Newton-Euler Modelling and Control of a Multicopter using Motor Algebra $\mathbf{G}^+_{3,0,1}$

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Abstract. In this work the dynamic model and the nonlinear control for a multi-copter have been developed using the geometric algebra framework specifically using the motor algebra $G^+_{3,0,1}$. The kinematics for the aircraft model and the dynamics based on Newton-Euler formalism are presented. Block-control technique is applied to the multi-copter model in combination with super twisting and including an internal dynamics estimator driven by maneuvers away from the origin. The stability of the presented control scheme is shown. The experimental analysis shows that our non-linear controller law is able to reject external disturbances and to deal with parametric variations.

Keywords. N-E modelling, motor algebra $G^+_{3,0,1}$, dynamic model, multi-copter control, adaptive estimator.

1. Introduction

Unmanned autonomous vehicles - or UAV's -, specially multi-copters, are becoming nowadays ubiquitous. Their popularity is due to their relative maneuverability for performing a wide range of applications; for example, monitoring roads or areas at risk, remote surveillance, inspection of power lines, etc. Nevertheless, some of these applications require more specific control tasks, even more, robust controllers because external dynamics or disturbances in general affect the flight performances. To this end, a knowledge of the dynamics of the system is necessary in many cases to improve the performance of the controller to be applied.

In most of the existing literature, the Newton-Euler formalism in used to obtain the dynamic model (e.g. [1]). The dynamic equations of motion contains some non-linearities and coupled terms which makes the design of a control law a challenging task. In addition, in many cases, these equations are separated into the translational and rotational motions, and consequently, the control problem is divided into the position control problem and the attitude control problem, respectively. Which later cases several representations of this dynamics have been discussed in the literature, such as Euler angles, Rodrigues parameters, axis/angle, and unit quaternion. Unit quaternion is a very efficient tool in attitude control due to its qualities, such as the use of the least possible number of parameters to represent orientation, its ability to describe rotations in 3D space, its computational stability, and lack of the gimbal-lock effect inherent to Eulers angles.

In order to eliminate the need of designing two separate controls for attitude and position, dual-quaternion or motor algebra can be used because of their ability to simultaneously deal in a most compact and efficient manner, with attitude and position control problems. There are some works using dual-quaternions for multi-copters ([2], [3]) in which a feedback control based on dual quaternion model is applied, but external disturbances are not considered and the proposed control laws are designed only for stabilization of the system.

Recent works in control of multi-copters address the problem of minimizing disturbance in multi-copters ([4], [5]), using non-linear control techniques such as sliding mode in order to track a three dimensional reference in the space. Nevertheless, as in most cases, the control problem is divided in position control and attitude control.

In an attempt to merge both concepts, namely to unify the dynamic model to simultaneously deal with the position and attitude control, and the design of a control law which handles the tracking problem of a multi-copter in presence of large disturbances. Additionally, in this work a block control technique [6] with second order sliding mode [7] is proposed for trajectory tracking in presence of external disturbances and model uncertainties using a motor algebra based model.

The rest of this work is organized as follows. In section 2, the mathematical preliminaries of motor algebra are introduced. In section 2.5, the rigid-body spatial velocity using motor algebra is presented. In section 3, the dynamic of multi-copter is developed considering aggressive maneuvers, this is adding an unknown internal dynamics estimator for large external perturbations. Section 4 defines a control law based on the model presented in previous sections. The stability analysis of the control law applied to the aircraft in closed-loop is given. In section 5, simulation results for a specific case of quad-rotor are shown. Finally, in section 6, the conclusions of this work are presented.

2. Mathematical preliminary

This section presents briefly the mathematical concepts about motor algebra.To begin with, this section outlines the motor algebra $G^+_{3,0,1}$.

2.1. The algebra of rotors

In geometric algebra a *rotor* (short name for rotator), $\boldsymbol{R}$, is an even-grade element of the Euclidean algebra $G_{3,0,0}$ of the 3D-space. If Q= $r_0 + r_1 i + r_2 j + r_3 k \in \mathbb{H}$ represents a unit quaternion, then the rotor which performs the same rotation is simply given by

$$\boldsymbol{R} = \underbrace{r_0}_{scalar} + \underbrace{r_1(I_E e_1) - r_2(I_E e_2) + r_3(I_E e_3)}_{bivectors} \in G^+_{3,0,0}. \tag{1}$$

where $I_E = e_1 e_2 e_3$ is the Euclidean pseudoscalar.

The rotor algebra $G^+_{3,0,0}$ is therefore a subset of the Euclidean geometric algebra of three-dimensional space.

The reversion and magnitude of a rotor $\boldsymbol{R}$ are, respectively, given by

$$\begin{aligned} \tilde{\boldsymbol{R}} &= r_0 - r_1 e_2 e_3 - r_2 e_3 e_1 - r_3 e_1 e_2 = r_0 - \boldsymbol{r} \\ ||\boldsymbol{R}||^2 &= \boldsymbol{R}\tilde{\boldsymbol{R}}. \end{aligned} \tag{2}$$

This implies that the unique multiplicative inverse of $\boldsymbol{R}$ is given by

$$\boldsymbol{R}^{-1} = \tilde{\boldsymbol{R}} \parallel \boldsymbol{R} \parallel^{-2}. \tag{3}$$

If a rotor $\boldsymbol{R}$ satisfies the equation

$$\boldsymbol{R}\tilde{\boldsymbol{R}} = \parallel \boldsymbol{R} \parallel^2 = r_0^2 - \boldsymbol{r} \cdot \boldsymbol{r} = 1, \tag{4}$$

then we say that this rotor is a unit rotor and its multiplicative inverse is simply $\boldsymbol{R}^{-1} = \tilde{\boldsymbol{R}}$.

It can be easily shown that the unit rotor corresponds to the geometric product of two unit 3D vectors,

$$\boldsymbol{R} = \boldsymbol{m}\boldsymbol{n} = \boldsymbol{m} \cdot \boldsymbol{n} + \boldsymbol{m} \wedge \boldsymbol{n}. \tag{5}$$

The two terms in the right hand side of equation (5) correspond to the scalar and bivector terms of an equivalent quaternion in $G_{3,0,0}$, and thus $\boldsymbol{R} \in G^+_{3,0,0}$. This even subalgebra contains to the algebra of rotors.

Considering the scalar and the bivector terms of the rotor of equation (5), we can further write the Euler representation of a 3D rotation with angle θ in the right-hand sense, as follows:

$$\begin{aligned} \boldsymbol{R} &= r_0 + \boldsymbol{r} = r_0 + r_1 e_2 e_3 + r_2 e_3 e_1 + r_3 e_1 e_2 \\ &= a_c + a_s \bar{\boldsymbol{r}}_n = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) \bar{\boldsymbol{r}}_n \\ &= e^{\frac{\theta}{2} \bar{\boldsymbol{r}}_n}, \end{aligned} \tag{6}$$

where $\bar{\boldsymbol{r}}_n$ is the unit rotation axis-vector spanned by the bivector basis $e_2 e_3$, $e_3 e_1$, and $e_1 e_2$, and the scalars a_c and $a_s \in \mathbb{R}$. The polar representation of a rotor given in equation (6) is possible, because rotors as a Lie group can be expressed in terms of the Lie algebra of bivectors: The orbits on the Lie group manifold describe the evolution of the actions of rotors. The bivector $\bar{\boldsymbol{r}}_n$ corresponds to the Lie operator tangent to an orbit or geodesic.

Since a rotor is a quaternion, we can embed quaternions in the more comprehensive mathematical system offered by geometric algebra. Different as in quaternion theory, in geometric algebra the quaternions or rotors have a clear geometric interpretation due to the representation in space of the rotations as described above by using two successive reflections with respect to planes crossing the origin.

2.2. Motor Algebra

The word *motor* is an abbreviation of "moment and vector." Clifford introduced motors with the name bi-quaternions [8]. Motors are dual quaternions with the necessary condition $I_m^2 = 0$. They can be found in the special 4D even sub-algebra of $G_{3,0,1}$. This even sub-algebra is denoted by $G_{3,0,1}^+$ and is only spanned via a bivector basis, as follows:

$$\underbrace{1}_{scalar}\;,\quad \underbrace{e_2e_3, e_3e_1, e_1e_2, e_4e_1, e_4e_2, e_4e_3}_{6\quad bivectors}\;,\quad \underbrace{I_m}_{unit\ pseudoscalar}\;. \tag{7}$$

This kind of basis allows us to represent spinors, which are composed of scalar and bivector terms. Motors, then, are also spinors, and as such, they represent a special kind of rotor. Because a Euclidean transformation includes both rotation and translation, we will show below that a spinor representation for both transformations in the definition of motors. But we must first show the relationship between motors and screw motion theory.

