A Stability Problem for Discrete-Time Linear Periodic Systems

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Abstract—Conditions to construct a periodic output feedback of a periodic system are given in terms of properties of rational matrices. It turns out that these conditions are analogous to the ones involved in the existence of periodic realization of a periodic collection of rational matrices. The obtained periodic output feedback is used in the stabilization problem.

Keywords—Discrete-time linear periodic system, Transfer matrix, Matrix fraction description, Output feedback, Stability.

1. INTRODUCTION

An important topic in the analysis of linear time-invariant control systems is the feedback problem, which is one of the most useful tools for the study of the pole assignment problem, the stability problem, . . . , of dynamical systems. Important results on state feedback and output feedback have been collected in [1–3], for invariant systems.

In the last years, many authors have discussed about the state feedback for pole-assignment and the stabilization problem for linear periodic systems (see [4–6]). The pole-assignment problem for linear invariant systems and for single periodic systems by means of periodic output feedback has been considered in [4]. In [5], it has been studied the output stabilization via state feedback and state-observer in the time-domain, then it is necessary information about the state, the input and the output of the original system. The results have been given in terms of structural properties of state-space systems. In [6], it has been solved the pole-assignment problem for discrete-time linear periodic systems via state feedback. In these works, the systems are considered in the temporal domain.

On the other hand, in linear multivariable theory, the matrix fraction description is commonly used since many of their physical systems are described by means of the transfer matrix (see [7,8]). In [9], the authors considered a periodic collection of rational matrices associated with a linear periodic system such that, with some conditions, the collection is called the transfer function of the periodic system. Furthermore, in [10] one constructs a matrix fraction description of the transfer function at time $s + 1$ from a matrix fraction description at time $s$. In this paper, the matrix fraction description of a periodic collection of rational matrices will be considered as a
new way to study periodic output feedbacks. We will use the transfer matrix model because this context has advantages in simplicity and directness, and helps to obtain straightforward proofs. Moreover, in this approach it is not necessary, the explicit knowledge of structural properties and the concept of state to solve the modification of the dynamics of a system by means of a feedback.

Our goal is to answer the following question: given a periodic collection of rational matrices associated with a periodic system, which are the conditions over the feedback in order for the new closed-loop collection becomes a periodic system? We answer this question in Section 2. In Section 3, the periodic set of output feedback will be used in the study of stability of periodic systems.

Since many real systems are causal systems, it is well known, see [2], then that the associated transfer function must be proper. Consequently, we use proper rational matrices in this paper.

An $N$-periodic system, $N \in \mathbb{Z}^+$ on the $z$-domain can be represented by an $N$-periodic collection of rational matrices [10]:

$$\{G_s(z), s \in \mathbb{Z}\}, \quad G_s(z) \in \mathbb{R}^{pN \times mN}(z), \quad G_{s+N}(z) = G_s(z), \quad (1)$$

satisfying the following conditions:

(i) $G_{s+1}(z) = S_{1,p}(z)G_s(z)S_{1,m}(z)$,

where $S_{s,t}(z) = \begin{bmatrix} O & I_{(N-s)t} \\ zI_{st} & O \end{bmatrix}$, \quad $t = p, m$; and $N$ is the period,

(ii) $G_0(z)$ is a proper rational matrix with a lower block-triangular polynomial part.

Then, the periodic system is given by

$$\tilde{y}_s(z) = G_s(z)\tilde{u}_s(z), \quad s \in \mathbb{Z}, \quad (2)$$

where $\tilde{y}_s(z)$ and $\tilde{u}_s(z)$ are the $z$-transform of the outputs and the inputs, respectively.

In [9], one obtains an $N$-periodic realization of (2) denoted by $(C(\cdot), A(\cdot), B(\cdot), D(\cdot))_N$. We shall assume that $D(\cdot) = O$, in this case the matrices given in (1) satisfy the following condition (iii) instead of (ii):

(iii) $G_0(z)$ is a proper rational matrix with a strictly lower block-triangular polynomial part.

Let us consider a periodic collection of rational matrices $F_s(z) \in \mathbb{R}^{mN \times pN}(z)$, $F_{s+N}(z) = F_s(z), \ s \in \mathbb{Z}$. We apply to the periodic system (2) the following output feedback: $\tilde{u}_s(z) = -F_s(z)\tilde{y}_s(z) + \tilde{v}_s(z)$. If we suppose that $[I + G_s(z)F_s(z)] \in \mathbb{R}^{pN \times pN}(z)$ is nonsingular then

$$\tilde{y}_s(z) = G_s^F(z)\tilde{v}_s(z), \quad \text{with } G_s^F(z) = [I + G_s(z)F_s(z)]^{-1}G_s(z), \quad (3)$$

represents the corresponding closed-loop system.

