A hierarchy of impicational (semilinear) logics: the propositional case

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Abstract

In Abstract Algebraic Logic the general study of propositional non-classical logics has been traditionally based on the abstraction of the Lindenbaum-Tarski process. In this kind of process one considers the Leibniz relation of indiscernible, i.e. logically equivalent, formulae. Such approach has resulted in a classification of logics partly based on generalizations of equivalence connectives: the Leibniz hierarchy. This paper performs an analogous abstract study of non-classical logics based on the kind of generalized implication connectives they possess. It yields a new classification of logics expanding Leibniz hierarchy: the hierarchy of impicational logics. In this framework the notion of impicational semilinear logic can be naturally introduced as a property of the implication, namely a logic L is an impicational semilinear logic iff it has an implication such that L is complete w.r.t. the matrices where the implication induces a linear order, a property which is typically satisfied by well-known systems of fuzzy logic. The hierarchy of impicational logics is then restricted to the semilinear case obtaining a classification of impicational semilinear logics that encompasses almost all the known examples of fuzzy logics and suggests new directions for research in the field. Moreover, the role of generalized disjunction connectives is considered in a similar abstract fashion and their relation with implications and semilinearity is studied. In particular, the classical law of Proof by Cases is shown to be equivalent to semilinearity of the logic under certain natural conditions.

Keywords: Abstract Algebraic Logic, Disjunctive logics, Hierarchy of impicational logics, Implicative logics, Leibniz hierarchy, Linearly ordered logical matrices, Mathematical Fuzzy Logic, Non-classical logics, Proof by Cases property, Semilinear logics.

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1 Introduction

Algebraic Logic is the branch of Mathematical Logic that studies logical systems by giving them a semantics based on some particular kind of algebraic structures. It can be traced
back to George Boole’s works when he studied classical propositional logic by means of a two-element algebra that became its canonical semantics. Thus, in a sense, it could be argued that it is the most ancient branch of Mathematical Logic. Tarski’s refinement of the proof by Linenbaum that classical logic was indeed complete with respect to the semantics given by Boolean algebras starts from a theory $T$ and a formula $\varphi$ such that $T \not \vdash_{\text{CPC}} \varphi$, i.e. $T$ does not prove $\varphi$ in the classical propositional calculus, and then it considers the following binary relation on the set of formulae: $\langle \alpha, \beta \rangle \in \Omega(T)$ iff $T \vdash_{\text{CPC}} \alpha \leftrightarrow \beta$. This relation is shown to be in fact a congruence in the algebra of formulae $\text{Fm}_L$ such that the formulae of $T$ constitute exactly one equivalence class. So, it is enough to take the corresponding quotient, $\text{Fm}_L/\Omega(T)$, and show that it is a Boolean algebra such that the class of $T$ is its top element, and hence in this algebra the elements of $T$ are interpreted as true while $\varphi$ is not (because $T \not \vdash_{\text{CPC}} \varphi$).

Analogous proofs were later used to show the completeness of non-classical logics with respect to their corresponding algebraic semantics (e.g. intuitionistic logic w.r.t. Heyting algebras) and thus it became a standard method in Algebraic Logic called the Lindenbaum-Tarski process. The fact that it could be analogously repeated in many propositional logics led to more general studies where it was used to show completeness theorems for broad classes of logics such as Rasiowa’s implicative logics (studied in her monograph [26]). Abstract Algebraic Logic (AAL) was born as the natural next step to be taken in this evolution: the abstract study of logical systems by generalizing the Lindenbaum-Tarski process to arbitrary logics.

The last decades have seen the florescence of this subfield of Algebraic Logic resulting in a deep theory on the link between logics and classes of algebras (or logical matrices defined over the algebras). The generalization of Lindenbaum-Tarski construction consisted in realizing that the congruence $\Omega(T)$ is actually the relation given by those formulae that, relatively to $T$, are substitutable in any context salva veritate, i.e.: $\langle \alpha, \beta \rangle \in \Omega(T)$ if, and only if, for every formula in at least one variable $\chi(x)$, $\chi(\alpha)$ is true relatively to $T$ iff $\chi(\beta)$ is true relatively to $T$. Thus, $\Omega(T)$ was the relation of logically equivalent formulae modulo $T$, and the quotient $\text{Fm}_L/\Omega(T)$ could be seen as the identification of indiscernible propositions, i.e. a formalization of the ancient Leibniz principle of equality of indiscernibles. Therefore, $\Omega(T)$ was called the Leibniz congruence of $T$. In classical logic this relation was easily defined by means of the connective $\leftrightarrow$, whereas in other logics the situation could be substantially more complicated even though Leibniz congruence could be still definable by means of some set of formulae in two variables (possibly infinite and with parameters). It gave rise to the class of protoalgebraic logics: logics where there is a set of formulae $E(p, q, \overline{\tau})$ defining Leibniz congruence. This set can be seen as a generalization of equivalence connective, and indeed it has been called equivalence set. By imposing several extra conditions, a number of subclasses of protoalgebraic logics were defined yielding a classification of logics called the Leibniz hierarchy. As it was based on properties of the congruence $\Omega$ and the sets $E(p, q, \overline{\tau})$, it was essentially an equivalence-based classification.