Note also that the dual of a scalar is the pseudoscalar I_m and that the duals of the first three basis bivectors are actually the next three bivectors, that is, $(e_2e_3)^* = I_m e_2e_3 = e_4e_1$.

We said in section 2.1 that a rotor relates two vectors in 3D space. According to Clifford [8], a motor operation is necessary to convert the rotation axis of a rotor into the rotation axis of a second rotor. Each rotor can be geometrically represented as a rotation plane with the rotation axis normal to this plane. Figure 1(a) depicts a motor action in detail. Note that the involved rotor axes are represented as line axes. In the figure, we first orient one axis parallel to the other by applying the rotor $\boldsymbol{R}_s$. Then, we slide the rotated axis a distance d along the connecting axis, so that it ends up overlapping the axis of the second rotor. Altogether, this operation can be described as forming a *twist* about a screw with line axis $\boldsymbol{l}$, whose pitch relationship *pitch* equals $\frac{d}{\theta}$ for $\theta \neq 0$. A motor, then, is specified only by its direction and the position of the *screw-axis line*, twist angular magnitude, and pitch. Figure 1(b) shows an action of a motor on a real object. In this case, the motor relates the rotation-axis line of the initial position of the object to the rotation-axis line of its final position. Note that in both figures the angle and sliding distance indicate how rigid displacement takes place around and along a screw-axis line $\boldsymbol{l}$, respectively. A *degenerated motor* can only rotate and not slide along the line $\boldsymbol{l}$ as figure 1(c) shows. In this case, therefore, the two axes are coplanar.

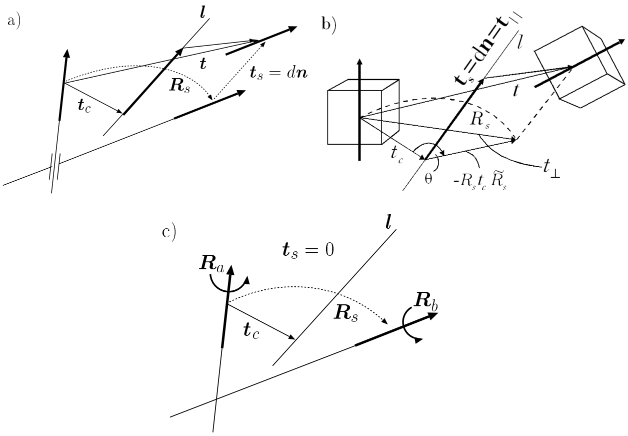

FIGURE 1. Screw motion about the line axis $\boldsymbol{l}$ ($\boldsymbol{t}_S$: longitudinal displacement by d and $\boldsymbol{R}_s$: rotation angle θ): (a) motor relating two axis lines, (b) motor applied to an object, (c) degenerated motor relating two coplanar rotors. (Note: indicated 3D vectors are represented as bivectors in text.)

2.3. Motors, rotors, and translators in $\mathbf{G}^+_{3,0,1}$

Since a rigid motion consists of the rotation and translation transformations, it should be possible to split a motor multiplicatively in terms of these two spinor transformations, which we will call a rotor and a *translator*. In the following discussion, we will denote all bivector components of a spinor by slant bold lowercase letters. Let us now express this procedure algebraically. First of all, let us consider a simple rotor in its *Euler representation* for a rotation with an angle θ,

$$\begin{aligned}\boldsymbol{R} &= a_0 + a_1 e_2 e_3 + a_2 e_3 e_1 + a_3 e_1 e_2 \\ &= a_0 + \boldsymbol{a} \\ &= \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\boldsymbol{n} \\ &= a_c + a_s \boldsymbol{n}, \end{aligned} \tag{8}$$

where $\boldsymbol{n}$ is the unit 3D bivector of the rotation axis spanned by the bivector basis e_2e_3, e_3e_1, e_1e_2, and a_c, $a_s \in \mathbb{R}$. Now, dealing with the rotor of a screw motion, the rotation-axis vector should be represented as a screw-axis line. For that, we must relate the rotation axis to a reference coordinate system at the distance $\boldsymbol{t}_c$. A 3D translation in motor algebra is represented by $\boldsymbol{T}_c$ called a translator. If we apply a translator from the left to rotor $\boldsymbol{R}$, and then apply the translator's reversion from the right, we get a modified rotor,

$$\begin{aligned}\boldsymbol{R}_s &= \boldsymbol{T}_c\boldsymbol{R}\widetilde{\boldsymbol{T}}_c \\ &= \Big(1+I_m\frac{\boldsymbol{t}_c}{2}\Big)(a_0+\boldsymbol{a})\Big(1-I_m\frac{\boldsymbol{t}_c}{2}\Big) \\ &= a_0+\boldsymbol{a}+I_m a_0\frac{\boldsymbol{t}_c}{2}+I_m\frac{\boldsymbol{t}_c}{2}\boldsymbol{a}-I_m a_0\frac{\boldsymbol{t}_c}{2}-I_m\boldsymbol{a}\frac{\boldsymbol{t}_c}{2} \\ &= a_0+\boldsymbol{a}+I_m\Big(\frac{\boldsymbol{t}_c}{2}\boldsymbol{a}-\boldsymbol{a}\frac{\boldsymbol{t}_c}{2}\Big) \\ &= a_0+\boldsymbol{a}+I_m(\boldsymbol{a}\times\boldsymbol{t}_c). \end{aligned} \tag{9}$$

Here, $\boldsymbol{t}_c$ is the 3D vector of translation spanned by the bivector basis e_2e_3, e_3e_1, e_1e_2. Then, expressing the last equation in Euler terms, we get the spinor representation,

$$\begin{aligned}\boldsymbol{R}_s &= a_0+a_s\boldsymbol{n}+I_m a_s\boldsymbol{n}\wedge\boldsymbol{t}_c \\ &= a_c+a_s(\boldsymbol{n}+I_m\boldsymbol{m}) \\ &= \cos\left(\frac{\theta}{2}\right)+\sin\left(\frac{\theta}{2}\right)(\boldsymbol{n}+I_m\boldsymbol{m}) \\ &= \cos\left(\frac{\theta}{2}\right)+\sin\left(\frac{\theta}{2}\right)\boldsymbol{l}. \end{aligned} \tag{10}$$

This result is indeed interesting because the new rotor $\boldsymbol{R}_s$ can now be applied with respect to an axis line $\boldsymbol{l}$ expressed in dual terms of direction $\boldsymbol{n}$ and moment $\boldsymbol{m}=\boldsymbol{n}\wedge\boldsymbol{t}_c$. Now, to define the motor finally, let us slide the distance $\boldsymbol{t}_S=d\boldsymbol{n}$ along the rotation-axis line $\boldsymbol{l}$. Since a motor is applied from the left and conjugated from the right, we should use the half of $\boldsymbol{t}_S$ in the spinor expression of $\boldsymbol{T}_S$ when we define the motor:

$$\begin{aligned}\boldsymbol{M} &= \boldsymbol{T}_s\boldsymbol{R}_s=\Big(1+I_m\frac{\boldsymbol{t}_S}{2}\Big)(a_0+\boldsymbol{a}+I_m\boldsymbol{a}\wedge\boldsymbol{t}_c) \\ &= \Big(1+I_m\frac{d\boldsymbol{n}}{2}\Big)(a_c+a_s\boldsymbol{n}+I_m a_s\boldsymbol{n}\wedge\boldsymbol{t}_c) \\ &= a_c+a_s\boldsymbol{n}+I_m a_s\boldsymbol{n}\wedge\boldsymbol{t}_c+I_m\frac{d}{2}a_c\boldsymbol{n}+I_m\frac{d}{2}a_s\boldsymbol{n}\boldsymbol{n} \\ &= \Big(a_c-I_m\frac{d}{2}a_s\Big)+\Big(a_s+I_m a_c\frac{d}{2}\Big)(\boldsymbol{n}+I_m\boldsymbol{n}\wedge\boldsymbol{t}_c) \\ &= \Big(a_c-I_m a_s\frac{d}{2}\Big)+\Big(a_s+I_m a_c\frac{d}{2}\Big)\boldsymbol{l}. \end{aligned} \tag{11}$$