2. OUTPUT FEEDBACK FOR A LINEAR PERIODIC SYSTEM

Given a periodic collection of rational matrices $\{G_s(z), s \in \mathbb{Z}\}$ satisfying (i) and (iii), we consider the new set of rational matrices $\{G_s^F(z), s \in \mathbb{Z}\}$ given by (3). The closed-loop system $\tilde{y}_s(z) = G_s^F(z)\tilde{v}_s(z), \ s \in \mathbb{Z}$ should admit a periodic realization $(C(\cdot), A(\cdot), B(\cdot))_N$ if the collection (3) satisfies (i) and (iii), (see [9]).

First, we study conditions on $\{F_s(z), s \in \mathbb{Z}\}$ in order to the collection $\{G_s^F(z), s \in \mathbb{Z}\}$ verifies condition (i). For that, let us consider a left factorization $\{D_s(z), N_s(z)\}$ of the rational matrix $G_s(z)$ at time $s$, that is,

$$G_s(z) = D_s^{-1}(z)N_s(z), \quad D_s(z) \in \mathbb{R}^{pN \times pN}[z], \quad N_s(z) \in \mathbb{R}^{pN \times mN}[z].$$
Since $G_s(z)$ satisfies (i), then
\[ G_{s+1}(z) = S_{1,p}D_{s}^{-1}(z)N_s(z)S_{1,m}^{-1}(z) = [D_s(z)S_{N-1,p}(z)]^{-1} N_s(z)S_{N-1,m}(z), \]
where we have used some properties of matrices $S_{s,t}(z)$ given in [10, Lemma 1]. Therefore, $D_{s+1}(z) = D_s(z)S_{N-1,p}(z)$ and $N_{s+1}(z) = N_s(z)S_{N-1,m}(z)$ is a left factorization of $G_{s+1}(z)$ (see [10, Proposition 1]). Since
\[ G^F_s(z) = [I + G_s(z)F_s(z)]^{-1} G_s(z) = [D_s(z) + N_s(z)F_s(z)]^{-1} N_s(z), \]
then we have
\[ G^F_{s+1}(z) = [D_{s+1}(z) + N_{s+1}(z)F_{s+1}(z)]^{-1} N_{s+1}(z) = [D_s(z)S_{N-1,p}(z) + N_s(z)S_{N-1,m}(z)F_{s+1}(z)]^{-1} N_s(z)S_{N-1,m}(z) = z^{-1}S_{1,p}(z) \left[D_s(z)S_{N-1,m}(z)F_{s+1}(z)S_{N-1,p}^{-1}(z) \right]^{-1} N_s(z)zS_{1,m}^{-1}(z). \]
Then, if the collection of rational matrices $\{F_s(z), s \in \mathbb{Z}\}$ satisfy
\[ F_s(z) = S_{N-1,m}(z)F_{s+1}(z)S_{N-1,p}^{-1}(z), \tag{4} \]
we have
\[ G^F_{s+1}(z) = S_{1,p}(z)[D_s(z) + N_s(z)F_s(z)]^{-1} N_s(z)S_{1,m}^{-1}(z) = S_{1,p}(z)G^F_s(z)S_{1,m}^{-1}(z). \]
Conversely, it is trivial to prove that if $G^F_s(z)$ satisfies (i), then $F_s(z)$ does not satisfy, in general, the equation (4).

Now, we study conditions on $\{F_s(z), s \in \mathbb{Z}\}$ so that the collection $\{G^F_s(z), s \in \mathbb{Z}\}$ satisfies (iii). Before, we prove the following result, for an arbitrary proper rational matrix.

**Lemma 1.** Let $G(z) \in \mathbb{R}^{pN \times mN}(z)$ be a proper rational matrix with polynomial part $J$, and let $F(z) \in \mathbb{R}^{mN \times pN}(z)$ be a proper rational matrix with polynomial part $L$. Then,
\[ G^F(z) = [I + G(z)F(z)]^{-1} G(z) \text{ is proper if and only if } I + JL \neq O. \]