Largely independent of these developments, another subfield of Algebraic Logic has been rapidly growing on recent times: the algebraic study of fuzzy logics. A number of logical systems have been proposed and intensively studied to deal with the reasoning with vagueness and the concept of graded truth. They include Gödel-Dummett logic [10], Łukasiewicz infinitely-valued logic [23], Product logic [20], Hájek’s BL logic [18], logics based on left-continuous t-norms such as MTL logic [12], and uninorm logic UL [24] among many others. All of them are many-valued logics that enjoy some algebraic semantics that can be generated from linearly ordered algebras, and thus they enjoy a completeness theorem with respect to
linearly ordered algebras. This common feature has led Běhounek and Cintula to argue in [2] that fuzzy logics are the logics of chains, i.e. the logics that have a complete semantics based on linearly ordered algebras. Moreover, these chains are typically ordered in a uniform way by means of some implication connective \( \rightarrow \) in the sense that in every algebra \( A \) there is a designated set \( F \subseteq A \) such that for every \( a, b \in A \), \( a \leq b \) iff \( a \rightarrow b \in F \). An implication connective like this, in addition, plays a central role in the Lindenbaum-Tarski process of these logics since the congruence \( \Omega(T) \) can be defined in the following way: \( \langle \alpha, \beta \rangle \in \Omega(T) \) iff \( T \vdash \alpha \rightarrow \beta \) and \( T \vdash \beta \rightarrow \alpha \). In other words, the symmetrized implication \( \{ p \rightarrow q, q \rightarrow p \} \) gives an equivalence set in these logics. Therefore, implication connectives play a double fundamental role in fuzzy logics: they define order in the algebras and, when symmetrized, they allow to define Leibniz congruence. From this point of view, implications are much more useful than just plain equivalence connectives.

Therefore, it makes sense to develop a finer classification in AAL based on implications instead of equivalences. It would be a more general approach because from every implication an equivalence can be retrieved (just by symmetrizing), while equivalencies do not have all the features of an implication (they define only the identity order). This is what we intend to do in this paper. We want to do it in the pure AAL style aiming at the most general possible framework. Thus, we allow implications to be definable connectives by means of possibly infinite and parameterized sets of formulae. This new approach to logical systems results in an implication-based classification of logics that expands Leibniz hierarchy and will be called the hierarchy of implicational logics. Its largest class coincides with that of protoalgebraic logics, the same largest class in Leibniz hierarchy, but it allows to distinguish more subclasses yielding a hierarchy finer than the traditional one. In particular, also fits well with some previously defined classes of logics: Rasiowa’s implicative logics [26], and Cintula’s weakly implicative logics [6]. In this framework of implicational logics we introduce in a natural way a very general notion of implicational semilinear logic: an implicational logic is semilinear if it has a semilinear implication, i.e. a generalized implication such that the logic is complete w.r.t. the models where it defines a linear order. In symbols: if \( L \) is a logic and \( \Rightarrow \) is an implication set, \( L \) is implicational semilinear w.r.t. \( \Rightarrow \) if \( \models_{\text{MOD}}^\ell(L) \).

The term ‘semilinear’ was introduced by Olson and Raftery in [25] (in much more specific context of residuated lattices) and it refers to the fact that in finitary semilinear logics the subdirectly irreducible matrices are linear (following the tradition in Universal Algebra of calling a class of algebras ‘semi\( X \)’ whenever its subdirectly irreducible members have the property \( X \); e.g. as in ‘semisimple’). This technical notion of implicational semilinear logic is a first (big) step towards a mathematical definition of what a fuzzy logic is in the sense of [2], because it describes the most usual way in which logical systems happen to be semantically based on chains. In principle this definition might not be intrinsic in the sense that different implications could induce different classes of linear models. Nevertheless, we will prove that for every (finite) \( \Rightarrow \), \( \text{MOD}^\ell_{\Rightarrow}(L) \) coincides with the class of relatively finitely subdirectly irreducible models of \( L \). Moreover, for every algebra \( A \) of a reduced model, all its logical filters are up-sets with respect to the order given by the implication set. When the order is total, the up-sets are linearly ordered by inclusion, and hence logical filters form also a chain. Thus, an easy method to show that an implicational logic is not semilinear consists just in giving a subdirectly irreducible algebra with two incomparable filters.

On the other hand, in the fuzzy logic literature the completeness of the logics with respect to chains is usually shown by means of linear filters, i.e. filters which induce total orders in...
the quotient. One usually proves the *Linear Extension Property* (LEP): for every filter $F$ and every $a 
otin F$, there is a linear filter $F'$ such that $F \subseteq F'$ and $a \notin F'$. We study this and some related properties in our general framework showing that it is equivalent to the semilinearity metarule and to the fact that matrix models are subdirect product of linear ones. Interestingly enough, in many well known fuzzy logics the notion of linear filter coincides with the traditional notion of prime filter coming from Boolean algebras (recall it: $F$ is prime if from $a \lor b \in F$, we infer that $a \in F$ or $b \in F$). Thus, the (LEP) turns out to be equivalent in many cases to the (PEP) (*Prime Extension Property*), and hence a property which involves just disjunction connective is also enough for the fuzziness of the logic. Disjunction is usually not a primitive connective in prominent fuzzy logics, for it can be typically defined as $\varphi \lor \psi = (\varphi \rightarrow \psi) \rightarrow \psi$ in Lukasiewicz logic, or as $\varphi \lor \psi = ((\varphi \rightarrow \psi) \rightarrow \psi) \land ((\psi \rightarrow \varphi) \rightarrow \varphi)$ in weaker systems. For logics without conjunction we realize that the set $\{(\varphi \rightarrow \psi) \rightarrow \psi, (\psi \rightarrow \varphi) \rightarrow \varphi\}$ would essentially do the job. Thus, we are again in the same situation as in the implication case: we have an important connective that may be defined by a set of formulae. It suggests that an analogous abstract treatment should be pursued for disjunction connective. We also do it in this paper. We consider arbitrary sets of parameterized formulae and study the kinds of disjunctions they define according to a set of usual properties of disjunction in classical logic. One of these properties, the Proof by Cases (PCP), turns out to define a generalization of the class of disjunctive logics (already considered in the literature) that we call *disjunctive logics*. In addition, we prove that the (PCP) is equivalent to the (PEP), and so to fuzziness of the logic. It generalizes the results in [31] where the considered disjunction connectives were primitive or defined by a single formula.