Note that this expression of the motor makes explicit the unit line bivector of the screw-axis line $\boldsymbol{l}$. For this article, it is useful to recall the notion of a function of a dual variable, in which a differentiable real function $f:\mathbb{R}\to\mathbb{R}$ with a dual argument $\alpha+\epsilon\beta$, where $\alpha,\beta\in\mathbb{R}$ and $\epsilon^2=0$, can be expanded using a Taylor series. Because $\epsilon^2=\epsilon^3=\epsilon^4=...=0$, the function reads

$$\begin{aligned}f(\alpha+\epsilon\beta) &= f(\alpha)+\epsilon f'(\alpha)\beta+\epsilon^2 f''(\alpha)\frac{\beta^2}{2!}+... \\ &= f(\alpha)+\epsilon f'(\alpha)\beta. \end{aligned} \tag{12}$$

A useful illustration of this expansion is the exponential function of a dual number,

$$e^{\alpha+\epsilon\beta} = e^{\alpha} + \epsilon e^{\alpha}\beta = e^{\alpha}(1+\epsilon\beta). \tag{13}$$

Now let us express a motor using Euler representation. By substituting the constants $a_c = \cos(\frac{\theta}{2})$ and $a_s = \sin(\frac{\theta}{2})$ in the motor equation (11) and using equation (12), we get

$$\begin{aligned} \boldsymbol{M} &= \boldsymbol{T}_s\boldsymbol{R}_s \\ &= \left(\cos\left(\frac{\theta}{2}\right) - I_m \sin\left(\frac{\theta}{2}\right)\frac{d}{2}\right) + \left(\sin\left(\frac{\theta}{2}\right) + I_m\cos\left(\frac{\theta}{2}\right)\frac{d}{2}\right)\boldsymbol{l} \\ &= \cos\left(\frac{\theta}{2} + I_m\frac{d}{2}\right) + \sin\left(\frac{\theta}{2} + I_m\frac{d}{2}\right)\boldsymbol{l}, \end{aligned} \tag{14}$$

which is a dual-number representation of the spinor. Now, let us analyze the resultant expressions,

$$\begin{aligned} \boldsymbol{R} &= \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\boldsymbol{n} \\ \boldsymbol{R}_s &= \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\boldsymbol{l} \\ \boldsymbol{M} &= \cos\left(\frac{\theta}{2} + I_m\frac{d}{2}\right) + \sin\left(\frac{\theta}{2} + I_m\frac{d}{2}\right)\boldsymbol{l}. \end{aligned} \tag{15}$$

We can see that the rotation axis $\boldsymbol{n}$ of the simple rotor $\boldsymbol{R}$ is changed to a rotation-axis line, so that $\boldsymbol{R}_s$ now rotates about an axis line. And in the motor expression, the information for sliding distance d is now made explicit in terms of dual arguments of the trigonometric functions. It is also interesting to note that the expression for the motor using dual angles simply extends the expression of $\boldsymbol{R}_s$.

If we expand the exponential function of the dual bivectors using a Taylor series, the result will follow the general expression of equation (13) as

$$e^{\alpha+I_m\beta} = e^{\alpha} + I_m e^{\alpha}\beta = e^{\alpha}(1+I_m\beta). \tag{16}$$

Hence, we obtain the motor expression as the spinor

$$\boldsymbol{M} = e^{\boldsymbol{l}\frac{\theta}{2}+I_m\frac{\boldsymbol{t}_S}{2}} = e^{\boldsymbol{l}\frac{\theta}{2}}\left(1+I_m\frac{\boldsymbol{t}_S}{2}\right) = \boldsymbol{R}_s\boldsymbol{T}_s, \tag{17}$$

where $I_m\frac{\boldsymbol{t}_S}{2} = I_m\frac{1}{2}(t_1e_2e_3 + t_2e_3e_1 + t_3e_1e_2) = \frac{1}{2}(t_1e_4e_1 + t_2e_4e_2 + t_3e_4e_3)$.

If we want to express the motor using only rotors in a dual spinor representation, we proceed as follows:

$$\begin{aligned} \boldsymbol{M} &= \boldsymbol{R}_s\boldsymbol{T}_s = \boldsymbol{R}_s(1+I_m\frac{\boldsymbol{t}_S}{2}) \\ &= \boldsymbol{R}_s + I_m\boldsymbol{R}_s\frac{\boldsymbol{t}_S}{2}. \end{aligned} \tag{18}$$

Let us consider carefully the resultant dual part of the motor. This is the geometric product of the bivector $\boldsymbol{t}_S$ and the rotor $\boldsymbol{R}_s$. Since both are expressed in terms of the same bivector basis, their geometric product will be also expressed in this basis, which can be considered as a new rotor $\boldsymbol{R}'_s$. Thus, we can further write

$$\boldsymbol{M} = \boldsymbol{R}_s + I_m \boldsymbol{R}_s \frac{\boldsymbol{t}_S}{2} = \boldsymbol{R}_s + I_m \boldsymbol{R}'_s. \tag{19}$$

In this equation the line axes of the rotors are skewed (see figure 1(a). That means that they represent the general case of non-coplanar rotors. If the sliding distance $\boldsymbol{t}_S$ is zero, then the motor will degenerate to a rotor

$$\boldsymbol{M} = \boldsymbol{R}_s \boldsymbol{T}_s = \boldsymbol{R}_s \Big(1 + I_m \frac{\boldsymbol{t}_S}{2}\Big) = \boldsymbol{R}_s \Big(1 + I_m \frac{0}{2}\Big) = \boldsymbol{R}_s. \tag{20}$$

In this case, that is, when the two generating axis lines of the motor are coplanar, we get the so-called *degenerated motor* (see figure 1(c).

Finally, the bivector $\boldsymbol{t}_S$ can be expressed in terms of the rotors using previous results

$$\widetilde{\boldsymbol{R}}_s \boldsymbol{R}'_s = \widetilde{\boldsymbol{R}}_s \Big(\boldsymbol{R}_s \frac{\boldsymbol{t}_S}{2}\Big); \tag{21}$$

therefore,

$$\boldsymbol{t}_S = 2\widetilde{\boldsymbol{R}}_s \boldsymbol{R}'_s. \tag{22}$$

Figure 1 shows that the 3D vector $\boldsymbol{t}$, expressed in the bivector basis, is referred to the rotation axis of the rotor, and that $\boldsymbol{t}_S$ is a bivector along the motor-axis line. Thus, $\boldsymbol{t}$, considered here as a bivector, can be computed in terms of the bivectors $\boldsymbol{t}_c$ and $\boldsymbol{t}_S$, as follows:

$$\begin{aligned} \boldsymbol{t} &= \boldsymbol{t}_\perp + \boldsymbol{t}_\parallel \\ &= (\boldsymbol{t}_c - \boldsymbol{R}_s \boldsymbol{t}_c \widetilde{\boldsymbol{R}}_s) + (\boldsymbol{t} \cdot \boldsymbol{n})\boldsymbol{n} = (\boldsymbol{t}_c - \boldsymbol{R}_s \boldsymbol{t}_c \widetilde{\boldsymbol{R}}_s) + d\boldsymbol{n} \\ &= \boldsymbol{t}_c - \boldsymbol{R}_s \boldsymbol{t}_c \widetilde{\boldsymbol{R}}_s + \boldsymbol{t}_S \\ &= \boldsymbol{t}_c - \boldsymbol{R}_s \boldsymbol{t}_c \widetilde{\boldsymbol{R}}_s + 2\widetilde{\boldsymbol{R}}_s \boldsymbol{R}'_s. \end{aligned} \tag{23}$$

So far, we have analyzed the motor from a geometrical point of view. Next, we will look at the motor's relevant algebraic properties.

