Projections onto Hyperplanes in Banach Spaces

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1. INTRODUCTION AND NOTATION

In this paper $X$ will always denote a real Banach space, $X^*$ its norm dual, $U, S$ ($U^*, S^*$) their unit balls and spheres. If $V$ is a closed subspace of $X$, a projection onto $V$ is a continuous linear operator $P: X \rightarrow V$ such that $Py = y$ if $y \in V$. A hyperplane in $X$ is a subspace $V$ of the form $V = f^{-1}(0)$, where $f \in S^*$. It is easy to see that any projection $P$ onto the hyperplane $V = f^{-1}(0)$ is of the form $P = x - f(x)z$, with $z \in f^{-1}(1)$; this projection will be denoted by $P_z$.

We clearly have $1 \leq \|P_z\| \leq 1 + \|z\|$. Let $\varepsilon > 0$. Since $\exists z_\varepsilon \in f^{-1}(1)$ with $\|z_\varepsilon\| < 1 + \varepsilon$, we can always find a projection $P$ with $\|P\| < 2 + \varepsilon$ and, when $X$ is reflexive, with $\|P\| \leq 2$. The relative projection constant $\lambda(V, X)$ of $V$ with respect to $X$ is defined by: $\lambda(V, X) = \inf\{\|P\|: P$ projects $X$ onto $V\}$; note that $1 \leq \lambda(V, X) \leq 2$; $P$ is a minimal projection onto $V$ if $\|P\| = \lambda(V, X)$. Reference [4] contains a very interesting and complete study of minimal projections and relative projection constants when $X$ is one of the sequence spaces $c_0, l_1$.

The aim of this paper is to present some results related to the projections onto a hyperplane and to point out the relationships among the norms of the projections, the shape of the unit ball and the metric properties of the hyperplanes. Section 2 contains the main result (Theorem 3): it is proved that an upper bound for the number $\lambda(V, X)$ leads to the characterization of those hyperplanes which are range of a projection with norm strictly less than 2 (Theorem 4). In Section 3 an application of the previous results gives a substantial improvement of an inequality proved in [9] between the Jung constant $J$ and the projection constants $\lambda_1$ of a Banach space (Theorem 6). In Section 4 a new parameter $F(X)$ of the Banach space $X$, depending on the collection of all hyperplanes of $X$, is considered and studied. Section 5 is devoted to a short investigation of the function $\rho$ (defined below) and other functions related to the norm of the projections onto a given hyperplane.

We list now some other definitions and notations.
For any real $a$ set $V_a = f^{-1}(a)$ (note that all the $V_a$ are isometric with $V = V_0$). For $0 \leq a < 1$ set $C_a = U \cap V_a$, $A(a) = \frac{1}{2} \text{diam } C_a$ and

$$\rho(a) = \rho_r(a) = r_{v_a}(C_a) = r_v(C_a);$$

$C_a$ is sometimes called a hypercircle, $A(a)$ is half the diameter of the set $C_a$ and $\rho(a)$ is the (Chebyshev) radius of $C_a$ relative to the set $V_a$, i.e., the number:

$$\rho(a) = \inf_{x \in V_a} \sup_{y \in C_a} \|x - y\|, \|x\| \leq 1, f(x) = a.$$

For $0 \leq \varepsilon$ set

$$E^e(a) = \{x \in V_a : \sup_{y \in C_a} \|x - y\| \leq \rho(a) + \varepsilon\};$$

for $\varepsilon = 0$, $E^0(a) = E(a)$ is the (possibly empty) set of the centers of $C_a$ relative to $V_a$. (Note that if $\varepsilon > 0$, $E^e(a)$ is always non-empty.) $C_a$, $\rho(a)$ and $E^e(a)$ are studied, in a slightly different situation, in [7].

2. Main Result

Let us begin with the following:

**Lemma 1.** Assume that $0 \leq a < 1$, $\varepsilon > 0$.

(i) For $a \in V_a$ we have $c \in E^e(a)$ if and only if

$$\|c - y\| \leq \|y\| + \rho(a) - 1 + \varepsilon$$

for any $y \in C_a$.

(ii) $1 - a \leq \rho(a) \leq 1 + a$. (2.2)

(iii) $\|c\| \leq 2a + \rho(a) - 1 + \varepsilon$ (2.3)

for any $c \in E^e(a)$.

(iv) $A(a) \leq \rho(a) \leq (1 + a) A(a)$.

(2.4)

**Proof.** (i) and (ii) are essentially Theorems 2 and 3 in [8]; for the sake of completeness we give here a new proof.

(i) Let $c \in V_a$. If (2.1) holds then clearly $\sup_{y \in C_a} \|c - y\| \leq \rho(a) + \varepsilon$ which means that $c \in E^e(a)$. Assume now that $c \in E^e(a)$ and that $x \neq c$ is a point in the relative interior of $C_a$; the line $\lambda c + (1 - \lambda) x$ meets the relative boundary of $C_a$ in two points $\xi_i = \lambda_i c + (1 - \lambda_i) x$, with $\|\xi_i\| = 1$. 


One of the $\lambda_i$, say $\lambda_1$, is strictly negative. We have $c - \zeta_i = (1 - \lambda_i)(c - x)$, so 
$$
(1 - \lambda_1)\|c - x\| \leq \rho(a) + \varepsilon = \rho(a) - 1 + \|\zeta_i\| + \varepsilon.
$$
Since $\|\zeta_i\| \leq \|x\| - \lambda_1\|x - c\|$ we get 
$$
(1 - \lambda_1)\|c - x\| \leq \rho(a) - 1 + \|x\| - \lambda_1\|x - c\|.
$$
So $\|x - c\| \leq \|x\| + \rho(a) - 1$ and this last inequality holds also for points $x$ in the relative boundary of $C_a$.