The outline of the paper is the following: after this introduction, Section 2 gives the necessary basic notions from AAL that will be needed. Section 3 presents our theory of implications and the hierarchy of implicational logics that they induce as an expansion of Leibniz hierarchy. The problem of showing mutual differences between the classes is almost solved by a series of examples. Then, we introduce the concept of semilinear implication and show some of its characterizations in general and important consequences in finitary logics. Section 4 presents the abstract study of disjunctions defined by arbitrary sets of parameterized formulae. Starting from the very weak notion of p-protodisjunction, it gives a hierarchy of disjunction-like connectives, it shows their mutual differences and ends with a generalization of the already known notion of disjunctive logics: the disjunctive logics. The main property defining disjunctive logics, the (PCP), is then studied from the semantical and also from the syntactical point of view. Section 5 is devoted to the interplay of disjunctions and implications and in particular shows the above mentioned equivalences between semilinearity, (LEP), (PEP) and (PCP). It also studies completeness of logics with respect to a semantics of densely ordered matrices by means of a meta-rule that combines disjunction and implication. Finally, Section 6 restricts the hierarchy of implicational logics to the semilinear case, thus obtaining a new hierarchy of implicational semilinear logics. Some classes are shown to collapse and others to be different. Well-studied classes of fuzzy logics are shown to lie on the top of the classification.
2 Preliminaries

2.1 Basic notions

For the development of the paper we need to recall the basic definitions and results of Abstract Algebraic Logic.\(^1\) We start with some syntactical definitions. The notion of propositional language \(\mathcal{L}\) is defined in the usual way (a set of connectives with finite arity). By \(\text{Fm}_{\mathcal{L}}\) we denote the free term algebra over a denumerable set of variables in the language \(\mathcal{L}\), by \(\text{Fm}_{\mathcal{L}}\) we denote its universe and we call its elements \(\mathcal{L}\)-formulae (we omit \(\mathcal{L}\) when it is clear from the context; analogously with other notions defined in this section). The set of finite sequences of \(\mathcal{L}\)-formulae is denoted by \(\text{Fm}_{\mathcal{L}}^\omega\), while the set all sequences (including infinite ones) of \(\mathcal{L}\)-formulae is denoted by \(\text{Fm}_{\mathcal{L}}^{<\omega}\). We denote by \(\text{Eq}_{\mathcal{L}}\) the set of \(\mathcal{L}\)-equations, i.e. formal expressions of the form \(\varphi \approx \psi\), where \(\varphi, \psi \in \text{Fm}_{\mathcal{L}}\). The endomorphisms of \(\text{Fm}_{\mathcal{L}}\) are called \(\mathcal{L}\)-substitutions.

A \(\mathcal{L}\)-consecution\(^2\) is a pair \(\Gamma \triangleright \varphi\), where \(\Gamma \subseteq \text{Fm}_{\mathcal{L}}\) and \(\varphi \in \text{Fm}_{\mathcal{L}}\). A consecution \(\Gamma \triangleright \varphi\) is finitary if \(\Gamma\) is finite. Notice that a set of consecutions \(\vdash_{\mathcal{L}}\) can be understood as a relation between sets of formulae and formulae; we often use an infix notation, i.e. we write \(L \vdash_{\mathcal{L}} \varphi\) instead of \(\Gamma \triangleright \varphi \in \vdash_{\mathcal{L}}\).

A propositional logic (also called sentential logic or just logic) is a pair \(L = \langle \mathcal{L}, \vdash_{\mathcal{L}} \rangle\) where \(\mathcal{L}\) is a propositional language and \(\vdash_{\mathcal{L}}\) is a set of \(\mathcal{L}\)-consecutions satisfying the following conditions:

1. Consequence relation:
   For every \(\Gamma \cup \Delta \cup \{\varphi, \psi\} \subseteq \text{Fm}_{\mathcal{L}}\),
   
   (a) \(\varphi \vdash_{L} \varphi\).
   
   (b) If \(\Gamma \vdash_{L} \varphi\) and \(\Gamma \subseteq \Delta\), then \(\Delta \vdash_{L} \varphi\).
   
   (c) If \(\Gamma \vdash_{L} \varphi\) and for every \(\psi \in \Gamma\), \(\Delta \vdash_{L} \psi\), then \(\Delta \vdash_{L} \varphi\).

2. Structural:
   For every \(\mathcal{L}\)-consecution \(\Gamma \triangleright \varphi\) and \(\mathcal{L}\)-substitution \(\sigma\) if \(\Gamma \vdash_{L} \varphi\), then \(\sigma[\Gamma] \vdash_{L} \sigma(\varphi)\).

