' \) by first deleting a vertex of degree \( \geq 6 \) (and some more vertices) and then adding an edge. But this is impossible as \( G_{x_1}' = C_3 + C_5 \) or \( K_2 + H_7 \).

There only remains the case where \( W \) has one vertex, namely \( v_0 \), in its interior. If \( k = 1 \), then there is at most one vertex distinct from \( u_1 \), say \( u_0 \), which is joined to both \( x \) and \( y \), by Lemma 2.2. If \( k = 2 \) we can interchange between \( \{ z_1, z_2 \} \) and \( \{ u_1, u_2 \} \). In any case we can assume that \( N(v_0) \) has vertex set \( \{ x, y, z_1, z_2, u_1 \} \) and we can apply Lemmas 5.2, 5.3 to 5-color \( G_0 \). This contradiction proves (9).
(10) If $x$ and $y$ are two nonadjacent vertices of $N(v_0)$, then $G'_{xy}$ is a $K_6$.

Proof of (10). Suppose (reductio ad absurdum) that $G'_{xy} = T_{11}$. We define the facial walk $W: yz_1z_2 \cdots z_mux_1u_2 \cdots u_ky$ as in the proof of (9). As $T_{11}$ is a triangulation, $1 \leq k \leq m \leq 2$. As $x$ and $y$ have degree $\geq 5$ in $G_0$ and they correspond to a vertex of degree 6 in $T_{11}$, it follows that there are at least two vertices of $G_0$ inside $W$. Using Proposition 4.1 we conclude that $k = m = 2$ and that there are precisely two vertices $v_0, v'_0$ inside $W$ with neighbor sets \{x, z_1, z_2, y, v'_0\} and \{x, y, u_1, u_2, v_0\}, respectively. Clearly, $G_{xy}$ contains no 6-regular graph with 11 vertices. As $T_{11} \not\subseteq K_5$, $G_{xy}$ does not contain $K_6$ either. This contradiction proves (10).

(11) Each vertex $v$ in $N(v_0)$ has degree at least 2 in $N(v_0)$ or else $N(v_0) \supseteq K_4$.

Proof of (11). Suppose $v$ has degree $\leq 1$ in $N(v_0)$. Then $N(v_0)$ has a vertex $u$ such that $u$ is not in $N(v)$, but $G_0 \cup \{uw\}$ is toroidal. The maximality property (iv) in Theorem 6.1 implies that $G \cup \{uw\}$ contains a subgraph $M$ which is isomorphic to $K_6$ or $C_3 + C_5$ or $K_2 + H_7$ or $T_{11}$. If $M$ does not contain $v_0$, then in any 5-coloring of $G - v_0$, $u$ and $v$ must have the same colors (because $M$ is not 5-colorable). But then $G_{uw}$ is 5-colorable, contradicting (5). Hence $M$ contains $v_0$. As $T_{11}$ is 6-regular, $M \neq T_{11}$. If $M = K_6$ or $C_3 + C_5$, then each vertex of $N(v_0)$ has degree $\geq 3$ in $N(v_0) \cup \{uw\}$ and, hence, degree $\geq 2$ in $N(v_0)$. So we can assume that $M = K_2 + H_7$ and that $v$ has degree 2 in $N(v_0) \cup \{uw\}$. Hence $u$ has degree 8 in $M$. By Proposition 4.1, $G_0$ consists of $M - uv$ and one more vertex joined to $u, v$ and three more vertices in $M$. But then it is easy to find a vertex $v_1$ of degree 5 in $G_0$ such that $N(v_1) \supseteq K_4$. Then also $N(v_0) \supseteq K_4$ by the maximality property (ii). This proves (11).

We let $v_0, x, y$ and the facial walk $W: xu_1u_2 \cdots u_ky z_1z_2 \cdots z_mx$ in $G_0 - v_0$ ($1 \leq k \leq m$) and $R_1, R_2$ be as in (9). As $G'_{xy} = K_6$, $G''_{xy}$ is obtained from $K_6$ by adding additional edges. It follows that $m \leq 5$ and that $m + k \leq 7$, by Lemma 2.2. Also, the vertices $x$, $y$, $z_1$, ..., $z_m$ are distinct and $x, y, u_1, ..., u_k$ are distinct by Lemma 2.4. Possibly, \{ $z_1$, ..., $z_m$ \} $\cap$ \{ $u_1$, ..., $u_k$ \} $\neq \emptyset$. Also note that $G(\{z_1, ..., z_m\}) = K_m$ and $G(\{u_1, ..., u_k\}) = K_k$. Let $z_0$ be the vertex in $G_{xy}$ obtained by identifying $x$ and $y$.