(ii) For $v \in V_a$ we have $\rho(a) \leq \sup\{|v - y|, y \in C_a\} \leq \|v\| + 1$ which implies $\rho(a) \leq 1 + a$ since $\inf\{|v|, v \in V_a\} = a$. For $c \in E^a(a)$, $y \in C_a$ by (2.1) we have $\rho(a) \geq 1 - \varepsilon + \|c - y\| - \|y\| \geq 1 - \varepsilon - \|y\|$. This implies $\rho(a) \geq 1 - a$. (Select $z$ such that $\|z\| = 1, f(z) \geq 1 - \varepsilon$ and take $y = az/f(z)$.)

(iii) (2.3) is just a consequence of (2.1).

(iv) $\Delta(a) \leq \rho(a)$ is trivial. Let $x \in C_a$ with $\|x\| = 1$, $v \in V_a$ with $\|v\| < a + \varepsilon$; there exists a $\lambda < 0$ such that $\|\lambda x + (1 - \lambda)v\| = 1$. So we have 
$$
1 \leq -\lambda + (1 - \lambda)(a + \varepsilon);
$$
therefore $1 - \lambda \geq 2/(1 + a + \varepsilon)$. $2\Delta(a) \geq \|x - (\lambda x + (1 - \lambda)v)\| = (1 - \lambda)\|x - v\|$. Taking sup on $x$ we get $2\Delta(a) \geq (1 - \lambda)\rho(a) \geq 2\rho(a)/(1 + a + \varepsilon)$ which completes the proof of (2.4).

Let us define

$$
c = c_v = \sup\{\rho_v(a), 0 \leq a < 1\}.
$$

By (2.1) we have $1 \leq c_v \leq 2$. Define also $\gamma_\varepsilon : [0, 1) \to [0, \|P_z\|] \text{ by }$

$$
\gamma_\varepsilon(a) = \sup\{|x - az|, x \in C_a\} = \sup\{|P_zx|, x \in C_a\},
$$

where $P_z$ is the usual projection defined by $P_zx = x - f(x)z$ ($f(z) = 1$). Clearly we have $\gamma_\varepsilon(0) = 1$, $\sup\{\gamma_\varepsilon(a), 0 \leq a < 1\} = \|P_z\|$; also, $\rho(a) = \inf_{\varepsilon \in V_a} \sup\{|x - v|, x \in C_a\} = \inf_{\varepsilon \in V_a} \sup\{|x - az|, x \in C_a\} = \inf_{\varepsilon \in V_a} \|P_z\| = \inf_{\varepsilon \in V_a} \sup_a \gamma_\varepsilon(a) \geq \inf_{\varepsilon} \sup_a \rho(a) = c_v = \inf_a \gamma_\varepsilon(a)$. We cannot in general interchange here inf sup with sup inf; i.e., in the inequalities

$$
1 \leq c_v \leq \lambda(V, X)
$$

it can happen that $c_v < \lambda(V, X)$. An example is given in Section 5. The parameter $c_v$ is considered also in [9] but is defined differently; in [13] a related parameter $v(V)$ is studied. In order to make a comparison possible we note that, using the notations of [8, 9, 13] and the ones introduced here, we have the equivalences $\rho(a) = r(d/a)/(d/a)$, $r_i/s = \rho(d/s)$, where $d$ is a fixed distance; see [8, 9, 13]. (This follows from the equality $C_i(s) = sC_1/(1)$, where $C_a(s) = sU \cap V_a$.) In particular, note that

$$
c_v = m(V, X) \quad [9, \text{Lemma, p. 42}],
$$

$$
\rho'_+(0) = v(V) = \bar{I}((V)/d) \quad [13, \text{p. 85}].
$$
here $\rho_1'(0) = \lim_{a \to 0^+} ((\rho(a) - \rho(0))/a) = \lim_{a \to 0^+} ((\rho(a) - 1)/a)$. (The right
derivative at the origin of $\rho$ exists since the ratio $(\rho(a) - 1)/a$ is non-increasing; see Section 5.) It is consequently easy to prove (see [13]) that

$$\lambda(V, X) \leq 1 + \rho_1'(0) = 1 + \phi(V).$$

(2.8)

We now want to prove a lemma on projections.

Let $V = f^{-1}(0), f \in S^*_\alpha, 0 < a < 1$ and $\varepsilon > 0$; select $z_e \in f^{-1}(1)$ such that $\|z_e\| < 1 + \varepsilon$ and $c_a^\alpha \in E^\alpha(\alpha)$. We define the projections $P_{z_e}$ and $Q_a^\varepsilon$ onto $V$

by

$$P_{z_e} x = x - f(x) z_e, \quad Q_a^\varepsilon x = x - f(x) c_a^\alpha e_\alpha.$$

Set also $A = \{x \in S: a \leq f(x) \leq 1\}, B = \{x \in S: 0 \leq f(x) < a\}$.