**Proof.** Decompose the strictly proper part of $G(z)$ by the trivial factorization, that is, $G(z) = (d(z)I)^{-1}N(z) + J$, with deg $(N(z)) < \text{deg} (d(z)) = n$. Analogously, we write $F(z) = \hat{B}(z)(a(z)I)^{-1} + L$, with deg $(\hat{B}(z)) < \text{deg} (a(z)) = r$. By construction of $G^F(z)$, we get the following factorization:
\[ G^F(z) = [I + G(z)F(z)]^{-1} G(z) = \left[I + \left((d(z)I)^{-1}N(z) + J\right) (\hat{B}(z)a(z)I)^{-1} + L\right]^{-1} (d(z)I)^{-1} N(z) + J \]
\[ = a(z) \left[d(z)a(z) [I + JL] + N(z)\hat{B}(z) + d(z)J\hat{B}(z) + N(z)La(z)\right]^{-1} (N(z) + d(z)J). \]
Note that in the factorization, the degree of the numerator matrix is
\[ \text{deg} \left(a(z) \left(N(z) + d(z)J\right)\right) = \text{deg} (a(z)) + \text{deg} \left(N(z) + d(z)J\right) \leq r + n, \]
while the degree of the denominator matrix is
\[ \text{deg} \left(d(z)a(z) [I + JL] + N(z)\hat{B}(z) + d(z)J\hat{B}(z) + N(z)La(z)\right) = \text{deg} (d(z)a(z) [I + JL]) = \text{deg} (d(z)) + \text{deg} (a(z)) = n + r, \]
if and only if $I + JL \neq O$. Then the proof follows.
Consider the \( N \)-periodic collection of rational matrices \( \{F_s(z), s \in \mathbb{Z}\} \) satisfying the recurrence equation (4). If we impose that \( F_0(z) \) is a proper matrix with a lower block-triangular polynomial part, then by [9] all matrices \( F_s(z) \), obtained by (4), are also proper matrices with a lower block-triangular polynomial part. Thus, we write \( F_s(z) = B_s(z)(a_s(z)I)^{-1} + L_s, \) where \( L_s \) is a lower block-triangular matrix, for all \( s \in \mathbb{Z} \). Since \( G_{s}(z) = (d_s(z)I)^{-1}N_s(z) + J_s \) satisfies (iii), then \( J_s \) is a strictly lower block-triangular matrix. Therefore, \( I + J_sL_s \neq O \) and by Lemma 1, \( G_{s}^F(z) \) is a proper matrix, for all \( s \in \mathbb{Z} \).

Finally, we consider the factorization of \( G_{s}^F(z) \) given in Lemma 1. Since the numerator matrix has degree less than or equal to \( r + n \), \( J_s \) is a strictly lower block-triangular matrix and \( \deg (\tilde{N}_{s}(z)) < n \), then degree of the \((i, j)\)-blocks satisfies
\[
\deg \left[ a_s(z) \left( \tilde{N}_{s}(z) + J_s d_s(z) \right) \right]_{ij} < n + r, \quad i \leq j.
\]
Since the denominator matrix has degree equal to \( n + r \), then \( G_{s}^F(z) \) has a strictly lower block-triangular polynomial part, for all \( s \in \mathbb{Z} \). We summarize the previous statements in the following theorem.

**Theorem 1.** Consider an \( N \)-periodic collection of rational matrices (3)
\[
G_{s}^F(z) = \left[ I + G_{s}(z)F_{s}(z) \right]^{-1}G_{s}(z), \quad s \in \mathbb{Z},
\]
where \( \{G_{s}(z), s \in \mathbb{Z}\} \) satisfies conditions (i) and (iii), and \( \{F_{s}(z), s \in \mathbb{Z}\} \) is an \( N \)-periodic collection of rational matrices. If \( F_{s+1}(z) = S_{s,m}(z)F_{s}(z)S_{s,p}(z), s \in \mathbb{Z}, \) and \( F_0(z) \) is a proper rational matrix with a lower block-triangular polynomial part, then \( \{G_{s}^F(z), s \in \mathbb{Z}\} \) is a periodic system satisfying conditions (i) and (iii).

### 3. Conditions for the Stability of a Closed-Loop Periodic System

Next, we study how to construct, if it is possible, a periodic collection of rational matrices \( \{F_s(z), s \in \mathbb{Z}\} \) such that the closed-loop periodic system can be stabilized.