(12) Every vertex $s$ in $G_{xy} - (G'_{xy} \cup N(v_0))$ is joined to four vertices of $G'_{xy} - z_0$ which induce a $K_4$ in $G_0$.

Proof of (12). By (7), $s$ is contained in a graph of the form $G'_{xy'}$, where $x'$ and $y'$ are nonadjacent vertices in $N(v_0)$. As $G'_{xy'} = K_6$, $s$ is in a $K_5 \subseteq G_0$. That $K_5$ has four vertices in $G'_{xy}$ because it is not possible to add $r$ vertices inside $R_1$ or $R_2$ (or any other cycle) and join them completely to $5 - r$ vertices of the same cycle unless $r = 1$. 
(13) \( G_{xy} - (G'_{xy} \cup N(v_0)) \) has at most one vertex \( s \).

Proof of (13). Suppose (reductio ad absurdum) that \( s, t \) are two distinct vertices in \( G_{xy} - (G'_{xy} \cup N(v_0)) \). By (12), each of \( s, t \) is joined to four vertices of some facial cycle of \( G'_{xy} \). If both \( s \) and \( t \) are inside \( R_2 \) then the two \( K_4 - s \) in \( G(R_2) \) that \( s \) and \( t \) are joined to have at most one vertex in common unless \( m = 5 \) and the notation can be chosen such that the two \( K_4 - s \) in (12), which \( s \) and \( t \) (respectively) are joined to, have vertex sets \( \{z_1, z_2, z_3, z_4\} \), \( \{z_4, z_5, y, z_1\} \) or \( \{y, z_1, z_2, z_3\} \), \( \{z_3, z_4, z_5, y\} \). In the former case we get a contradiction to Lemma 2.2 and in the latter case \( \{y, z_1, z_2, z_3, z_4, z_5\} \) induces a \( K_6 \) in \( G \), contradicting the initial assumption of the proof of Theorem 6.1. So we can assume that \( t \notin \text{int}(R_2) \). Now \( N(t) \subseteq G'_{xy} \). For otherwise \( t \) would have a neighbor which is in a face of the graph induced by \( (G'_{xy} - z_0) \cup \{x, y, v_0, t\} \) bounded by a 3-cycle of a 4-cycle. (If \( t \in \text{int}(R_1) \) and \( k = 3 \), then \( N(t) \cap G(R_1) = K_4 \) which implies that \( u_1y \) or \( u_1x \) is a double edge in \( G'_{xy} \). Then \( m = 3 \), \( s \in \text{int}(R_2) \), and \( d(v_0) \leq 4 \), a contradiction.) As \( G_0 \) minus the vertices in that face is 5-colorable, \( G_0 \) is also 5-colorable by Proposition 4.1. So \( N(t) \subseteq G'_{xy} \). If \( t \) has degree \( \geq 6 \) in \( G_0 \), then all facial cycles of the graph induced by \( (G'_{xy} - z_0) \cup \{x, y, v_0, t\} \) are 3-cycles or 4-cycles (except possibly \( R_1 \) and \( R_2 \) which may be 5-cycles) and since that graph does not contain \( s \) we again obtain a contradiction by Proposition 4.1. So \( t \) has degree 5. If \( t \notin \text{int}(R_1) \), then \( k \leq m \leq 3 \) by Lemma 2.2(a). And if \( t \in \text{int}(R_2) \), then \( 3 = k \leq m \leq 4 \). Also, \( N(t) \) contains a \( K_4 \). Hence \( N(v_0) \) contains a \( K_4 \), by the maximality property (ii) of \( N(v_0) \). At most one of \( x, y \) is in this \( K_4 \) in \( N(v_0) \). That \( K_4 \) contains at most one vertex \( q \) in \( \text{int}(R_1) \cup \text{int}(R_2) \). The fifth neighbor of \( v_0 \) is either \( x \) or \( y \). The subgraph of \( G_0 \) induced by \( (G'_{xy} - z_0) \cup \{x, y, v_0, t, q\} \) (or \( (G'_{xy} - z_0) \cup \{x, y, v_0, t\} \) if \( q \) does not exist) has a 5-coloring. That 5-coloring can be extended to a 5-coloring of \( G_0 \) by Proposition 4.1. This contradiction proves (13).