**Lemma 2.** We have

$$\sup_{x \in A} \|P_{z_e} x\| < 2 + \varepsilon,$$

$$\sup_{x \in B} \|P_{z_e} x\| < 1 + \alpha + \varepsilon,$$

$$\|P_{z_e}\| < 2 + \varepsilon,$$

(2.9)

$$\sup_{x \in A} \|Q_a^\varepsilon x\| \leq 1 + (\rho(a) - 1)/a + \varepsilon,$$

$$\sup_{x \in B} \|Q_a^\varepsilon x\| \leq 2a + \rho(a) + \varepsilon,$$

$$\|Q_a^\varepsilon\| \leq \max(1 + (\rho(a) - 1)/a, 2a + \rho(a)) + \varepsilon.$$

(2.10)

**Proof.** (2.9) is trivial. Let us prove (2.10). If $x \in A$ then $ax/f(x) \in C_\alpha$; therefore

$$\|Q_a^\varepsilon x\| = \left\| \frac{f(x)}{a} \left[ \frac{a}{f(x)} x - c_a^\alpha \right] \right\|.$$

Using (2.1) we obtain

$$\|Q_a^\varepsilon x\| \leq \frac{f(x)}{a} \left[ \frac{a}{f(x)} \|x\| + \rho(a) - 1 + a\varepsilon \right]$$

$$\leq 1 + \frac{f(x)}{a} |\rho(a) - 1 + a\varepsilon|.$$

If $x \in B$, using (2.3) we obtain $\|Q_a^\varepsilon x\| \leq 1 + \|c_a^\alpha\| \leq 2a + \rho(a) + \varepsilon$. |

Let us consider for $0 < a < 1$ the function $\psi$ defined by

$$\psi(a) = \left(1 + \frac{\rho(a) - 1}{a}\right) - (2a + \rho(a)) - \frac{(\rho(a) - 1)(1 - a)}{a} - 2a.$$

(2.11)
Since \( p(a) \leq 1 + a \) we have \( \psi(a) \leq 1 - 3a \); hence \( \psi(a) < 0 \) if \( a > \frac{1}{3} \). Also \( \psi(0) = p'_+(0) \). By (2.8) we have \( \psi(0) > 0 \) if \( \lambda(V, X) > 1 \); therefore in this case there exists a \( \beta \in (0, \frac{1}{3}) \) such that \( \psi(\beta) = 0 \). Recalling that \( c = \sup p(a) \) we also have \( \psi(a) \leq (c - 1)(1 - a)/a - 2a \). Assume that \( 1 < c < 2 \); then \( \psi((c - 1)/c) \leq (8 - 6c - c^2)/2c \) and therefore \( \psi((c - 1)/c) < 0 \) if \( c > \sqrt{17} - 3 \approx 1.123 \). We shall use this last fact in proving Theorem 2.

Now consider the problem: when does a projection \( P: X \to V \) exist with \( \|P\| \leq 2 \)? If \( \lambda(V, X) < 2 \) this is obviously the case; when \( \lambda(V, X) = 2 \) this is still the case if \( X \) is reflexive. We shall prove a more general result.

Recall that it is said that a Banach space \( X \) admits centers if for every bounded subset \( A \) of \( X \) the set of the (absolute) centers of \( A \) is non-empty. Examples of such spaces are: dual (hence reflexive) spaces, \( L^1(\mu) \) (\( \mu \sigma \)-finite) and \( C(Q) \) (\( Q \) Hausdorff compact) but the class is wider; see [2] for new examples and a survey of the classical existence theorems.

Let us consider \( V \) as a Banach space in itself; noting that \( E(a) \) is the set of the centers of \( C_a \) in \( V \), which is isometric with \( V^* \), we see that \( E(a) \) is non-empty if \( V \) admits centers.

**Theorem 1.** If \( V \) admits centers there exists a projection \( P: X \to V \) such that \( \|P\| \leq 2 \).

**Proof.** Since \( V \) admits centers \( E(a) \) is non-empty for \( 0 \leq a < 1 \); the projection \( Q^a \) considered in Lemma 2 is defined also for \( \varepsilon = 0 \) by any \( c_a \in E(a) \) \( (Q^a x = x - c_a f(x)/a) \). Using (2.10) we get \( \|Q^a\| \leq \max(1 + (\rho(a) - 1)/a, 2a + \rho(a)) \). Since we may assume that \( \lambda(V, X) > 1 \) there exists a \( \beta \in (0, \frac{1}{3}) \) such that \( \psi(\beta) > 0 \). For this \( \beta \) we have \( \|Q^\beta\| \leq 1 + (\rho(\beta) - 1)/\beta \); hence \( \|Q^\beta\| \leq 2 \) since \( \rho(\beta) \leq 1 + \beta \). \( \square \)

**Example 1.** Take \( X = 1^1 \), \( V = f^{-1}(0) \), where \( f \in S^* \) is the element of \( 1^\infty \) defined by \( f = (1/2, 2/3, \ldots, (n - 1)/n, \ldots) \). We have \( \lambda(V, X) = 2 \) (see [4, Corollary, p. 224]). On the other hand, it is easy to see that there is no norm 2 projection onto \( V \); therefore for any \( P: X \to V \) we have \( \|P\| > 2 \) (no projection is minimal). This counterexample is due to Grünbaum [15, p. 199]. By the preceding theorem \( V \) does not admit centers. For a similar negative example see [10, p. 41].

We now prove our main result.