A logic \(L\) is finitary if for every \(\Gamma \cup \{\varphi\} \subseteq \text{Fm}_{\mathcal{L}}\) such that \(\Gamma \vdash_{L} \varphi\) there is a finite \(\Gamma_0 \subseteq \Gamma\) such that \(\Gamma_0 \vdash_{L} \varphi\). We write \(\Gamma \vdash_{L} \Delta\) when \(\Gamma \vdash_{L} \varphi\) for every \(\varphi \in \Delta\). We write \(\Gamma \vdash_{L} \Delta\) when \(\Gamma \vdash_{L} \Delta\) and \(\Delta \vdash_{L} \Gamma\). A theory of a logic \(L\) is a set of formulae \(T\) such that if \(T \vdash_{L} \varphi\) then \(\varphi \in T\). By \(Th(L)\) we denote the set of all theories of \(L\). Each propositional logic \(L\) defines a closure system over the set \(\text{Fm}_{\mathcal{L}}\) whose closed sets are the theories of \(L\) and the corresponding closure operator \(C\) over \(\text{Fm}_{\mathcal{L}}\) is defined as: \(C(\Gamma) = \bigcap\{T \in Th(L) \mid \Gamma \subseteq T\} = \{\varphi \in \text{Fm}_{\mathcal{L}} \mid \Gamma \vdash_{L} \varphi\}\).

A logic can be presented by means of several kinds of proof systems. In this paper we consider mainly Hilbert-style systems. Given a finitary logic \(L = \langle \mathcal{L}, \vdash_{L} \rangle\), we say that a set \(\mathcal{A}\mathcal{S}\) of \(\mathcal{L}\)-consecutions whose left member is finite is a presentation of \(L\) if the relation \(\vdash_{L}\) coincides with the provability relation given by \(\mathcal{A}\mathcal{S}\) as a Hilbert-style axiomatic system: for every \(\Gamma \cup \{\varphi\} \subseteq \text{Fm}_{\mathcal{L}}\), \(\Gamma \vdash_{L} \varphi\) iff there is a proof of \(\varphi\) from \(\Gamma\), i.e. a finite sequence of formulae \(\langle \psi_0, \psi_1, \ldots, \psi_n \rangle\) such that \(\psi_n = \varphi\) and for every \(i < n\) either \(\psi_i \in \Gamma\) or for some \(\Delta \triangleright \alpha \in \mathcal{A}\mathcal{S}\) there is a substitution \(\sigma\) such that \(\sigma(\alpha) = \psi_i\) and \(\sigma[\Delta] \subseteq \{\psi_0, \ldots, \psi_{i-1}\}\). In the

\(^1\)The reader can find comprehensive presentations of the field in the monographs [8, 13] and in the survey [14]. Any necessary background on Universal Algebra can be found in [4].

\(^2\)This term is borrowed from [1]; however, we use it in a very simplified version. The term ‘sequent’ is sometimes used instead.
case of infinitary logics we would need to consider proofs as founded trees labeled by formulæ satisfying analogs of the conditions above.

Traditionally propositional logics are given a semantics in terms of matrices. Given a language $L$, an $L$-matrix is a pair $A = \langle A, D \rangle$ where $A$ is an $L$-algebra and $D$ is a subset of $A$ called the filter of $A$. A matrix is trivial if its algebra has only one element and its filter is the singleton of this element. A homomorphism from $Fm_L$ to $A$ is called an $A$-evaluation. The semantical consequence with respect to a class of matrices $K$ is defined as: $\Gamma \models_K \varphi$ iff for each $A \in K$ and each $A$-evaluation $e$ we obtain $e(\varphi) \in D$ whenever $e[\Gamma] \subseteq D$. Clearly, $(L, \models_K)$ is a logic. We say that a matrix $A$ is a model of $L$ if $\vdash_L \subseteq \models_A$. Let $\text{MOD}(L)$ be the class of all models of $L$. Each logic is complete with respect to the semantics given by all its models:

**Theorem 2.1.** Let $L = (L, \vdash_L)$ be a logic. For every $\Gamma \cup \{ \varphi \} \subseteq Fm_L$, $\vdash_L \varphi$ if, and only if, $\Gamma \models_{\text{MOD}(L)} \varphi$.

Given an $L$-algebra $A$, a subset $F \subseteq A$ is an $L$-filter if $\langle A, F \rangle \in \text{MOD}(L)$. Let $\mathcal{F}_iL(A)$ be the set of all $L$-filters over $A$. Observe that for every set of formulæ $T$, we have $T \in \text{Th}(L)$ iff $\langle Fm_L, T \rangle \in \text{MOD}(L)$, i.e. $\mathcal{F}_iL(Fm_L) = \text{Th}(L)$: these models are called the Lindenbaum matrices for $L$. It is straightforward to check that $\mathcal{F}_iL(A)$ is closed under arbitrary intersections and hence it is a closure system. Let us recall some basic notions in closure systems.

**Definition 2.2.** Let $C$ be a closure system. A family $B \subseteq C$ is called a basis of $C$ if for every $X \in C \setminus \{ A \}$ there is a $D \subseteq C$ such that $X = \bigcap D$.