2.4. Properties of motors

A general motor can be expressed as

$$\boldsymbol{M}_\alpha = \alpha \boldsymbol{M}, \tag{24}$$

where $\alpha \in \mathbb{R}$ and $\boldsymbol{M}$ is a unit motor, as explained in the previous sections. In this section, we will employ unit motors. The norm of a motor $\boldsymbol{M}$ is defined as follows:

$$\begin{aligned} |\boldsymbol{M}| &= \boldsymbol{M}\widetilde{\boldsymbol{M}} = \boldsymbol{R}_s \boldsymbol{T}_s \widetilde{\boldsymbol{T}}_s \widetilde{\boldsymbol{R}}_s = \boldsymbol{R}_s \Big(1 + I_m \frac{\boldsymbol{t}_S}{2}\Big)\Big(1 - I_m \frac{\boldsymbol{t}_S}{2}\Big) \widetilde{\boldsymbol{R}}_s \\ &= \boldsymbol{R}_s \widetilde{\boldsymbol{R}}_s = 1, \end{aligned} \tag{25}$$

where $\widetilde{\boldsymbol{M}}$ is the conjugate motor and 1 is the identity of the motor multiplication. Now, using equation (19) and considering the unit motor magnitude, we find two useful properties, expressed by

$$\begin{aligned} |\boldsymbol{M}| &= \boldsymbol{M}\widetilde{\boldsymbol{M}} = (\boldsymbol{R}_s + I_m \boldsymbol{R}'_s)(\widetilde{\boldsymbol{R}}_s + I_m \widetilde{\boldsymbol{R}}'_s) \\ &= \boldsymbol{R}_s\widetilde{\boldsymbol{R}}_s + I_m(\boldsymbol{R}'_s\widetilde{\boldsymbol{R}}_s + \boldsymbol{R}_s\widetilde{\boldsymbol{R}}'_s) = 1. \end{aligned} \tag{26}$$

These equations require the following constraints:

$$\boldsymbol{R}_s\widetilde{\boldsymbol{R}}_s = 1 \tag{27}$$

$$\boldsymbol{R}'_s\widetilde{\boldsymbol{R}}_s + \boldsymbol{R}_s\widetilde{\boldsymbol{R}}'_s = 0. \tag{28}$$

Now we can show that the combination of two rigid motions can be expressed using two consecutive motors. The resultant motor describes the overall displacement, namely,

$$\begin{aligned} \boldsymbol{M}_c &= \boldsymbol{M}_a\boldsymbol{M}_b = (\boldsymbol{R}_{s_a} + I_m\boldsymbol{R}'_{s_a})(\boldsymbol{R}_{s_b} + I_m\boldsymbol{R}'_{s_b}) \\ &= \boldsymbol{R}_{s_a}\boldsymbol{R}_{s_b} + I_m(\boldsymbol{R}_{s_a}\boldsymbol{R}'_{s_b} + \boldsymbol{R}'_{s_a}\boldsymbol{R}_{s_b}) \\ &= \boldsymbol{R}_{s_c} + I_m\boldsymbol{R}'_{s_c}. \end{aligned} \tag{29}$$

Note that, on the one hand, pure rotations combine multiplicatively and, on the other hand, the dual parts containing the translation combine additively.

Using equation (19), let us express a motor in terms of dual spinors:

$$\begin{aligned} \boldsymbol{M} &= \boldsymbol{R}_s\boldsymbol{T}_s = \boldsymbol{R}_s + I_m\boldsymbol{R}'_s \\ &= (a_0 + a_1e_{23} + a_2e_{31} + a_3e_{12}) + I_m(b_0 + b_1e_{23} + b_2e_{31} + b_3e_{12}) \\ &= (a_0 + \boldsymbol{a}) + I_m(b_0 + \boldsymbol{b}). \end{aligned} \tag{30}$$

A motor expressed in terms of a translator and a rotor is manipulated similarly as in the case of a rotor, from the left and its conjugate from the right. These left and right operations, called *motor reflections*, are used to build an automorphism equivalent to the screw. Yet, by conjugating only the rotor or only the translator for the second reflection, we can derive different types of automorphisms.

By changing the sign of the scalar and bivector in the real and dual parts of the motor, we get the following variations:

$$\begin{aligned} \boldsymbol{M} &= (a_0 + \boldsymbol{a}) + I_m(b_0 + \boldsymbol{b}) = \boldsymbol{R}_s\boldsymbol{T}_s \\ \widetilde{\boldsymbol{M}} &= (a_0 - \boldsymbol{a}) + I_m(b_0 - \boldsymbol{b}) = \widetilde{\boldsymbol{T}}_s\widetilde{\boldsymbol{R}}_s \\ \bar{\boldsymbol{M}} &= (a_0 + \boldsymbol{a}) - I_m(b_0 + \boldsymbol{b}) = \widetilde{\boldsymbol{T}}_s\boldsymbol{R}_s \\ \widetilde{\bar{\boldsymbol{M}}} &= (a_0 - \boldsymbol{a}) - I_m(b_0 - \boldsymbol{b}) = \boldsymbol{T}_s\widetilde{\boldsymbol{R}}_s. \end{aligned} \tag{31}$$

The first, second, and fourth versions will be used for modeling the motion of points, lines, and planes, respectively.

Using equation (31), it is straightforward to compute the expressions for the individual components:

$$\begin{aligned} a_0 &= \frac{1}{4}(\boldsymbol{M} + \widetilde{\boldsymbol{M}} + \bar{\boldsymbol{M}} + \widetilde{\bar{\boldsymbol{M}}}) \\ I_m b_0 &= \frac{1}{4}(\boldsymbol{M} + \widetilde{\boldsymbol{M}} - \bar{\boldsymbol{M}} - \widetilde{\bar{\boldsymbol{M}}}) \\ \boldsymbol{a} &= \frac{1}{4}(\boldsymbol{M} - \widetilde{\boldsymbol{M}} + \bar{\boldsymbol{M}} - \widetilde{\bar{\boldsymbol{M}}}) \\ I_m \boldsymbol{b} &= \frac{1}{4}(\boldsymbol{M} - \widetilde{\boldsymbol{M}} - \bar{\boldsymbol{M}} + \widetilde{\bar{\boldsymbol{M}}}). \end{aligned} \tag{32}$$

Using equation (17), the logarithm of a motor can be written as

$$\begin{aligned} \log(M) &= \log(R_s T_s) \\ &= \log(\mathrm{e}^{\frac{1}{2}(\theta \mathbf{n} + I_m \mathbf{t_s})}) \\ &= \tfrac{\theta}{2}\mathbf{n} + \tfrac{\mathbf{t_s}}{2} I_m \end{aligned} \tag{33}$$

2.5. Rigid-body spatial velocity using motor algebra

In this section we present rigid motions of lines. Lines are represented in terms of two dual bivector bases one for the line orientation and the second for the line moment. Considering the case of the pure rotor motion, we represent the relationship of the moving frames as follows:

$$l_k'(t) = \boldsymbol{M}(t) l_k \widetilde{\boldsymbol{M}}(t), \tag{34}$$

where l_k' and l_k are the coefficients of the two frames. Taking its time derivative

$$\dot{l}_k' = \dot{\boldsymbol{M}} l_k \widetilde{\boldsymbol{M}} + \boldsymbol{M} l_k \dot{\widetilde{\boldsymbol{M}}}, \tag{35}$$

and substituting $\boldsymbol{M} l_k = l_k' \boldsymbol{M}$, one gets

$$\dot{l}_k' = \dot{\boldsymbol{M}} \widetilde{\boldsymbol{M}} l_k' + l_k' \boldsymbol{M} \dot{\widetilde{\boldsymbol{M}}}. \tag{36}$$

Taking the time derivative of the identity $\boldsymbol{M}\widetilde{\boldsymbol{M}} = 1$, we get the useful relations

$$\begin{aligned} \partial_t(\boldsymbol{M}\widetilde{\boldsymbol{M}}) &= 0, \\ \dot{\boldsymbol{M}}\widetilde{\boldsymbol{M}} + \boldsymbol{M}\dot{\widetilde{\boldsymbol{M}}} &= 0, \end{aligned}$$

substituting the last equation in equation (36), we get

$$\dot{l_k}'(t) = \dot{\boldsymbol{M}}\widetilde{\boldsymbol{M}} l_k' - l_k' \dot{\boldsymbol{M}}\widetilde{\boldsymbol{M}}. \tag{37}$$

The right side of the last equation can be written as the inner product between a bivector and the vector basis coefficients, we rewrite it as follows:

$$\dot{l_k}' = (2\dot{\boldsymbol{M}}\widetilde{\boldsymbol{M}}) \cdot l_k'. \tag{38}$$