**Theorem 2.** Let \( V = f^{-1}(0) \), \( f \in S^* \). For every \( \sigma > 0 \) there exists a projection \( P_\sigma : X \to V \) such that

\[
\|P_\sigma\| \leq g(c) + \sigma, \tag{2.12}
\]
where \( c \) is defined by (2.5) and \( g: [1, 2] \to [1, 2] \) by

\[
g(c) = 1 + \frac{1}{4} \left( (c - 1) + \sqrt{(c - 1)^2 + 8(c - 1)} \right) \quad \text{if} \quad 1 \leq c \leq \sqrt{17} - 3
\]

\[
= 1 + \frac{8(c - 1)}{c^2 + 4(c - 1)} \quad \text{if} \quad \sqrt{17} - 3 < c \leq 2.
\]

\( (2.13) \)

**Proof:** By (2.10) we have for the projection \( Q^a_\alpha \): \( \| Q^a_\alpha \| \leq \max(1 + (\rho(a) - 1)/a, 2a + \rho(a)) + \sigma \leq \max(1 + (c - 1)/a, 2a + c + \sigma). \)

Computing the optimal value for \( a \) we find a projection \( Q^a_\alpha \) such that \( \| Q^a_\alpha \| \leq 1 + \frac{1}{4} \left( (c - 1) + \sqrt{(c - 1)^2 + 8(c - 1)} \right) + \sigma. \) (This computation was done in [9, Theorem 4]. We will give here a much better result when \( c > \sqrt{17} - 3. \))

Set \( P^\alpha_\lambda = \lambda P^\alpha_{z_1} + (1 - \lambda) Q^a_\alpha \). \( P^\alpha_\lambda \) is of course a projection and for \( 0 \leq \lambda \leq 1, \) using Lemma 2, we have

\[
\| P^\alpha_\lambda x \| \leq \lambda \| P^\alpha_{z_1} x \| + (1 - \lambda) \| Q^a_\alpha x \|
\]

\[
\leq \lambda(2 + \sigma) + (1 - \lambda) \left( 1 + \frac{\rho(a) - 1}{a} + \sigma \right) \quad \text{if} \quad x \in A
\]

\[
\leq \lambda(1 + a + \sigma) + (1 - \lambda)(2a + \rho(a) + \sigma) \quad \text{if} \quad x \in B
\]

hence we obtain

\[
\| P^\alpha_\lambda \| \leq \max \left( 2\lambda + (1 - \lambda) \frac{a + \rho(a) - 1}{a}, (1 + a) \lambda + (1 - \lambda)(2a + \rho(a)) + \sigma \right).
\]

When \( \psi(a) < 0 \) (\( \psi \) is defined by (2.11)) a possible and optimal choice for \( \lambda \) in \( [0, 1] \) is

\[
\lambda = \lambda_a = -\frac{\psi(a)}{(1 - \rho(a))/a + a + \rho(a)} = \frac{2a^2 (1 - a)/(1 - \rho(a))}{1 + a^2 - \rho(a)(1 - a)}.
\]

With such a choice we get

\[
\| P^\alpha_{z_1} \| \leq 1 + \frac{2a^2}{1 + a^2 - (1 - a) \rho(a)} \quad \| \sigma \| \leq 1 + \frac{2a^2}{1 + a^2 - (1 - a) c} \quad \| \sigma \|.
\]

We have seen that \( \psi((2c - 2)/c) < 0 \) if \( c > \sqrt{17} - 3; \) therefore the choice \( a = (2c - 2)/c \) is permitted if \( \sqrt{17} - 3 < c < 2 \) (it must be \( a < 1 \)) and we obtain a projection \( R^a \) such that \( \| R^a \| \leq 1 + 8(c - 1)/(c^2 + 4(c - 1)) + \sigma. \) Using \( Q^a \) and \( R^a \) the proof of this theorem is completed (note that when \( c = 2, \) \( g(c) = 2 \) and (2.12) holds).
Remark. The function $g$ has the following properties: $g \in C^1(1, 2)$; $g(1) = 1$, $g(2) = 2$; $c \leq g(c)$; $g$ is strictly increasing and concave; $g'(1) = \infty$, $g'(2) = 0$. In the point $c_0 = \sqrt{17} - 3$ we have $g(c_0) = (\sqrt{17} - 1)/2$, $g'(c_0) = (\sqrt{17} + 1)/2$.

**Theorem 3.** We have

$$1 \leq c_v \leq \lambda(V, X) \leq g(c_v) \leq 2,$$

(2.14)

where the function $g$ is defined by (2.13).

**Proof.** This is (2.7) and an obvious consequence of Theorem 2.

**Theorem 4.** We have

$$\lambda(V, X) = 1 \iff c_v = 1 \iff \forall a \in (0, 1): \rho(a) \leq 1,$$

(2.15)

$$\lambda(V, X) < 2 \iff c_v < 2 \iff \exists a \in (0, 1): \rho(a) < 1 + a.$$

(2.16)

**Proof.** (2.15) follows from (2.14) since $g(1) = 1$. By (2.14) and the properties of $g$ it follows that $c_v < 2 \Rightarrow \lambda(V, X) < 2$. Also, $c_v < 2 \Rightarrow \exists a: \rho(a) < 1 + a$. Assume now that for $a, \beta \in (0, 1)$ we have $\rho(\beta) < 1 + \beta$. If $\psi(\beta) > 0$ for $\sigma$ small enough the projection $Q_{\alpha}^a$ used in Theorem 2 has norm $\|Q_{\alpha}^a\| < 2$ for $a = \beta$. If $\psi(\beta) < 0$ the projection $P_{\lambda, a}^\alpha$ (see Theorem 2) has norm $\|P_{\lambda, a}^\alpha\| \leq 1 + \beta^2 + \sigma$.