**Proposition 2.3.** Let $C$ be a closure system over a set $A$ and $B \subseteq C$. Then, $B$ is a basis of $C$ iff for every $Y \in C$ and every $a \in A \setminus Y$ there is $Z \in B$ such that $Y \subseteq Z$ and $a \notin Z$.

A crucial notion for the classification of logical systems in Abstract Algebraic Logic is the so-called Leibniz congruence of a matrix. Given a matrix $A = \langle A, F \rangle$, a binary relation $\Omega_A(F) \subseteq A^2$ is defined as $\langle a, b \rangle \in \Omega_A(F)$ if, and only if, for every sequence of parameters $\overline{x}$, $L$-formula $\varphi(\overline{x}, \overline{a})$, and $\overline{c} \in A^{<\omega}$ we have $\varphi^A(a, \overline{c}) \in F$ iff $\varphi^A(b, \overline{c}) \in F$. Thus, we have defined the indiscernibility relation in $A$. This relation has an important characterization. Recall that given an algebra $A$ and a subset $F \subseteq A$, a congruence $\theta \in \text{Co}(A)$ is said to be compatible with $F$ if for every $a, b \in A$ such that $a \in F$ and $\langle a, b \rangle \in \theta$, we have $b \in F$.

**Theorem 2.4.** $\Omega_A(F)$ is the maximum congruence of $A$ compatible with $F$.

Observe that when $A$ is the algebra of formulæ $Fm_L$, the Leibniz congruence of a theory $T$ is given by the pairs $\langle \alpha, \beta \rangle$ such that for every formula in at least one variable $\chi(x), \chi(\alpha) \in T$ iff $\chi(\beta) \in T$. Inspired by the famous Leibniz’s principle of equality of indiscernibles, $\Omega_A(F)$ is called the Leibniz congruence of $\langle A, F \rangle$. A matrix is said to be reduced if its Leibniz congruence is the identity relation. Given an arbitrary matrix $A = \langle A, F \rangle$, one can always produce a reduced one factorizing by the Leibniz congruence, i.e. $A^* = \langle A/\Omega_A(F), F/\Omega_A(F) \rangle$. Given a logic $L$, the class of its reduced models is denoted by $\text{MOD}^*(L)$, and the class of algebraic reducts of $\text{MOD}^*(L)$ is denoted by $\text{ALG}^*(L)$. Reduced models are enough to provide a complete semantics for the logic:

**Theorem 2.5.** Let $L = (L, \vdash_L)$ be a logic. For every $\Gamma \cup \{ \varphi \} \subseteq Fm_L$, $\vdash_L \varphi$ if, and only if, $\Gamma \models_{\text{MOD}^*(L)} \varphi$. 

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Matrices can be regarded as first-order structures where the filter corresponds to a unary predicate. In this context one can define the usual notions of substructure (now called submatrix), isomorphism, homomorphism, strict homomorphism, direct product, reduced product and ultraproduct for matrices. Given a class of matrices $K$, we will denote by $S(K), I(K), H(K), H_S(K), P(K), P_R(K)$ and $P_U(K)$ the closure of $K$ under the mentioned operations.

The notion of subdirect product from Universal Algebra is also generalized to matrices. A matrix $M$ is said to be representable as a subdirect product of the family of matrices $\{M_i | i \in I\}$ if there is an injective homomorphism $\alpha$ from $M$ into their direct product $\prod_{i \in I} M_i$ such that for every $i \in I$, the composition of $\alpha$ with the $i$-th projection, $\pi_i \circ \alpha$, is surjective. In this case $\alpha$ is called a subdirect representation, and it is called finite if $I$ is finite.

Let $L$ be a logic and $K \subseteq \text{MOD}^*(L)$. A non-trivial matrix $M \in K$ is (finitely) subdirectly irreducible relatively to $K$ if for every (finite) subdirect representation $\alpha$ of $M$ with a family $\{M_i | i \in I\} \subseteq K$ there is $i \in I$ such that $\pi_i \circ \alpha$ is an isomorphism. The class of all relatively (finitely) subdirectly irreducible matrices is denoted as $K_{R(F)SI}$.

**Theorem 2.6.** If $L$ is a finitary logic, then every matrix in $\text{MOD}^*(L)$ is representable as a subdirect product of matrices in $\text{MOD}^*(L)_{RSI}$.

**Corollary 2.7.** Let $L = \langle L, \vdash_L \rangle$ be a finitary logic. For every $\Gamma \cup \{\varphi\} \subseteq \text{Fm}_L$, $\Gamma \vdash_L \varphi$ if, and only if, $\Gamma \models \text{MOD}^*(L)_{RSI} \varphi$.

**Proposition 2.8.** Let $L$ be a logic and $A = \langle A, F \rangle \in \text{MOD}^*(L)$. Then:

1. $A \in \text{MOD}^*(L)_{RFSI}$ if, and only if, $F$ is finitely meet-irreducible.
2. $A \in \text{MOD}^*(L)_{RSI}$ if, and only if, $F$ is meet-irreducible.

### 2.2 Leibniz hierarchy

Notice that $\Omega_A$ can be seen as a mapping from $\text{Fi}_L(A)$ to $\text{Co}(A)$ and then it is called the Leibniz operator. Some classes of logics are defined according to the behavior of this operator. Let $L$ be a logic in a language $\mathcal{L}$. Then:

1. $L$ is called protoalgebraic if $\Omega_{\text{Fm}_L}$ is monotone on $\text{Th}(L)$, i.e. for every $T_1, T_2 \in \text{Th}(L)$, if $T_1 \subseteq T_2$ then $\Omega_{\text{Fm}_L}(T_1) \subseteq \Omega_{\text{Fm}_L}(T_2)$.