Let us call the bivector

$$\boldsymbol{V}_S = 2\dot{\boldsymbol{M}}\widetilde{\boldsymbol{M}} \tag{39}$$

the spatial velocity; as we will show next it comprises of a bivector for the linear velocity and a bivector for the angular velocity. Recalling that $\boldsymbol{M} = \boldsymbol{T}\boldsymbol{R}$, we proceed as follows

$$\begin{aligned}
\boldsymbol{V}_S &= 2\dot{\boldsymbol{M}}\widetilde{\boldsymbol{M}} = 2(\dot{\boldsymbol{T}}\boldsymbol{R} + \boldsymbol{T}\dot{\boldsymbol{R}})\widetilde{\boldsymbol{R}}\widetilde{\boldsymbol{T}} = 2\dot{\boldsymbol{T}}\boldsymbol{R}\widetilde{\boldsymbol{R}}\widetilde{\boldsymbol{T}} + 2\boldsymbol{T}\dot{\boldsymbol{R}}\widetilde{\boldsymbol{R}}\widetilde{\boldsymbol{T}} \\
&= 2\dot{\boldsymbol{T}}\widetilde{\boldsymbol{T}} + 2\Big(1 + I_m\frac{\boldsymbol{t}_S}{2}\Big)\Big(\frac{1}{2}\boldsymbol{\Omega}_S\Big)\Big(1 - I_m\frac{\boldsymbol{t}_S}{2}\Big) \\
&= I_m\dot{\boldsymbol{t}}_s\Big(1 - I_m\frac{\boldsymbol{t}_S}{2}\Big) + \Big(\boldsymbol{\Omega}_S + I_m\frac{\boldsymbol{t}_S}{2}\boldsymbol{\Omega}_S\Big)\Big(1 - I_m\frac{\boldsymbol{t}_S}{2}\Big) \\
&= I_m\dot{\boldsymbol{t}}_s + \Big(\boldsymbol{\Omega}_S + I_m\boldsymbol{\Omega}_S\frac{\boldsymbol{t}_S}{2} - I_m\frac{\boldsymbol{t}_S}{2}\boldsymbol{\Omega}_S\Big) \\
&= \boldsymbol{\Omega}_S + I_m\left(\boldsymbol{v}_S + \boldsymbol{\Omega}_S\frac{\boldsymbol{t}_S}{2} - \frac{\boldsymbol{t}_S}{2}\boldsymbol{\Omega}_S\right) \qquad (40)
\end{aligned}$$

where the angular-velocity bivector $\boldsymbol{\Omega}_S$ is the dual of the linear-velocity bivector $\boldsymbol{v}_S$.

Multiplying equation (39) from the left by $\boldsymbol{M}$ and dividing by 2, we get the dynamic motor equation

$$\dot{\boldsymbol{M}} = \frac{1}{2}\boldsymbol{M}\boldsymbol{V}_S. \qquad (41)$$

Assuming that the screw motion of the body is constant through time, the dynamic motor equation (41) can be integrated to give

$$\boldsymbol{M}(t) = \boldsymbol{M}(0)e^{-\frac{\boldsymbol{V}_S}{2}} = \boldsymbol{M}(0)e^{-\frac{\left(\boldsymbol{\Omega}_S + I_m\left[\boldsymbol{v}_S + \boldsymbol{\Omega}_S \underline{\times} \boldsymbol{t}_S/2\right]\right)}{2}}. \qquad (42)$$

This represents a motor that rotates with a constant-frequency rotation in the right-hand sense and has a constant linear velocity as well.

3. Mathematical model based on motor algebra to multi-copter

Based on the Newton-Euler formalism and the forces transformed into the space of pure bivectors, it comes out

$$\begin{aligned}
\sum F &= \frac{d}{dt}(m^{-1}\boldsymbol{v}_S) + \boldsymbol{\Omega}_S \underline{\times} (m^{-1}\boldsymbol{v}_S) \\
\sum T &= \frac{d}{dt}\mathscr{S}^{-1}(J^{-1}\mathscr{S}\{\boldsymbol{\Omega}_S\}) + J^{-1}\boldsymbol{\Omega}_S \underline{\times} \mathscr{S}^{-1}(J\mathscr{S}\{\boldsymbol{\Omega}_S\}) \qquad (43)
\end{aligned}$$

where $\underline{\times}$ is the commutator product, $J = \mathrm{diag}(J_x,\ J_y,\ J_z)$ the inertia tensor, m the mass, and the transformation $\mathscr{S}$ relates the bivectorial magnitude and vector in $\mathbb{R}^3$ as follows $\boldsymbol{v}_S \in \mathbb{R}^3, \boldsymbol{v}_S = v_x e_1 + v_y e_2 + v_z e_3, \mathscr{S}\{\boldsymbol{v}_S\} = \boldsymbol{v}_S I_E$ and $I_E = e_1 e_2 e_3$ the Euclidean pseudo-scalar. Besides, additional terms are expressed as follows:

$$\begin{aligned}
\sum F &= F_{prop} - F_{aero} - F_{grav} \\
\sum T &= T_{prop} - T_{aero} - T_{gyro} \qquad (44)
\end{aligned}$$

where $T_{prop}, T_{aero}, T_{gyro} \in \mathbb{R}^3$ are aerodynamic and gyroscopic torques exerted by the propellers, respectively. F_{prop}, F_{aero} are bivector forces exerted by the propellers and aerodynamic forces, respectively. $F_{grav} = m\widetilde{\boldsymbol{R}}_s G \boldsymbol{R}_s$ is the force effected by gravity, with $G = (0, [0,\ 0,\ g]^T)m/s^2$ and the rotor R_s. These variables are defined as

$$
\begin{aligned}
F_{prop} &= \left(0, \left[0\ 0 \sum_{i=1}^{n} F_i\right]^T\right) \\
T_{prop} &= \begin{bmatrix} d(\sum F_y^- - \sum F_y^+) \\ d(\sum F_x^- - \sum F_x^+) \\ c\sum_{i=1}^{n}(-1)^i F_i \end{bmatrix} \\
T_{gyro} &= \sum_{i=1}^{n} J_R(\mathscr{S}\{\boldsymbol{\Omega}_S\} \times e_3^B)(-1)^{i+1}\omega_i
\end{aligned}
\tag{45}
$$

for n motors, where d is the distance from center of mass to rotor axis, c is the drag factor, J_R is the rotor inertia, ω_i is the i-th propeller velocity, $\sum F_y^-$, $\sum F_y^+$ are the sum of the breakdown along the y axis negative and positive, respectively; analogously to the x axis. Note that here the 6-vectors are represented in terms of the 6 bivectors of $G^+_{3,0,1}$, i.e. $e_{23}, e_{31}, e_{12}, e_{41}, e_{42}, e_{43}$.

The equations describing the dynamics multi copter is obtained using (44) and (45) in (44), note that the centrifugal force $\boldsymbol{R}_s[\boldsymbol{\Omega}_S \underline{\times} (m\boldsymbol{v}_S)]\widetilde{\boldsymbol{R}}_{\boldsymbol{S}}$ is nullified as the reference system of the earth is fixed.

$$
\begin{aligned}
\mathscr{S}\{\dot{\boldsymbol{\Omega}}_S\} &= J^{-1}(T_{prop} - T_{aero} - T_{gyro} - \mathscr{S}\{\boldsymbol{\Omega}_S\} \times J\mathscr{S}\{\boldsymbol{\Omega}_S\}) \\
\dot{\boldsymbol{v}}_S &= m^{-1}\boldsymbol{R}_{\boldsymbol{s}}(F_{prop} - F_{aero})\widetilde{\boldsymbol{R}}_{\boldsymbol{S}} - G
\end{aligned}
$$

Then

$$
\dot{\boldsymbol{V}}_S = \widehat{F} + \widehat{U} + \widehat{w} \tag{46}
$$

where

$$
\begin{aligned}
\widehat{F} &= \mathscr{S}^{-1}\{J^{-1}(-T_{gyro} - \mathscr{S}\{\boldsymbol{\Omega}_S\} \times J\mathscr{S}\{\boldsymbol{\Omega}_S\})\} + \\
&\quad + I_m\left[-G + \boldsymbol{v}_S \underline{\times} \boldsymbol{\Omega}_S + \right. \\
&\quad \left. + \boldsymbol{t}_S \underline{\times} \mathscr{S}\{J^{-1}(-T_{gyro} - \mathscr{S}\{\boldsymbol{\Omega}_S\} \times J\mathscr{S}\{\Omega_s\})\}\right]. \\
\widehat{U} &= \mathscr{S}^{-1}\{J^{-1}T_{prop}\} + I_m\left[m^{-1}\boldsymbol{R}_{\boldsymbol{s}} F_{prop}\widetilde{\boldsymbol{R}}_{\boldsymbol{S}} + \right. \\
&\quad \left. + \boldsymbol{t}_S \underline{\times} \mathscr{S}\{J^{-1}T_{prop}\}\right] \\
\widehat{w} &= \mathscr{S}^{-1}\{-J^{-1}T_{aero}\} + I_m\left[m^{-1}\boldsymbol{R}_{\boldsymbol{s}}(-F_{aero})\widetilde{\boldsymbol{R}}_{\boldsymbol{S}} \right. \\
&\quad \left. + \boldsymbol{t}_S \underline{\times} \mathscr{S}\{-J^{-1}T_{aero}\}\right]
\end{aligned}
\tag{47}
$$

so that, $\hat{w}$ is the aerodynamic wrench depending of the multi-copter velocity respect to air speed [9].