3. The Parameters $J$ and $\lambda_1$

We first recall briefly some well known definitions and properties of certain projection constants. Assume that $V$ is a real Banach space: we say that $V \in \mathcal{P}_\lambda$ ($\lambda \geq 1$) if for every subspace $Z$ there is a projection $P: Z \to V$ such that $\|P\| \leq \lambda$. The (absolute) projection constant of $V$ is $\lambda(V) = \inf \{r: V \in \mathcal{P}_r\}$. We say that $V \in E_\lambda$ ($\lambda \geq 1$) if for every subspace $Z$ with $\dim Z/V = 1$ there is a projection $P: Z \to V$ such that $\|P\| \leq \lambda$. The constant $\lambda_1(V)$ is defined by $\lambda_1(V) = \inf \{r: V \in E_r\}$.

Note that $\lambda(V) = \sup \{\lambda(V, Z): V \subset Z\}$ and

$$\lambda_1(V) = \sup \{\lambda(V, Z): V \subset Z, \dim Z/V = 1\}.$$

(3.1)

It is easily seen that $1 \leq \lambda_1(V) \leq \lambda(V) \leq \infty$, $\lambda_1(V) \leq 2$. We recall also the definition of the Jung constant of $V$, $J(V)$:

$$J(V) = \sup \{r(A)/A(A), A \subset V, A \text{ bounded}\};$$

here $r(A)$ is the (absolute Chebyshev) radius of $A$ and $A(A) = \frac{1}{2} \text{diam}(A)$. 
Clearly $1 \leq J(V) \leq 2$. References on all these parameters are found in [9] where especially the relationship between $J$ and $\lambda_1$ is investigated.

We now give some applications of the results of Section 2.

We note that when $V$ is a hyperplane in $X$, for greater precision one should write $c(V, X)$ instead of $c_V$ or $c$ and $\rho_{V, X}(a)$ instead of $\rho_V(a)$ or $\rho(a).

Theorem 3 in [9] can be stated as:

**Theorem 5 (see [9]).**

$$J(V) = \sup \{c(V, X), \ V \subset X, \ \dim X/V = 1 \}. \quad (3.2)$$

The following is the main application, in this context, of Theorem 3.

**Theorem 6.** We have

$$1 \leq J(V) \leq \lambda_1(V) \leq g(J(V)) \leq 2. \quad (3.3)$$

**Proof.** In (2.14) take $\dim X/V = 1$, use (3.1), (3.2) and the fact that $g$ is strictly increasing. \[\]

**Corollary (see [7]).**

$$J(V) = 1 \iff \lambda_1(V) = 1.$$  

This was first proved in [7]; see [9] for other equivalences and references. Theorem 7 follows immediately from Theorem 6.

**Theorem 7.** $J(V) = 2 \iff \lambda_1(V) = 2.$

This is a new result. This theorem has motivations in Banach space theory; see, for example, Theorem 8. The interest in describing situations where the Jung constant and the projection constant $\lambda_1$ have the same value goes back to Grünbaum (see [14, 15]).

We remark that Theorem 6 is a substantial improvement of Theorem 4 (formula (4)) in [9] since the new bound $\lambda_1(V) \leq g(J(V))$ is now significant for every value of $J(V)$. This fact gives a parallel improvement of Theorem 5 in [9]. In fact we have:

**Theorem 8.** Let $C(Q)$ be the space of real continuous functions on the compact $Q$ with the usual sup norm. We have

$$J(C(Q)) < 2 \iff C(Q) \in \mathcal{F}_1.$$  

**Proof.** If $J(C(Q)) < 2$ by (3.3), $\lambda_1(C(Q)) < 2$ and this implies that $C(Q) \in \mathcal{F}_1$ by a theorem of Amir; see [1]. \[\]
This last result has been proved independently by Professor Amir who communicated it at the 1981 meeting on Approximation Theory in Oberwolfach.

Note that $C(Q) \in \mathcal{P}_1$ if and only if $Q$ is stonian.

4. THE PARAMETER $F$

We now discuss the relevance of the previous results from a different point of view. For a given (real) Banach space $X$ let us define

$$F(X) = \sup \{ \lambda(V, X), \text{ } V \text{ is a hyperplane in } X \}.$$ 

If $\dim X = 1$, $F(X) = 0$ and if $\dim X = 2$, $F(X) = 1$. To avoid trivialities we assume in this section that $\dim X > 2$.

$F$ is a parameter of the space which satisfies $1 \leq F(X) \leq 2$. If $X$ is a Hilbert space of course $F(X) = 1$. For the converse observe that the classical Kakutani's theorem ($X$ is Hilbert if and only if every hyperplane $V$ in $X$ is range of a norm one projection) is not applicable here since the condition $\lambda(V, X) = 1$ does not imply, in general, the existence of a norm one projection onto $V$; however, still the condition $F(X) = 1$ implies that $X$ is a Hilbert space. This fact was pointed out to me by Professor Amir and can be proved using the Garkavi-Klee characterization of Hilbert spaces via Chebyshev centers.

How to evaluate $F(X)$? Again Theorem 3 turns out to be useful. Define

$$C(X) = \sup \{ c_v, \text{ } V \text{ is a hyperplane of } X \}.$$ 

We easily obtain the analog of Theorems 3 and 4, namely.