(14) If the vertex \( s \) in (12), (13) does not exist for any choice of \( x, y \), then \( |V(G_0)| = 11 \).

Proof of (14). If \( s \) does not exist, then \( G'_{xy} - z_0 \) contains at least four vertices not in \( N(v_0) \), by (8). On the other hand \( G'_{xy} \) is a \( K_6 \), so it has at most five vertices not in \( N(v_0) \). So \( 10 \leq |V(G_0)| \leq 11 \). Suppose (reductio ad absurdum) that \( |V(G_0)| = 10 \). Let \( N(v_0) = \{x, y, t_1, t_2, t_3\} \), and the remaining vertices of \( G_0 - v_0 \) are called \( w_1, w_1, w_3, w_4 \). We can assume that \( V(G'_{xy}) = \{z_0, t_1, w_1, w_2, w_3, w_4\} \). By (8) we can assume that \( t_1 \) has degree 9 in \( G_0 \). As \( G(\{t_1, w_1, w_2, w_3, w_4\}) \) is a \( K_5 \), \( N(v_0) \) contains no \( K_4 \) by Lemma 2.5. Hence each of \( w_1, \ldots, w_4 \) has degree \( \geq 5 \) by the maximality property (ii) of \( N(v_0) \). Also, if \( v_0u, v_0v \) are consecutive in the clockwise ordering around \( v_0 \), then \( uv \) is an edge of \( G_0 \) by (the proof of) (11). So we
can assume that \( N(v_0) \) contains the path \( xt_2t_3y \) or \( xt_2yt_3 \) (say the latter, since otherwise we rename the vertices in \( N(v_0) \)) and all four edges from \( t_i \) to \( N(v_0) - t_1 \). As \( \{ v_0 \} \cup N(v_0) \) induces a planar graph, by Lemma 2.5, \( N(v_0) \) contains no additional edges. There are at least 12 edges from \( \{ w_1, w_2, w_3, w_4 \} \) to \( N(v_0) \). Then an easy count shows that \( G_0 \) has at least 12 + 6 + 7 + 30 = 55 edges. Hence \( G_0 \) triangulates the torus and each vertex \( w_i \) has degree 6 in \( G_0 \). Any vertex in \( G_0 \) which has degree 5 can play the role of \( v_0 \). So we have proved the following: \( G_0 \) has a vertex of degree 9, and if \( v \) is a vertex of degree 5 in \( G_0 \), then the four vertices not joined to \( v \) have degree 6. So the vertices of degree 5 induce a complete graph \( K_1, K_2, \) or \( K_3 \). Since \( G_0 \) has precisely 30 edges, the vertices of degree 5 must induce a \( K_3 \). It follows that \( G_0 \) is a triangulation of the torus with one vertex of degree 9, six vertices of degree 6, and three vertices of degree 5 which induce a \( K_3 \). Suppose \( a_1a_2a_3a_1 \) is the 3-cycle whose vertices have degree 5. As \( G_0 - \{ a_1, a_2, a_3 \} \) has 7 vertices and 18 edges it is nonplanar, and, hence, \( a_1a_2a_3a_1 \) is contractible. Since \( G_0 - \{ a_1, a_2, a_3 \} \) is connected, \( a_1a_2a_3a_1 \) is facial. Let \( b_1b_2\ldots b_6b_1 \) be the unique facial 6-walk in \( G_0 - \{ a_1, a_2, a_3 \} \). Then we can assume that \( G_0 \) contains the walk \( a_1b_2a_2b_3a_3b_4a_1 \) and the edges \( a_1b_1, a_2b_2, a_3b_3 \). Without loss of generality we can assume that \( b_2 \) has degree 9. Hence \( b_2 = b_5 \), and \( b_1, b_2, b_3, b_4, b_6 \) are distinct. The two vertices \( c_1, c_2 \) in \( G_0 - \{ a_1, a_2, a_3, b_1, b_2, b_3, b_4, b_6 \} \) are neighbors and are both joined to \( b_1, b_2, b_3, b_4, b_6 \). As \( b_1, b_3, b_4, b_6 \) have degree 6, \( G_0 \) contains the edge \( b_1b_3 \). Now we contract \( a_2b_3 \) into \( b_3 \) and \( a_3b_4 \) into \( b_4 \). The only missing edges in the resulting graph are \( a_1c_1, a_1c_2, b_1b_4, b_3b_6 \). Hence that graph and \( G_0 \) are nonoriental by Proposition 2.3. This contradiction proves (14).