The model is described by (40) with state $M = \boldsymbol{R}_s + I_m\boldsymbol{R}'_s$ and (46) with state $\boldsymbol{V}_S = \boldsymbol{\Omega}_S + I_m\boldsymbol{\Omega}'_S$.

4. Control design

The control objective is to track the output $\boldsymbol{t}_S = t_1e_2e_3 + t_2e_3e_1 + t_3e_1e_2$ and the orientation of multi-copter ψ, with the reference $\boldsymbol{t}_r$, ψ_r. Thus the reference motor is $M_r = R_d(1 + \frac{1}{2}\boldsymbol{t}_r I_m)$, with $\boldsymbol{t}_r$ known and $R_d = r_{0d} + r_{1d}e_2e_3 + r_{2d}e_3e_1 + r_{3d}e_1e_2$ as solution of the equation system

$$\begin{aligned}\psi_r &= \tan^{-1}\left(\frac{2\,(r_{0d}r_{3d} + r_{1d}r_{2d})}{1 - 2(r_{2d}^2 + r_{3d}^2)}\right) \\ 1 &= r_{0d}^2 + r_{1d}^2 + r_{2d}^2 + r_{3d}^2 \qquad (48)\end{aligned}$$

and other three equation deduced later.

The tracking error is defined as $M_e = M_r\widetilde{M}$. With origin $O = 1+0I_m = 1$, and take asymptotically to zero $\hat{z}_1 = \log(M_e) = z_1 + I_m z_1'$. One works out the error dynamics $\hat{z}_1$ results

$$\begin{aligned}\hat{z}_1 &= \log(R_d\widetilde{\boldsymbol{R}}_{\boldsymbol{s}}) + (\boldsymbol{t}_r - \boldsymbol{t}_S)I_m \\ \dot{\hat{z}}_1 &= \frac{d}{dt}\log(R_d) - 1/2\,\Omega_s + I_m[\dot{\boldsymbol{t}}_r - \Omega_T + \Omega_s \underline{\times}(\widetilde{\boldsymbol{R}}_{\boldsymbol{s}}\boldsymbol{R}_s')].\end{aligned}$$

where according equation (22) $\boldsymbol{t}_S = 2\widetilde{\boldsymbol{R}}_{\boldsymbol{s}}\boldsymbol{R}_{\boldsymbol{s}}'$. It should be noted that the derivative of reference rotor logarithm is $d/dt\log(R_d) = \mathscr{S}^{-1}\{B_2\mathscr{S}\{\Omega_s\}+\bar{f}\}$, where the matrix B_2 and the vector $\bar{f}$ are unknown functions.

The following assumption is instrumental to get the desired result:

Assumption 1. *The unknown matrix B_2 and vector $\bar{f}$ are bounded and can be expressed as Kronecker series dependent on the state $\boldsymbol{R}_{\boldsymbol{s}} = r_0 + r_1e_2e_3 + r_2e_3e_1 + r_3e_1e_2$, this is*

$$\begin{aligned}B_{23} &= \sum_{l=0}^{\infty}\frac{1}{l!}C_l I_3 \otimes \mathscr{S}\{R_s^l\} \\ \bar{f} &= \sum_{l=0}^{\infty}\frac{1}{l!}D_l\mathscr{S}\{R_s^l\}\end{aligned}$$

where $C_l \in \mathbb{R}^{3\times 3(4^l)}$ and $D_l \in \mathbb{R}^{3\times 4^l}$ are constants, I_3 is the identity matrix of order 3. Besides $\boldsymbol{R}_{\boldsymbol{s}}^{\boldsymbol{l}} = \boldsymbol{R}_{\boldsymbol{s}}{}^{l-1}\boldsymbol{R}_{\boldsymbol{s}}$ with $\boldsymbol{R}_{\boldsymbol{s}}{}^0 = 1$ and the operator $\otimes$ as the Kronecker product.

Then an adaptive estimator for $log(R_d)$ is defined

$$\dot{\hat{\gamma}} = \widehat{B}_2\mathscr{S}\{\Omega_s\} + \widehat{\bar{f}} + \epsilon \qquad (49)$$

where

$$\epsilon = \alpha_1\tilde{\gamma} + \alpha_2\text{sign}\,(\tilde{\gamma}) \qquad (50)$$

is used to ensure stability for $\tilde{\gamma} = \mathscr{S}\{\log(R_d)\} - \hat{\gamma}$. And

$$\begin{aligned} \widehat{B}_2 &= \sum_{l=0}^{N} \frac{1}{l!} \widehat{C}_l I_3 \otimes \mathscr{S}\{\boldsymbol{R_s}^l\} \\ \widehat{\bar{f}} &= \sum_{l=0}^{N} \frac{1}{l!} \widehat{D}_l \mathscr{S}\{\boldsymbol{R_s}^l\}, \end{aligned} \tag{51}$$

are the estimate functions, with $\widehat{C}_l$ and $\widehat{D}_l$ as constants to be adapt with dynamic

$$\begin{aligned} \dot{\widehat{C}}_l &= \frac{\gamma_2}{l!} \tilde{\gamma} \mathscr{S}\{\Omega_s\}^T (I_3 \otimes \mathscr{S}\{\boldsymbol{R_s}^l\})^T, \\ \dot{\widehat{D}}_l &= \frac{\gamma_1}{l!} \tilde{\gamma} \mathscr{S}\{\boldsymbol{R_s}^l\}^T. \end{aligned} \tag{52}$$

The error dynamics become

$$\dot{\tilde{\gamma}} = \widetilde{B}_2 \mathscr{S}\{\Omega_s\} + \widetilde{\bar{f}} + \bar{e} - \epsilon \tag{53}$$

with

$$\begin{aligned} \widetilde{B}_2 &= \sum_{l=0}^{N} \frac{1}{l!} \widetilde{C}_l I_3 \otimes \mathscr{S}\{\boldsymbol{R_s}^l\} \\ \widetilde{\bar{f}} &= \sum_{l=0}^{N} \frac{1}{l!} \widetilde{D}_l \mathscr{S}\{\boldsymbol{R_s}^l\} \\ \bar{e} &= \sum_{l=N+1}^{\infty} \frac{1}{l!} C_l I_3 \otimes \mathscr{S}\{\boldsymbol{R_s}^l\} \mathscr{S}\{\boldsymbol{\Omega}_s\} + \sum_{l=N+1}^{\infty} \frac{1}{l!} D_l \mathscr{S}\{\boldsymbol{R_s}^l\} \end{aligned}$$

and the adaptive errors

$$\begin{aligned} \widetilde{C}_l &= C_l - \widehat{C}_l \\ \widetilde{D}_l &= D_l - \widehat{D}_l. \end{aligned}$$

The following result shows the convergence of the error to the equilibrium point.