**Theorem 9.** For the Banach space $X$ we have:

$$1 \leq C(X) \leq F(X) \leq g(C(X)) \leq 2, \quad (4.1)$$

$$F(X) = 1 \Leftrightarrow C(X) = 1, \quad (4.2)$$

$$F(X) < 2 \Leftrightarrow C(X) < 2. \quad (4.3)$$

We give now, in a particular case, a more precise evaluation. Recall that in a Banach space $X$ the modulus of convexity of $X$ is the function $\delta_X : [0, 2] \to [0, 1]$ defined by $\delta_X(\varepsilon) = \inf \{ 1 - \| x + y \|/2 : x, y \in S, \| x - y \| \geq \varepsilon \}$. $X$ is uniformly convex (u.c.) if and only if $\delta_X(\varepsilon) > 0$ for $\varepsilon > 0$; in this case $\delta_X$ is invertible and we denote by $\eta_X$ the inverse function. Assume
that $X$ is u.c., $V=f^{-1}(0)$, $f \in S^*$, $0 \leq a \leq 1$ and set $\Gamma_a = \{ x \in S : f(x) \geq a \} \supset C_a$. We have the following simple result:

$$\Delta(a) \leq \text{diam } \Gamma_a / 2 \leq \eta_X (1 - a)/2. \quad (4.4)$$

In fact, assume that $x, y \in \Gamma_a \cap S$; then $\|x + y\|/2 \leq 1 - \delta_X(\|x - y\|)$, that is, $\delta_X(\|x - y\|) \leq 1 - \|x + y\|/2 \leq 1 - a$; hence $\delta_X(\text{diam } \Gamma_a) \leq 1 - a$ which implies (4.4).

**Theorem 10.** If $V$ is a hyperplane in a u.c. space $X$ we have

$$\rho_V(a) \leq \eta_X (1 - a)(1 + a)/2. \quad (4.5)$$

Consequently

$$C(X) \leq \sup_a \eta_X (1 - a)(1 + a)/2 = D(X) < 2, \quad (4.6)$$

$$F(X) \leq g(D(X)) < 2.$$  

**Proof.** (4.5) follows from (2.4) and (4.4), then observe that the right hand side of (4.5) does not depend on $V$; hence (4.6) follows immediately using Theorem 3. \[\Box\]

Note that from (4.4) it follows the well known fact that in a u.c. space we have $\lim_{a \to 1^-} \rho(a) = \lim_{a \to 1^-} \Delta(a) = 0$.

The fact that $F(X) < 2$ in a u.c. space $X$ is contained in a more general result that we will prove in Theorem 12. We need first to recall some other facts on Banach spaces.

A Banach space $X$ is uniformly non-square (u.n.s.) if there exists an $\varepsilon > 0$ such that $\min(\|x + y\|, \|x - y\|) \leq 2 - \varepsilon$ for $x, y \in U$. It is easily seen that if $X$ is u.c. then $X$ is u.n.s.

The radial projection $R : X \to U$ is defined by

$$Rx = x \quad \text{if } x \in U$$
$$= x/\|x\| \quad \text{if } x \notin U.$$

The radial constant $k(X)$ of the real Banach space $X$ is defined by

$$k(X) = \sup \left\{ \frac{\|Rx - R_y\|}{\|x - y\|}, \quad x, y \in X, x \neq y \right\}.$$  

It is well known that $1 \leq k(X) \leq 2$; see, for example, [11] where other properties of $k$ are also described. Thiele proved in [18] the interesting fact that $k(X) < 2 \iff X$ is u.n.s.
Smith introduced in [16] the metric projection bound \( MPB(X) \) of the space \( X \) by \( MPB(X) = \sup \{ \| P_M \|, M \text{ is a proximinal subspace of } X \} \), where \( P_M(x) \subset M \) is the set of best approximations of \( x \) in \( M \) (non-empty by definition when \( M \) is proximinal) and \( \| P_M \| = \sup \{ \| y \|, y \in P_M(x), \| x \| \leq 1 \} \).

Baronti proved in [3] that \( MPB(X) = k(X) \).

Collecting all these facts we are able to prove:

**Theorem 11.** For any real Banach space \( X \) we have

\[
F(X) \leq k(X).
\] (4.7)

**Proof.** Set \( MPB(X) = \sup \{ \| P_V \| : \text{dim } X/V = 1, V \text{ proximinal} \} \). Obviously \( MPB(X) \leq MPB(X) \). (4.7) will be proved showing that \( F(X) \leq MPB(X) \). First note that u.n.s. Banach spaces are reflexive (this is a well known result due to R. C. James) so that the condition \( k(X) < 2 \) implies reflexivity: (4.7) is therefore trivially true if \( X \) is not reflexive since \( k(X) = 2 \).

Assume that \( X \) is reflexive and consequently that any hyperplane \( V \) is proximinal in \( X \): the (multivalued) best approximation operator \( P_V \) admits always a continuous linear selection which is therefore a projection. The inequality \( F(X) \leq MPB(X) \) will follow from the definitions of \( MPB(X) \) and of \( k(X) \).

We recall now a useful result of Bohnenblust (see [5]): let \( V \) be a hyperplane in an \( n \)-dimensional space \( X \). There always exists a projection \( P : X \to V \) such that \( \| P \| \leq 2(n - 1)/n \). This means that

\[
\text{dim } X = n \Rightarrow F(X) \leq 2 - 2/\text{dim } X.
\] (4.8)

Combining (4.7), (4.8) and Thiele's theorem already mentioned we get:

**Theorem 12.** We have \( F(X) < 2 \) in the following cases: \( X \) is finite dimensional, \( X \) is uniformly non square.