(15) The vertex \( s \) in (12), (13) does exist for some two nonadjacent vertices \( x, y \) in \( N(v_0) \).

Proof of (15). Suppose (reductio ad absurdum) that \( s \) does not exist. Then \( |V(G_0)| = 11 \) by (14). For any nonadjacent vertices \( x', y' \) in \( N(v_0) \), \( G_{x',y'} \) is a \( K_6 \) whose vertices are \( \{ x, y \} \) and the five vertices in \( G_0 - (N(v_0) \cup \{ v_0 \}) \) which we denote by \( w_1, \ldots, w_5 \). As in the proof of (14) we show that \( N(v_0) \nsubseteq K_4 \), and if \( v_0x', v_0y' \) are consecutive in the ordering around \( v_0 \), then \( x'y' \) is an edge of \( G_0 \).

Hence \( N(v_0) \) contains a 5-cycle which we denote by \( Q: q_1q_2q_3q_4q_5 \). As \( N(v_0) \nsubseteq K_4 \), we conclude that \( w_i \) is joined to at least two vertices of \( Q \) for \( i = 1, 2, \ldots, 5 \). Counting the edges mentioned until now we obtain 30. As \( G \) has at most 33 edges, we conclude that if \( Q \) has \( q \) chords, then there are at most \( 13 - q \) edges from \( Q \) to \( \{ w_1, \ldots, w_5 \} \). For any two nonadjacent vertices \( x', y' \) in \( Q \) and for each \( i \in \{ 1, \ldots, 5 \} \), \( w_i \) is joined to at least one of \( x', y' \). So if \( q = 0 \), \( w_i \) is joined to at least three vertices of \( Q \). This implies the presence of at least 15 edges from \( Q \) to \( \{ w_1, \ldots, w_5 \} \), a contradiction.
Also, $Q$ cannot have two chords having no end in common, since that would imply the existence of two disjoint nonplanar subgraphs of $G_0$, contradicting Lemma 2.5. So we can assume that $Q$ has the chord $q_1q_3$ and possibly $q_1q_4$ and no other chord. No vertex $w_i (1 \leq i \leq 5)$ is joined to both $q_2$ and a vertex in $\{q_4, q_5\}$, because then $G_0(\{v_0, w_i\} \cup N(v_0))$ would be nonplanar, contradicting Lemma 2.5. So, if $G_0$ contains $q_1q_4$ we conclude that, for each $i \in \{1, 2, \ldots, 5\}$, $w_i$ is joined to either $\{q_2, q_3\}$ or to $\{q_4, q_5\}$ (and possibly to $q_1$, too). As each vertex in $G_0$ has degree $\geq 5$, we can assume that each of $w_1, w_2, w_3$ is joined to $q_4$ and $q_5$, and that each of $w_4, w_5$ is joined to $q_3$ and $q_2$. But this results in a nontoroidal graph. For if $G_0$ is on the torus, then $\{w_4, w_5, q_3, q_2\}$ is a $K_4$ inside a face of $G_0(\{w_1, w_2, w_3, q_4, q_5\})$ which is a $K_5$. That face is homeomorphic to a disc. Hence one of $w_4, w_5, q_3, q_2$ is inside the $C_3$ induced by the three others. But any vertex of $w_4, w_5, q_3, q_2$ is joined by an edge or a path $q_2q_1q_5$ to $G_0(\{w_1, w_2, w_3, q_4, q_5\})$. This is a contradiction, so assume that $q_1q_4$ is not present. Then we can assume that $q_2$ is joined to $w_1$, $w_2$ because $q_2$ has degree at least 5. Each vertex in $\{w_1, \ldots, w_5\}$ is joined to two consecutive vertices of $q_1q_3q_4q_5q_1$. So each of $w_1, w_2$ is joined to each of $q_1, q_2, q_3$, and each of $w_3, w_4, w_5$ is joined to each of $q_4$ and $q_5$. But then $G_0(\{v_0, w_1, w_2, q_1, q_2, q_3\})$ and $G_0(\{q_4, q_5, w_3, w_4, w_5\})$ are nonplanar, contradicting Lemma 2.5. This proves (15).