Theorem 1. *Let Assumption* 1 *hold and consider the adaptive estimator* (49), (52) *for the system*

$$\mathscr{S}\{d/dt \, \log(R_d)\} = B_2 \mathscr{S}\{\boldsymbol{\Omega}_s\} + \bar{f}$$

Then the equilibrium point $\tilde{\gamma} = 0$ *of the closed loop system* (53), (50) *is asymptotically stable.*

Proof. Let a Lyapunov function

$$\mathcal{V}_\gamma = \frac{1}{2} \tilde{\gamma}^T \tilde{\gamma} + \frac{1}{2\gamma_2} \sum_{l=0}^{\infty} \sum_{r=1}^{3} \sum_{t=1}^{3(4^l)} \tilde{c}_{rtl}^2 + \frac{1}{2\gamma_1} \sum_{l=0}^{\infty} \sum_{r=1}^{3} \sum_{t=1}^{4^l} \tilde{d}_{rtl}^2, \tag{54}$$

where the terms $\widetilde{c}_{rtl}$ and $\widetilde{d}_{rtl}$ are the element r, t of the matrix $\widetilde{C}_l$ and $\widetilde{D}_l$, respectively. From (54) and (53), one obtains

$$\begin{aligned}\dot{\mathcal{V}}_\gamma = \quad & -\tilde{\gamma}^T\epsilon + \tilde{\gamma}^T\left(\sum_{k=0}^{N}\tfrac{1}{l!}\widetilde{C}_l I_3 \otimes \mathscr{S}\{\boldsymbol{R_s}^l\}\right)\mathscr{S}\{\boldsymbol{\Omega}_s\} + \\ & + \tfrac{1}{\gamma_2}\sum_{l=0}^{N}\sum_{r=1}^{3}\sum_{t=1}^{3(4^l)}\widetilde{c}_{rtl}\dot{\widetilde{c}}_{rtl} + \\ & + \sum_{l=0}^{N}\left(\tilde{\gamma}^T\tfrac{1}{l!}D_l\mathscr{S}\{\boldsymbol{R_s}^l\} + \tfrac{1}{\gamma_1}\sum_{r=1}^{3}\sum_{t=1}^{4^l}\widetilde{d}_{rtl}\dot{\widetilde{d}}_{rtl}\right) + \tilde{\gamma}^T\bar{e},\end{aligned}$$

notice that $\dot{\widetilde{c}}_{rtl} = -\dot{\widehat{c}}_{rtl}$ and $\dot{\widetilde{d}}_{rtl} = -\dot{\widehat{d}}_{rtl}$. Then, from equations (52), this results

$$\dot{\mathcal{V}}_\gamma = -\tilde{\gamma}^T\epsilon + \tilde{\gamma}^T\bar{e}.$$

Since B_2 and $\bar{f}$ are bounded, then

$$\|\bar{e}\| \le \delta_1\|\tilde{\gamma}\| + \delta_2$$

is also bounded. So if $v = \alpha_1\tilde{\gamma} + \alpha_2\mathrm{sign}(\tilde{\gamma})$, it follows that

$$\begin{aligned}\dot{\mathcal{V}}_\gamma &= -\tilde{\gamma}^T\alpha_1\tilde{\gamma} - \tilde{\gamma}^T\alpha_2\mathrm{sign}(\tilde{\gamma}) + \bar{e} \\ &\le -\alpha_1\|\tilde{\gamma}\|^2 - \alpha_2\|\tilde{\gamma}\| + \delta_1\|\tilde{\gamma}\|^2 + \delta_2\|\tilde{\gamma}\| \end{aligned} \tag{55}$$

so that, $\alpha_1 > \delta_1$ and $\alpha_2 > \delta_2$ make the system (53) asymptotically stable. Also using Barbalat's lemma and LaSalle's theorem stability in subsystem (52) is also ensured. □

Now, from Theorem 1, the closed loop system (53) with the adaptable law (52), has a stable equilibrium point $\tilde{\gamma} = 0$. Then the desired states are proposed as pseudo-control Ω as shown in [6], this is

$$\begin{aligned}\boldsymbol{V}_d &= \boldsymbol{\Omega}_{sd} + I_m\boldsymbol{\Omega}'_{sd} \\ \boldsymbol{\Omega}_{sd} &= 2\mathscr{S}^{-1}\left\{\left(\widehat{B}_2 - 1/2\, I_3\right)^{-1}\left(\widehat{\bar{f}} - 2k_{1R}\mathscr{S}\{z_1\}\right)\right\} \\ \boldsymbol{\Omega}'_{sd} &= \dot{\boldsymbol{t}}_r + 2\boldsymbol{\Omega}_{sd}\underline{\mathrm{x}}(\widetilde{\boldsymbol{R}}_{\boldsymbol{s}}\boldsymbol{R}'_{\boldsymbol{s}}) + \mathscr{S}^{-1}\{k_{1T}\mathscr{S}\{z'_1\}\}\end{aligned}$$

where k_{1q} and k_{1T} are the gain matrices control to stabilize $\hat{z}_1$. Now, defining $\hat{z}_2 = \boldsymbol{V}_d - \boldsymbol{V}_S$, results

$$\dot{\hat{z}}_2 = \dot{\boldsymbol{V}}_d - \widehat{F} - \widehat{U} - \widehat{w}$$

Otherwise, let define $\dot{\hat{z}}_2 = z_2 + z_{2T}$, then

$$\dot{\hat{z}}_2 = \dot{\boldsymbol{\Omega}}_{sd} - U_T + I_m(\dot{\boldsymbol{\Omega}}'_{sd} - m^{-1}R_d F_{prop}\widetilde{R}_d + U_T\underline{\mathrm{x}}\boldsymbol{t}_S) - \hat{F}$$

therefore $\hat{z}_2$ is chosen as sliding surface. The control is designed through the super-twisting algorithm [7]:

$$\begin{aligned}\widehat{U} &= \dot{\widehat{\boldsymbol{\Omega}}}_d - \widehat{F} + k_{2R}\|z_2\|\,\mathrm{sign}(z_2)) + \\ &\quad + I_m k_{2T}\|z'_2\|\,\mathrm{sign}(z'_2) - \widehat{V} \\ \dot{\widehat{V}} &= -\alpha_R\mathrm{sign}(z_2) - I_m\alpha_T\mathrm{sign}(z'_2)\end{aligned} \tag{56}$$

with $\widehat{U} = U_T + I_m U_F$, also $k_{2R}, k_{2T}, \alpha_R, \alpha_T$ are scalars to stabilize the sliding surface $\hat{z}_2 = z_2 + I_m z'_2$.

Finally we have

$$m^{-1}R_dF_{prop}\widetilde{R}_d = U_F - I_mU_T\underline{\mathbf{x}}\boldsymbol{t}_S = m^{-1}(2(r_{1d}r_{3d} + r_{0d}r_{2d})e_{23} + 2(r_{0d}r_{1d} + r_{2d}r_{3d})e_{31} + (r_{0d}^2 - r_{1d}^2 - r_{2d}^2 + r_{3d}^2)e_{12}) \sum_i^n F_i.$$

Where $R_d = r_{0d} + r_{1d}e_{23} + r_{2d}e_{31} + r_{3d}e_{12}$ and $T_{prop} = JU_T$, which, in conjunction with equations (48), solves the system of non-linear equations for R_d and for the forces of each motor F_i.

To show the stability of the closed-loop system (40), (46), (56), (49), (50), (52) suppose that the external perturbation is bounded by $\|\widehat{w}\| \leq b\|\hat{z}_2\|^{1/2}$, where the closed loop system is

$$\begin{aligned}\dot{\hat{z}}_1 &= -(1/2I_3 + \widehat{B}_2)^{-1}(k_{1R}\mathscr{S}\{z_1\} + k_{1T}\mathscr{S}\{z_1'\}) + \widetilde{\bar{f}} + \widetilde{B}_2\mathscr{S}\{\boldsymbol{\Omega}_s\} - \hat{z}_2 \\ \dot{\hat{z}}_2 &= STA(\hat{z}_2) + \widehat{w}.\end{aligned}$$

Where $STA(\hat{z}_2)$ is the super-twisting algorithm applying to $\hat{z}_2$ as in (47). After the system enters the sliding regime, we obtain $\hat{z}_2 = 0$, and thus consider the Lyapunov function

$$\mathcal{V} = \frac{1}{2}(\|z_1\|^2 + \|z_1'\|^2) + \mathcal{V}_\gamma.$$

Then we get

$$\begin{aligned}\dot{\mathcal{V}} &= -\mathscr{S}\{z_1\}^T(1/2I + \widehat{B}_2)^{-1}(k_{1R}\mathscr{S}\{z_1\} + k_{1T}\mathscr{S}\{z_1'\}) + \\ &\quad +\mathscr{S}\{z_1 + z_1'\}^T\left(\widetilde{\bar{f}} + \widetilde{B}_2\mathscr{S}\{\Omega_s\}\right) - \\ &\quad -(\alpha_1 - \delta_1)\|\tilde{\gamma}\|^2 - (\alpha_2 - \delta_2)\|\tilde{\gamma}\|\end{aligned} \tag{57}$$

Assuming that the estimation error for $\widetilde{\bar{f}} + \widetilde{B}_2\mathscr{S}\{\boldsymbol{\Omega}_s\}$ is bounded as $\widetilde{\bar{f}} + \widetilde{B}_2\mathscr{S}\{\boldsymbol{\Omega}_s\} \leq \delta\|z_1\|$, it comes that

$$\begin{aligned}\dot{\mathcal{V}} \leq\ & -(\lambda_{min}(k_{1R}, k_{1T}) - \delta)\|\hat{z}_1\|^2 - (\alpha_2 - \delta_2)\|\tilde{\gamma}\| - \\ & -(\alpha_1 - \delta_1 - \delta)\|\tilde{\gamma}\|^2.\end{aligned}$$

Using (54), (58) we finally obtain

$$\dot{\mathcal{V}} \leq\ -\theta\|\hat{z}_1\| - (\alpha_1 - \delta_1)\|\tilde{\gamma}\| - (\alpha_2 - \delta_2)\|\tilde{\gamma}\|^2$$

with θ a positive constant. Thus, the Lyapunov function is negative definite, and thus the solution $\|\hat{z}_1(t)\|$ tends asymptotically to zero.