5. The Functions \( \rho, \gamma, \text{ and } \Delta \)

Let the hyperplane \( V = f^{-1}(0) \) be fixed in \( X \), \( z \in f^{-1}(1) \), \( P_z : X \to V \) defined by \( P_z x = x - f(x) z \) and \( 0 \leq a < 1 \). This section is devoted to a short study of the following functions:

\[
\rho(a) = \inf_{v \in V_a} \sup_{x \in C_a} \| x - v \| : x \in C_a = r_{V_a}(C_a) = r_{V}(C_a),
\]

\[
\gamma_z(a) = \sup \{ \| x - az \| : x \in C_a \},
\]

\[
\Delta(a) = \frac{1}{2} \sup \{ \| x - y \| : x, y \in C_a \} = \frac{1}{2} \text{ diam}(C_a).
\]
Recall that
\[ \Delta(a) \leq \rho(a) = \inf_{x \in V_i} \gamma_x(a); \quad \|P_x\| = \sup_{a} \gamma_x(a). \]

Denote now by \( \phi \) any of the functions \( \rho, \gamma_x, \Delta \). We shall prove below that \( a \rightarrow \phi(a)/a \) is non-increasing in \((0, 1)\); therefore we can define \( \phi(1) = \lim_{a \rightarrow 1} \phi(a) \).

From now on we will consider \( \phi \) as defined in the closed interval \([0, 1]\).
Note that \( \phi(0) = 1 \) and that when \( X \) is u.c. \( \phi(1) = 0 \) (see (4.4)). Let us prove:

**Theorem 13.** For \( \alpha \leq \beta, \alpha \neq 1, \) we have
\[ -\frac{\alpha}{1 - \alpha} (\beta - \alpha) \leq \phi(\beta) - \phi(\alpha) \leq \frac{\alpha - 1}{\alpha} (\beta - \alpha). \]  

Moreover the function \( \phi \) is continuous in \([0, 1]\) and Lipschitz in every interval \([0, 1 - \varepsilon]\) with \( \varepsilon > 0 \).

**Proof.** For \( s > 0 \) set \( C_a^s = V_a \cap sU \). We generalize the functions \( \phi \) by putting \( \phi^s(a) = \phi(C_a^s) \) (to be defined in the natural way). Note that \( \phi^1 = \phi \). It is easy to see that for \( h > 0 \) we have
\[ \phi^{1 + h}(a) \geq \phi^1(a) + h. \]  

Let \( T \) be the map \( x \rightarrow a/\beta x; \) then \( TC_a^1 \subseteq C_a^{a/\beta} \) since \( \|Tx - Ty\| - a/\beta \|x - y\| \). Using (5.2) we get \( \alpha/\beta \phi^1(\beta) \leq \phi^{a/\beta}(\alpha) \leq \phi^1(\alpha) - (1 - \alpha/\beta), \) that is, \( \phi(a) \geq (1 - \alpha/\beta) + \alpha/\beta \phi(\beta) \) which is the right hand side of (5.1). Let \( Z \) be the map \( x \rightarrow \lambda x + (1 - \lambda) z, \lambda \in [0, 1], f(z) = 1 \). We have \( ZC_a^1 \subseteq C_{a + (1 - \lambda)}^1 \) and, taking the infimum on the \( z \) with \( f(z) = 1 \), also \( ZC_a^1 \subseteq C_{\lambda a + (1 - \lambda)} \). Since \( \|Zx - Zy\| = \lambda \|x - y\| \) we obtain \( \lambda \phi(a) \leq \phi(\lambda a + (1 - \lambda)) \) which gives, for \( \beta = \lambda a + (1 - \lambda), ((1 - \beta)/(1 - \alpha)) \phi(\alpha) \leq \phi(\beta) \) which is the left hand side of (5.1).

The other conclusions of the theorem follow immediately from (5.1). [\( \square \)]

**Remarks.** The right hand side of (5.1) may be written \( \phi(a) \geq (1 - \alpha/\beta) + \alpha/\beta \phi(\beta) \) which in particular means that the hypograph of \( \phi \) is convex with respect to the point \((0, \phi(0)) = (0, 1)\): we will say that \( \phi \) is concave with respect to \( 0 \).

We also have \( \phi(a)/a - \phi(\beta)/\beta \geq (\beta - \alpha)/a\beta, \) i.e., \( \alpha \rightarrow \phi(\alpha)/a \) is non-increasing, and, more significantly, we also have
\[ \frac{\phi(a) - 1}{\alpha} - \frac{\phi(\beta) - 1}{\beta} \geq -\frac{1}{\alpha} + \frac{1}{\beta} + \frac{\beta - \alpha}{a\beta} = 0. \]
i.e., \( a \to (\varphi(a) - 1)/a = (\varphi(a) - \varphi(0))/a \) is non-increasing, or equivalently

\[
\frac{\varphi(\beta) - \varphi(\alpha)}{\beta - \alpha} \leq \frac{\varphi(\alpha) - 1}{\alpha} \quad \text{for} \quad \alpha \leq \beta.
\]

We set \( \lim_{\alpha \to 0^+} ((\varphi(\alpha) - 1)/\alpha) = \varphi'_+(0) \). Note that \( (\varphi(\beta) - 1)/\beta \leq \varphi'_+(0) \), i.e., \( \varphi(\beta) \leq \varphi'_+(0) \beta + 1 \) and \( \varphi(\beta) - \varphi(\alpha) \leq (\beta - \alpha) \varphi'_+(0) \). We can also see from (5.1) that \( a \to \varphi(a)/(1 - a) \) is non-decreasing, that \( \varphi \) is concave with respect to 1 if \( \varphi(1) = 0 \) (for example, when \( X \) is u.c.) and finally that \( \varphi \) is non-increasing in the set \( \{ x \in [0, 1]: \varphi(x) \leq 1 \} \) and \( \varphi(x) \geq 1 \) in \( [0, \xi], \varphi(x) < 1 \) in \( (\xi, 1] \), where \( \xi = \sup \{ x: \varphi(x) \geq 1 \} \).