2. $L$ is called equivalential if $\Omega_{\text{Fm}_L}$ is monotone and commutes with inverse substitutions on $\text{Th}(L)$, i.e. for every $T \in \text{Th}(L)$ and every $\mathcal{L}$-substitution $\sigma$, $\Omega_{\text{Fm}_L}(\sigma^{-1}[T]) = \sigma^{-1}[\Omega_{\text{Fm}_L}(T)]$.

3. $L$ is called weakly algebraizable if $\Omega_{\text{Fm}_L}$ is monotone and injective on $\text{Th}(L)$.

4. $L$ is called algebraizable if $\Omega_{\text{Fm}_L}$ is monotone, injective and it commutes with inverse substitutions on $\text{Th}(L)$.

All of them have been intensively studied and several nice characterizations have been obtained.

**Theorem 2.9.** Let $L = \langle L, \vdash_L \rangle$ be a logic. The following are equivalent:

1. $L$ is protoalgebraic.
2. $L$ is equivalential.
3. $L$ is weakly algebraizable.
4. $L$ is algebraizable.
1. \(L\) is protoalgebraic.
2. For every \(\mathcal{L}\)-algebra \(A\), \(\Omega_A\) is monotone on \(\mathcal{F}_i(L)(A)\).
3. \(\text{MOD}^*(L)\) is closed under formation of subdirect products.
4. There exists a set of formulae in two variables and possibly with parameters \(E(p,q,\overline{r})\) such that:
   - \(\vdash_L E(p,p,\overline{r})\)
   - \(p, \bigcup_{\overline{\alpha} \in \text{Fm}_{\overline{L}}} E(p,q,\overline{\alpha}) \vdash_L q\)
   - \(\bigcup_{\overline{\alpha} \in \text{Fm}_{\overline{L}}} E(p,q,\overline{\alpha}) \vdash_L E(c(s_1,\ldots,s_{i-1},p,\ldots,s_n),c(s_1,\ldots,s_{i-1},q,\ldots,s_n),\overline{\beta})\)
     for every \(\overline{\beta} \in \text{Fm}_{\overline{L}}\), for each \(\langle c,n \rangle \in L\) and each \(i \leq n\).
5. There exists a set of formulae in two variables and possibly with parameters \(E(p,q,\overline{r})\) such that it defines Leibniz congruence on every model of \(L\), i.e. for every \(A = \langle A,F \rangle \in \text{MOD}(L)\) and every \(a,b \in A\), \(\langle a,b \rangle \in \Omega_A(F)\) iff \(E^A(a,b,\overline{c}) \subseteq F\) for every \(\overline{c}\) in \(A\).

Any set \(E(p,q,\overline{r})\) satisfying part 4. also satisfies part 5. and vice versa. These sets are called \(\text{parameterized equivalence sets}\).

**Theorem 2.10.** Let \(L = \langle \mathcal{L}, \vdash_L \rangle\) be a logic. The following are equivalent:

1. \(L\) is equivalential.
2. For every \(\mathcal{L}\)-algebra \(A\), \(\Omega_A\) is monotone and it commutes with inverse images by homomorphisms, that is, for every \(\mathcal{L}\)-algebra \(B\), every homomorphism \(h : A \rightarrow B\) and every \(F \in \mathcal{F}_i(L)(B)\), \(\Omega_A(\sigma^{-1}[F]) = \sigma^{-1}[\Omega_B(F)]\).
3. \(\text{MOD}^*(L)\) is closed under formation of submatrices and direct products.
4. There exists a set of formulae in two variables \(E(p,q)\) such that:
   - \(\vdash_L E(p,p)\)
   - \(p, E(p,q) \vdash_L q\)
   - \(E(p,q) \vdash_L E(c(s_1,\ldots,s_{i-1},p,\ldots,s_n),c(s_1,\ldots,s_{i-1},q,\ldots,s_n))\) for each \(\langle c,n \rangle \in \mathcal{L}\) and each \(i \leq n\).
5. There exists a set of formulae in two variables \(E(p,q)\) such that it defines Leibniz congruence on every model of \(L\), i.e. for every \(A = \langle A,F \rangle \in \text{MOD}(L)\) and every \(a,b \in A\), \(\langle a,b \rangle \in \Omega_A(F)\) iff \(E^A(a,b) \subseteq F\).

Again, any set \(E(p,q)\) satisfying part 4. also satisfies part 5. and vice versa. These sets are called \(\text{equivalence sets}\). It is clear that all equivalential logics are protoalgebraic. Moreover, all the possible (parameterized) equivalence sets are mutually interderivable:

**Proposition 2.11.** Let \(L = \langle \mathcal{L}, \vdash_L \rangle\) be a logic. Then:

\[\text{The remaining properties of equivalence (i.e. symmetry and transitivity) which are not explicitly required in the syntactical conditions in 4., follow either directly from 5. or from 4. by syntactical arguments.}\]
• If \( L \) is equivalential and \( E(p, q), E'(p, q) \subseteq \text{Eq}_{L} \) are equivalence sets for \( L \), then \( E(p, q) \models_{L} E'(p, q) \).