(16) The vertex $s$ defined in (12), (13) has degree 5, and $N(v_0)$ contains a $K_4$.

Proof of (16). The last statement follows from (12) and the maximality property (ii) of $N(v_0)$. So let us assume that $s$ has degree at least 6. By (12) $s$ is joined to a $K_4$ in $G_{xy} - z_0$. Now $s$ cannot have five neighbors in $G_{xy} - z_0$ because then $G_0$ would contain a $K_6$. It is easy to see that $s$ is inside $W$. So either $s$ has precisely four neighbors on $R_1$ or $R_2$ (which induce a $K_4$) and at least two neighbors $t_1$, $t_2$ not in $W$, or else $s$ has precisely five neighbors on $R_1$ or $R_2$, one of which is either $x$ or $y$ and at least one neighbor $t_1$ not in $W$. By (13), $v_0$ is joined to $t_1$ (and to $t_2$ if $t_2$ exists). Note that $s$ is not joined to both $x$ and $y$, since then $t_1$ would be inside $syv_0xs$, contradicting Corollary 4.2. As $k + m \leq 7$, and $k \leq m$, $s$ is inside $R_2$ (and $m \geq 4$).

Suppose now that $t_2$ exists. By Corollary 4.2, the notation can be chosen such that $s$ is joined to $z_1$. Assume without loss of generality that $t_1$ is inside $st_2v_0yz_1s$. By Proposition 4.1, $t_1$ is the only vertex inside $t_2v_0y_1zts_2$. Now $s$ has no neighbor $t_3$ inside $st_2v_0xz_3\ldots s$, because then $t_3$ would be in $N(v_0)$ and $t_2$ would be inside $st_1v_0t_3s$, contradicting Corollary 4.2. By Proposition 4.1, there is no vertex inside $st_2v_0xz_3\ldots s$. By Lemma 2.2 at least one of $z_2, z_3$ is not joined to both $x$ and $y$. (Since $m \geq 4$, at most one vertex of $G'_{xy}$ is joined to both $x$ and $y$.) So any 5-coloring $c$ of $G'_{xy} - z_0$ can
be extended to \( x, y \) such that \( \{c(x), c(y)\} \cap \{c(z_2), c(z_3)\} \neq \emptyset \). We extend this 5-coloring to \( s \). (This is possible because \( G_0 \not\cong K_6 \). Note that \( s \) is joined to none of \( x, y \).) Then we give either \( t_1 \) or \( t_2 \) a color in \( \{c(x), c(y)\} \cap \{c(z_2), c(z_3)\} \). (Note that none of \( t_1, t_2 \) are joined to \( z_2 \) or \( z_3 \).) Then we color \( v_0 \) and finally we color the uncolored vertex in \( \{t_1, t_2\} \). (This is possible because we have at least two choices for \( c(v_0) \).) This 5-coloring can be extended to a 5-coloring of \( G_0 \) by Proposition 4.1 (because no vertex inside \( s t_2 v_0 x z_m \cdots s \) is joined to \( s \)), a contradiction which proves that \( t_2 \) does not exist. Hence \( s \) has five neighbors on \( R_2 \). As \( G_0 \not\cong K_6 \), \( s \) is joined to one of \( x, y \), say \( x \).

Consider first the case where \( s \) is not joined to \( z_1 \). Then \( m = 5 \) and \( s \) is joined to \( x, z_5, z_4, z_3, z_2 \). There exists a 5-coloring \( c \) of \( G \) minus the interior of \( y z_1 z_2 s x v_0 y \). We must have \( c(x) \neq c(y) \) because \( G'_{xy} = K_6 \). Also \( c(s) = c(z_1) \) because \( z_1, z_2, z_3, z_4, z_5 \) induce a \( K_5 \). As \( c \) cannot be extended to a 5-coloring of \( G_0 \) it follows by Proposition 4.1 and the assumption that \( t_2 \) does not exist that there is only one vertex inside \( y z_1 z_2 s x v_0 y \), namely \( t_1 \), and that \( V(G_0) = \{z_1, z_2, z_3, z_4, z_5, y, x, v_0, s, t_1\} \). By (8), one of these vertices has degree 9. That vertex must be one of \( z_1, z_2 \). But this contradicts Lemma 2.2, so we can assume that \( s \) is joined to \( z_1 \).