For \( a = 1 \) the set \( C_1 = \{ x \in S: f(x) = 1 \} \) may of course be empty (for this reason the functions \( \varphi \) where defined originally only in \( [0, 1) \)). If we assume that \( C_1 \neq \emptyset \) we can define \( \varphi_1 = \varphi(C_1) \). It is easy to see that \( \varphi_1 \leq \varphi(1) \). We give an example where the inequality is strict.

**Example 2.** Let \( X \) be \( l^1 \), \( V_p = f_p^{-1}(0) \) with

\[
f_p = (1, 1, ..., 1, 1/2, 2/3, ..., (n - 1)/n, ....).
\]

By [4, Corollary, p. 224], we have \( \lambda(V_p, X) = 2 \); consequently by (2.16) \( \sup \rho(a) = 2 \) and \( \rho(1) = 2 \). However, one can see that \( \rho_1 = 0 \) for \( p = 1 \) and \( \rho_1^p \leq 1 \) for \( p = 2 \); here \( \rho_1 = r_{V_p}(C_1) \).

It could be asked whether the functions \( \varphi \) are concave. We will show with an example that this is not the case when \( \varphi = \gamma_\varepsilon \). It is, in general, difficult to compute explicitly the functions \( \varphi \); however, when \( X \) is a space of continuous functions, this is sometimes possible using an interesting and useful formula due to Smith and Ward [17].

**Theorem 14** (see [12, 17]).

Let \( T \) be a topological space, \( Y \) a subset of \( C(T) \), and \( A \) a bounded subset of \( C(T) \). Then

\[
r_Y(A) = r(A) + d(Y, E(A)).
\]

Here \( r(A) \), \( r_Y(A) \) are, respectively, the absolute radius and the radius with respect to \( Y \) of the set \( A \), \( d(Y, E(A)) \) is the distance from \( Y \) of the (non-empty) set of the absolute centers of \( A \).

The formula (5.3) was proved by Smith and Ward for \( T \) paracompact; the extension to any topological \( T \) is given in [12], where also a different proof and several applications of this formula are given. For the classical formulas
for \( r(A) \) and \( E(A) \) in \( C(T) \) see, for example, [12]. Note that for \( Y = V = f^{-1}(0), A = C_a, 0 \leq a \leq 1 \), we have

\[
\rho(a) = r(C_a) + d(V_a, E(C_a)).
\] (5.4)

**Example 3.** Let \( X = l^\infty(3), V = f^{-1}(0), f = (\frac{1}{3}, \frac{1}{4}, \frac{1}{5}) \). One can see that

\[
d(V_a, E(C_a)) = a \quad \text{for} \quad 0 < a < \frac{1}{2}
\]

\[
= \frac{1}{2} - a \quad \text{for} \quad \frac{1}{2} < a \leq \frac{1}{4}
\]

\[
= 0 \quad \text{for} \quad \frac{1}{2} < a \leq 1,
\]

\[
r(C_a) = 1 \quad \text{for} \quad 0 \leq a \leq \frac{1}{4}
\]

\[
= 2 - 2a \quad \text{for} \quad \frac{1}{2} < a \leq 1.
\]

By (5.4) we obtain

\[
\rho(a) = 1 + a \quad \text{for} \quad 0 \leq a \leq \frac{1}{4}
\]

\[
= \frac{1}{2} - a \quad \text{for} \quad \frac{1}{4} < a \leq \frac{1}{2}
\]

\[
= 2 - 2a \quad \text{for} \quad \frac{1}{2} < a \leq 1;
\]

hence \( \sup \rho(a) = c_r = \rho(1/4) = 5/4 \).

On the other hand, we have (see [4, Theorem 2], also [6, Theorem 3]): \( \lambda(V, X) = 9/7 \), so this is a case of strict inequality in (2.7). Note that in this example the function \( \rho \) is concave.

We consider now a minimal projection: let \( z = (8/7, 4/7, 8/7) \) (note that \( \|z\| = 8/7 > 1 \)). The projection \( P_x P_x = x - f(x)z \) is minimal since \( \|P_x\| = \sup_a \gamma_x(a) = 9/7 \). In fact: \( \gamma_x(a) = \sup \{\|x - az\|, f(x) = a, \|x\| \leq 1 \} \). Letting \( x = (x_1, x_2, x_3) \) we have \( x - az = (x_1 - a8/7, x_2 - a4/7, x_3 - a8/7) \) with the conditions \( |x_1| \leq 1, 3x_1 + 2x_2 + 3x_3 = 8a \). For \( 0 \leq a \leq 1/4 \) choosing \( x = (-1, 4a, 1) \) we get \( \gamma_x(a) = 1 + a8/7 \); for \( a = 1/2 \) choosing \( x = (1, -1, 1) \) we get \( \gamma_x(1/2) = 9/7 \). Also \( \gamma_x(1) = 3/7 \). Finally note that \( x - az = 1/14(8x_1 - 4x_2 - 6x_3, -3x_1 + 12x_2 - 3x_3, -6x_1 - 4x_2 + 8x_3) \); therefore \( \gamma_x(a) \leq 9/7 \) and equality is possible only for \( x \) of the form \( \pm (1, -1, -1), \pm (-1, 1, -1), \pm (-1, 1, 1) \). Since \( 8a \geq 0 \) the choice reduces to \( (-1, 1, 1), (1, -1, -1), (1, 1, -1) \) corresponding to the values \( 1/4, 1/2, 1/4 \) for \( a \). We conclude that \( \gamma_x(a) < 9/7 \) if \( a \notin \{1/4, 1/2\} \). We have shown that the function \( \gamma_x \) cannot be concave.

**References**