Consider an \( N \)-periodic system \( \{G_{s}(z), s \in \mathbb{Z}\} \) satisfying (i) and (iii). For each \( s \in \mathbb{Z} \), we consider a left coprime factorization \( \{D_{s}(z), N_{s}(z)\} \) and a right coprime factorization \( \{R_{s}(z), P_{s}(z)\} \) of \( G_{s}(z), \)
\[
G_{s}(z) = R_{s}(z)P_{s}^{-1}(z) = D_{s}^{-1}(z)N_{s}(z).
\]
Then, there exist four polynomial matrices \( \tilde{X}_{s}(z), \tilde{Y}_{s}(z), X_{s}(z), Y_{s}(z) \) such that
\[
D_{s}(z)\tilde{X}_{s}(z) + N_{s}(z)\tilde{Y}_{s}(z) = I,
X_{s}(z)P_{s}(z) + Y_{s}(z)R_{s}(z) = I,
\begin{bmatrix}
-N_{s}(z) & D_{s}(z) \\
X_{s}(z) & Y_{s}(z)
\end{bmatrix}
\begin{bmatrix}
-\tilde{Y}_{s}(z) & P_{s}(z) \\
\tilde{X}_{s}(z) & R_{s}(z)
\end{bmatrix}
= \begin{bmatrix}
I & O \\
O & I
\end{bmatrix}, \quad s \in \mathbb{Z}.
\]
With those factorizations at time \( s \), one can construct the following left coprime factorization \( F_{s}(z) = C_{s}^{-1}(z)E_{s}(z) \), where \( C_{s}(z) \) and \( E_{s}(z) \) are obtained using the following relation given in [7]:
\[
\begin{bmatrix}
C_{s}(z) & E_{s}(z)
\end{bmatrix}
= \begin{bmatrix}
K_{s}(z) & L_{s}(z)
\end{bmatrix}
\begin{bmatrix}
-N_{s}(z) & D_{s}(z) \\
X_{s}(z) & Y_{s}(z)
\end{bmatrix},
\]
for some \( L_{s}(z) \) nonsingular and stable matrix, and for some \( K_{s}(z) \) polynomial matrix. Thus,
\[
C_{s}(z) = -K_{s}(z)N_{s}(z) + L_{s}(z)X_{s}(z),
E_{s}(z) = K_{s}(z)D_{s}(z) + L_{s}(z)Y_{s}(z).
\]
In [7], using this feedback matrix, one obtains a stable left coprime factorization of $G^F_s(z)$, that is, this closed-loop system has all its poles in the open unit disk in the complex plane $|z| < 1$, and they are given by the solutions of

$$\det (C_s(z)P_s(z) + E_s(z)R_s(z)) = 0.$$ 

(6)

On the other hand, from (5) we obtain

$$[C_s(z) E_s(z)] \begin{bmatrix} S_{N-1,m}(z) & O \\ O & S_{N-1,p}(z) \end{bmatrix} = [K_s(z) L_s(z)] \begin{bmatrix} -N_s(z)S_{N-1,m}(z) & D_s(z)S_{N-1,p}(z) \\ X_s(z)S_{N-1,m}(z) & Y_s(z)S_{N-1,p}(z) \end{bmatrix},$$

and by (4), we consider the feedback $F_{s+1}(z) = C_{s+1}^{-1}(z)E_{s+1}$ at time $s + 1$, with

$$C_{s+1}(z) = -K_s(z)N_s(z)S_{N-1,m}(z) + L_s(z)X_s(z)S_{N-1,m}(z),$$

$$E_{s+1}(z) = K_s(z)D_s(z)S_{N-1,p}(z) + L_s(z)Y_s(z)S_{N-1,p}(z).$$

From that output feedback, the denominator of the new matrix $G^F_{s+1}(z)$ is

$$C_{s+1}(z)P_{s+1}(z) + E_{s+1}(z)R_{s+1}(z) = C_s(z)S_{N-1,m}(z)S_{1,m}(z)P_s(z) + E_s(z)S_{N-1,p}(z)S_{1,p}(z)R_s(z) = z(C_s(z)P_s(z) + E_s(z)R_s(z)),$$

hence,

$$\det (z(C_s(z)P_s(z) + E_s(z)R_s(z))) = z^{mN} \det (C_s(z)P_s(z) + E_s(z)R_s(z)),$$

and by (6), we conclude that its poles belong to the open unit disk in the complex plane.

Note that the degree of the characteristic polynomial of $G^F_{s+1}(z)$ has been increased. For eliminating this problem, it is sufficient to construct a coprime factorization of $G^F_{s+1}(z)$ using the usual techniques (see [1]). We summarize the above results in the following theorem.

**Theorem 2.** Consider an $N$-periodic system $\{G_s(z), s \in \mathbb{Z}\}$ satisfying conditions (i) and (iii). Then, there exists an $N$-periodic collection of output feedback $\{F_s(z), s \in \mathbb{Z}\}$ such that the closed-loop periodic system $\{G^F_s(z), s \in \mathbb{Z}\}$ can be stabilized.

Note that, the transfer matrix approach lead us not only to obtain results equivalent to those obtained by use of an observer but also is an appropriate way to study and to construct more general feedbacks.

**REFERENCES**