• If \( L \) is protoalgebraic and \( E(p, q, \bar{a}), E'(p, q, \bar{a}) \subseteq \text{Eq}_{L} \) are parameterized equivalence sets for \( L \), then \( \bigcup \{E(p, q, \bar{a}) \mid \bar{a} \in \text{Eq}_{L}^{\leq \omega} \} \models_{L} \bigcup \{E'(p, q, \bar{a}) \mid \bar{a} \in \text{Eq}_{L}^{\leq \omega} \} \).

For finitary protoalgebraic logics, the relatively finitely subdirectly irreducible models can be described in the following way:

**Theorem 2.12.** Let \( L \) be a finitary protoalgebraic logic complete with respect to a class \( K \subseteq \text{MOD}^{*}(L) \). Then, \( \text{MOD}^{*}(L)_{\text{RFSI}} \subseteq \text{HSP}_{U}(K^{+}) \), where \( K^{+} \) is the class \( K \) plus the trivial matrix.

**Definition 2.13.** The equational consequence relative to a class of algebras \( K \) is defined in the following way: \( \Pi \models_{K} \varphi \Leftrightarrow \psi \) if, and only if, for every \( A \in K \) and every \( A \)-evaluation \( e \), if \( e(\alpha) = e(\beta) \) for every \( \alpha \approx \beta \in \Pi \), then \( e(\varphi) = e(\psi) \).

Given a collection of equations \( \Pi \subseteq \text{Eq}_{L} \) and a parameterized set of formulae in two variables \( E(p, q, \bar{a}) \), \( E[\Pi] \) denotes the set of formulae \( \bigcup \{E(\varphi, \psi, \bar{a}) \mid \varphi \approx \psi \in \Pi, \bar{a} \in \text{Eq}_{L}^{\leq \omega} \} \). Using this notation, the following theorem shows that an equivalence set provides a translation of the equational consequence relative to \( \text{ALG}^{*}(L) \) into the logic \( L \).

**Theorem 2.14.** Let \( L \) be a logic in a language \( L \) and \( E(p, q, \bar{a}) \subseteq \text{Eq}_{L} \). The following are equivalent:

1. \( E(p, q, \bar{a}) \) is a parameterized equivalence set for \( L \).
2. \( p, \bigcup \{E(p, q, \bar{a}) \mid \bar{a} \in \text{Eq}_{L}^{\leq \omega} \} \models_{L} q \) and for every \( \Pi \cup \{ \varphi \approx \psi \} \subseteq \text{Eq}_{L} \) we have: \( \Pi \models_{\text{ALG}^{*}(L)} \varphi \approx \psi \) if, and only if, \( E[\Pi] \models_{L} E(\varphi \approx \psi) \).

**Theorem 2.15.** Let \( L = \langle L, \models_{L} \rangle \) be a logic. The following are equivalent:

1. \( L \) is weakly algebraizable.
2. For every \( L \)-algebra \( A \), \( \Omega_{A} \) is monotone and injective on \( F_{iL}(A) \).
3. For every \( L \)-algebra \( A \), \( \Omega_{A} \) is a lattice isomorphism between \( F_{iL}(A) \) and \( \text{Co}_{\text{ALG}^{*}(L)}(A) \) (the complete sublattice of congruences giving a quotient in \( \text{ALG}^{*}(L) \)).
4. \( L \) is protoalgebraic and for every \( L \)-algebra \( A \) and every \( F \in F_{iL}(A) \), \( F/\Omega_{A}(F) \) is the least \( L \)-filter on \( A/\Omega_{A}(F) \).
5. There exists a parameterized set of formulae in two variables \( E(p, q, \bar{a}) \) and a set of equations in one variable \( E(p) \subseteq \text{Eq}_{L} \) such that:
   - For every \( \Pi \cup \{ \varphi \approx \psi \} \subseteq \text{Eq}_{L} \) we have: \( \Pi \models_{\text{ALG}^{*}(L)} \varphi \approx \psi \) if, and only if, \( E[\Pi] \models_{L} E(\varphi \approx \psi) \)
   - \( p \models_{L} E[E(p)] \).

Given a set of equations in one variable \( E(p) \subseteq \text{Eq}_{L} \) and a set of formulae \( \Gamma \subseteq \text{Eq}_{L} \), \( E[\Gamma] \) denotes the set of equations \( \bigcup \{E(\gamma) \mid \gamma \in \Gamma \} \).

**Theorem 2.16.** Let \( L = \langle L, \models_{L} \rangle \) be a logic. The following are equivalent:
1. $L$ is algebraizable.

2. For every $L$-algebra $A$, $\Omega_A$ is injective and it commutes with inverse images by homomorphisms.

3. For every $L$-algebra $A$, $\Omega_A$ is a lattice isomorphism between $\mathcal{F}_{iL}(A)$ and $\text{Co}_{\text{ALG}^*(L)}(A)$ that commutes with inverse images by homomorphisms.

4. There exists a set of formulae in two variables $E(p, q)$ and a set of equations in one variable $E(p) \subseteq \text{Eq}_L$ such that:
   - For every $\Gamma \cup \{\varphi\} \subseteq \text{Fm}_L$ we have: $\Gamma \vdash_L \varphi$ iff $E[\varphi] \models_{\text{ALG}^*(L)} E(p, q)$.
   - $p \approx q \models_{\text{ALG}^*(L)} E(p, q)$ and $E(p, q) \models_{\text{ALG}^*(L)} p \approx q$.
   - For every $\Pi \cup \{\varphi \approx \psi\} \subseteq \text{Eq}_L$ we have: $\Pi \models_{\text{ALG}^*(L)} \varphi \approx \psi$ iff $E[\Pi] \vdash_L E(\varphi, \psi)$.
   - $p \vdash_L E(p)$ and $E(p) \vdash_L p$.