5. Simulation results

To verify the effectiveness and robustness of the proposed feedback control law detailed above, a simulation of a quad-rotor flight system is presented. In order to apply this control law, it is necessary to compute the angular velocities ω_i of the motors, which are obtained from the approximation of $F_i = b\omega_i^2$ to i-th actuator, with b the thrust factor and $F_i = c\omega_i^2$ for the tangential force and c as the drag factor. It follows that

$$\begin{bmatrix} \omega_1^2 \\ \omega_2^2 \\ \omega_3^2 \\ \omega_4^2 \end{bmatrix} = \begin{bmatrix} 0 & -db & 0 & db \\ -db & 0 & db & -db \\ -c & c & -c & c \\ b & b & b & b \end{bmatrix}^{-1} u$$

where d is the distance from center of mass to rotor axis, ω_i the i-th propeller velocity for $i = \{1, 2, 3, 4\}$ as defined previously, also $u = [F_1, F_2, F_3, F_4]^T$. Figure 2 depicts the forces and the movements of the quad-rotor.

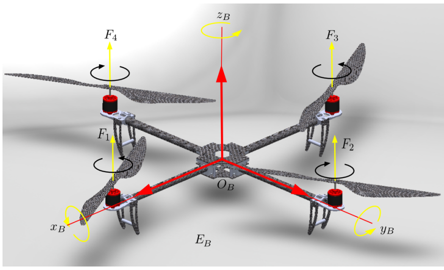

FIGURE 2. Forces and movements of a quadrotor.

The reference for the linear position is created by using way points to following in the coordinates (x_r, y_r, z_r) in an earth reference frame which are given by the points: $(0, 0, 1)$, $(1, 0, 1)$, $(1, 1, 1)$, $(0, 1, 1)$ and $(0, 0, 1)$ in meters at intervals of 4 s. Table 1 shows the parameters of the quadrotor used in the simulation.

TABLE 1. Parameters of quadrotor.

Parameter	Value
m	4.900
$I_{x,y}$	21.6×10^{-3}
I_z	43.2×10^{-3}
J_R	3.357×10^{-5}
d	0.45
c	1.140×10^{-7}
b	2.98×10^{-5}

Table 1.

To demonstrate the robustness of the proposed control law, the mass m and the inertial tensor I are increased by 30% of the initial value at $t = 10$ s. In addition, external disturbances as aerodynamic accelerations are included as shown in Figure 3.

The Figure 4 and 5 depict the change of position and the angles of the quadrotor respectively. It can be seen that at $t = 10\,s$ the system is perturbed causing a small transitory behavior in the both translational and rotational

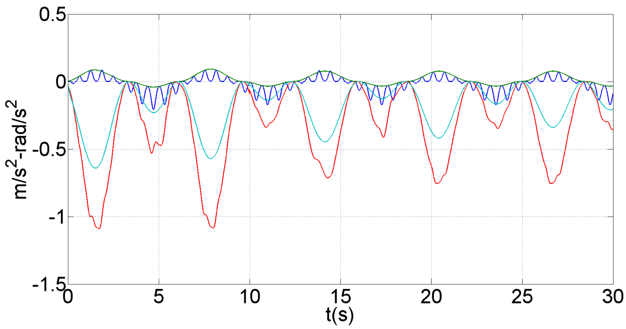

FIGURE 3. Aerodynamic acceleration

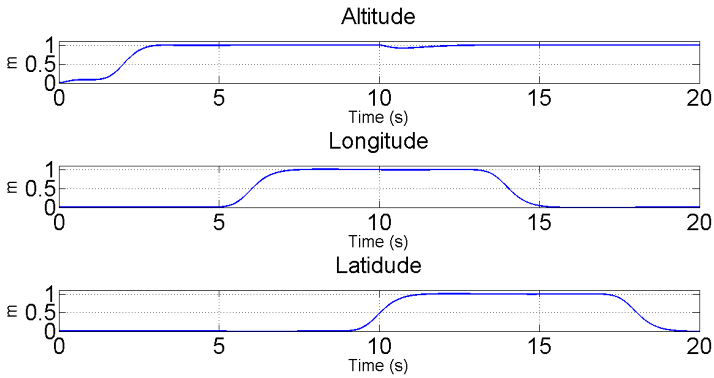

FIGURE 4. Translational motion.

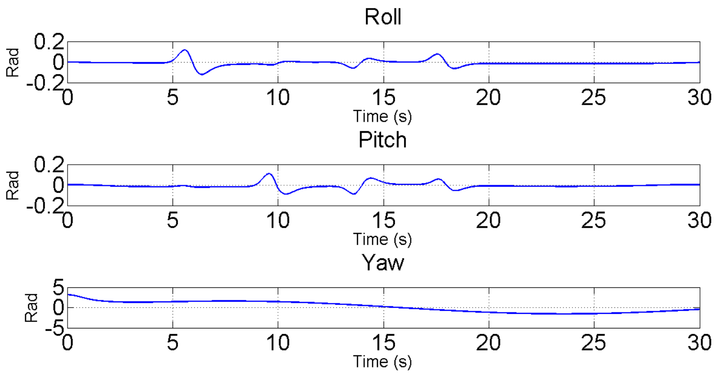

FIGURE 5. Rotational motion (angles).

motions due to parametric variation, however the control laws manage to keep the tracking of the reference despite of this transients and the external disturbances.

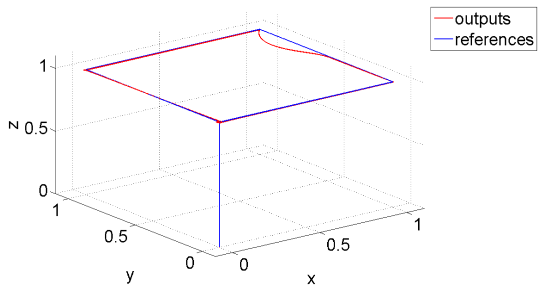

FIGURE 6. Trajectory in earth reference frame.

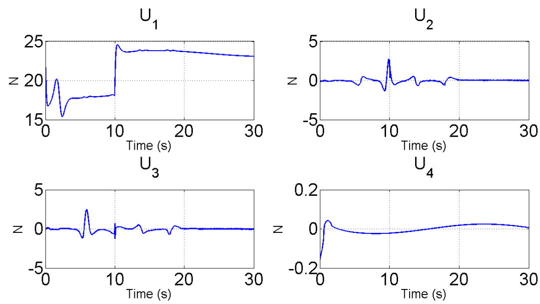

FIGURE 7. Control law.

The Figure 6 displays the position in the earth reference frame. It can be observed that the the quadrotor tracks satisfactorily the given reference. The slight deviation next to the third corner is due to the added perturbation at t=10 sec., but as we see the control manage to return to the expected trajectory.

Finally, the control signal is presented in Figure 7, which has a smooth behavior and it is within the range allowed by the actuators despite the external disturbances and parametric variations. It is shown at t=10 sec. how the control laws react under the added perturbation.

6. Conclusion

In this work the dynamic model and the nonlinear control for a multi-copter have been developed using the geometric algebra framework specifically using the motor algebra $G^+_{3,0,1}$. The kinematic for the aircraft model and the dynamic based on Newton-Euler formalism are presented and the block-control technique is applied in combination with super twisting including an internal dynamics estimator driven by maneuvers away from the origin. Finally, the stability of the presented control scheme is shown.

It has been also shown that our non-linear controller law is able to reject external disturbances and to deal with parametric variations. This control has the computational advantages inherited from geometric algebra compared to the classical use of Euler angles. Specifically, motor algebra is used due to its excellent ability to deal with the need of two separate control laws for attitude and position, allowing to solve both problems simultaneously in a more compact and efficient manner.

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