By Proposition 4.1, \( t_1 \) is the only vertex inside \( y z_1 s x v_0 y \), and \( t_1 \) is joined to all vertices of that cycle. As \( v_0 \) has degree \( \geq 5 \), we have \( k \geq 2 \), and \( v_0 \) is joined to two vertices in \( \{u_1, \ldots, u_k\} \). By Corollary 4.2, \( s, t_1, x, y, v_0 \) are the only vertices of \( G_0 \) which are not in \( G'_{xy} - z_0 = K_5 \). Hence \( |V(G_0)| = 10 \). By (8), \( G_0 \) has some vertex joined to all other vertices. That vertex must be \( z_1 = u_1 \). By Lemma 2.2 applied to \( G'_{xy} = K_6 \), \( m = 4, k = 2 \), and no vertex distinct from \( z_1, v_0 \) is joined to both \( x \) and \( y \). In any 5-coloring \( c \) of \( G_0 - \{v_0, t_1\} \) we must have \( c(s) = c(u_2) \). Hence \( u_2 \not\in \{z_2, z_3, z_4\} \). One of \( z_2, z_3 \) is joined to \( y \) because \( y \) has degree \( \geq 5 \). The other is joined to either \( x \) or \( y \). So, if we contract \( y t_1 \) into \( t_1 \) and \( v_0 x \) into \( x \), then the only missing edges in the resulting graph are \( su_2 \) and edges from \( \{z_2, z_3, z_4\} \) to \( \{x, t_1\} \) forming a \( K_5 \cup P_3 \). Hence \( G \) is nontoroidal by Proposition 2.3. This proves (16).

By (13), (15), \( G_0 \) is induced by \( \{v_0\} \cup N(v_0) \cup (G'_{xy} - z_0) \cup \{s\} \). By (8), \( G'_{xy} \) has at least three vertices not in \( N(v_0) \). If \( G'_{xy} \) has four vertices not in \( N(v_0) \), then \( G_0 \) has a \( K_5 \) with only one vertex in \( N(v_0) \). As \( N(v_0) \) has a \( K_4 \), by (16), we obtain a contradiction to Lemma 2.5. Hence \( G'_{xy} \) contains precisely three vertices, say \( w_1, w_2, w_3 \), not in \( N(v_0) \). The vertices of \( N(v_0) \) are denoted \( x, y, t_1, t_2, t_3 \). By (16), we can assume that \( G_0(\{y, t_1, t_2, t_3\}) = K_4 \). By (8) we can assume that \( t_1 \) has degree 9 in \( G_0 \). We can also assume that \( G'_{xy} \) contains \( t_2 \).

Let \( H_1 = G_0(\{v_0, y, t_1, t_2, t_3\}) \) and \( H_2 = G_0(\{t_1, t_2, w_1, w_2, w_3\}) \). Then each of \( H_1, H_2 \) is a \( K_5 \). Now we consider the facial walk \( u_1 u_2 \cdots u_q u_1 \) of
$H_1$ whose interior contains $w_1, w_2, w_3$. By Euler's formula $q \leq 8$. Hence no vertex occurs three times in $u_1 u_2 \cdots u_q$. Both $t_1$ and $t_2$ must occur twice and they must occur as $\cdots t_1 \cdots t_2 \cdots t_1 \cdots t_2$. Hence the edge $t_1 t_2$ does not occur twice in $u_1 u_2 \cdots u_q$. Those facial walks of $H_1 \cup H_2$, inside which $G_0$ has at least one (respectively two) vertices, have length at least 5 (respectively 6) by Proposition 4.1. All these conditions can only be met if $q = 8$ and $G_0(H_1 \cup H_2)$ has a facial 6-cycle or two facial 5-cycles whose interior contain $x$ and $s$. Having 5-colored $H_1$, the colors of $w_1, w_2, w_3$ can be chosen such that the coloring can be extended to $x, s$ too. The proof of Theorem 6.1 is complete.

REFERENCES