   In this case $\text{ALG}^*(L)$ is called the equivalent quasivariety semantics of $L$. If $L$ is finitary, we can add:

5. $L$ is weakly algebraizable and $\text{ALG}^*(L)$ is a quasivariety.

We say that a possibly parameterized set of formulae in two variables $E(p, q, \overrightarrow{r}) \subseteq \text{Fm}_L$ satisfies the $G$-rule in the logic $L$ if $p, q \vdash_L E(p, q, \overrightarrow{r})$. This property and the finiteness of the corresponding sets $E$ allow to define some other classes of logics. Let $L = \langle L, \vdash_L \rangle$ be a logic. Then:

1. $L$ is called finitely equivalential (algebraizable) if it is equivalential (algebraizable) with a finite equivalence set.

2. $L$ is called regularly weakly algebraizable if it has a parameterized equivalence set satisfying the $G$-rule.

3. $L$ is called regularly (finitely) algebraizable if it has a (finite) equivalence set satisfying the $G$-rule.

All those classes constitute the so-called Leibniz hierarchy. They are depicted in Figure 1 together with their subsumption order (with the largest class at the bottom). The intersection of any two classes of the Leibniz hierarchy is exactly their infimum w.r.t. the subsumption order. There are examples in the literature showing that they are mutually different.

We end with a characterization of regularly weakly algebraizable logics. A matrix $A = \langle A, D \rangle$ is called unital if $D$ is a singleton, and a class of matrices is called unital if all its members are.

**Theorem 2.17.** Let $L$ be a protoalgebraic logic with a non-empty parameterized equivalence set. The following are equivalent:

1. $L$ is regularly weakly algebraizable.

2. $\text{MOD}^*(L)$ is unital.
3 Implications and semilinear implications

We start by introducing some useful notation. Let $L$ be a propositional language and let $\Rightarrow(p, q, \overrightarrow{r}) \subseteq Fm_L$ be a set of formulae in two variables and, possibly, with parameters $\overrightarrow{r}$ (a sequence of variables). Given formulae $\varphi, \psi \in Fm_L$ and a sequence of formulae $\overrightarrow{a}, \Rightarrow(\varphi, \psi, \overrightarrow{a})$ denotes the set obtained by substituting the variables in $\Rightarrow(p, q, \overrightarrow{r})$ by the corresponding formulae, and $\varphi \Rightarrow L \psi$ denotes the set $\bigcup \{ \Rightarrow(\varphi, \psi, \overrightarrow{a}) \mid \overrightarrow{a} \in Fm_L^\omega \}$. Again, we omit the parameter $L$ when clear from the context. When there are no parameters in the set $\Rightarrow(p, q)$ and it is unitary, we write $\varphi \rightarrow \psi$ instead of $\varphi \Rightarrow \psi$.

3.1 A hierarchy of implications

In this section we consider a collection of properties typically satisfied by implication connectives and generalize them to sets of (parameterized) formulae, yielding a hierarchy of generalized implications.

**Definition 3.1.** Let $L$ be a logic and $\Rightarrow(p, q, \overrightarrow{r}) \subseteq Fm_L$ be a parameterized set of formulae. We say that $\Rightarrow$ is a weak p-implication in $L$ if:

- (R) $\vdash_L \varphi \Rightarrow \varphi$
- (MP) $\varphi, \psi \vdash_L \psi \Rightarrow_L \psi$
- (T) $\varphi \Rightarrow_L \psi, \psi \Rightarrow_L \chi \vdash_L \varphi \Rightarrow_L \chi$
- (sCng) $\varphi \Rightarrow_L \psi, \psi \Rightarrow \varphi \vdash_L c(\chi_1, \ldots, \chi_{i-1}, \varphi, \ldots, \chi_n) \Rightarrow c(\chi_1, \ldots, \chi_{i-1}, \psi, \ldots, \chi_n)$ for each $\langle c, n \rangle \in L$ and each $i \leq n$.

We change the prefix ‘weak’ to ‘algebraic’ if there is a set $E(p)$ of equations in one variable such that
We change the prefix ‘weak’ to ‘regular’ if:

\[(\text{Reg}) \quad \phi, \psi \vdash_L \psi \Rightarrow \phi\]

We change the prefix ‘weak’ to ‘Rasiowa’ if:

\[(W) \quad \phi \vdash_L \psi \Rightarrow \phi\]

Finally, if \(\Rightarrow\) is parameter-free we drop the prefix ‘p-’.

The properties (R), (MP), (T), (sCng), (Alg), (Reg), and (W) are, strictly speaking, properties of the pair \(\langle L, \Rightarrow \rangle\), however we will often do some slight terminological abuses by saying that just a logic or a set of formulae have a property when the other element is clear from the context. They correspond to usual properties fulfilled by implication connectives: reflexivity, modus ponens, transitivity, symmetrized congruence and weakening. The condition (Alg) (resp. (Reg)) corresponds to the class of (resp. regularly) algebraizable logics as will be justified later.

**Proposition 3.2.**

- All the properties defined above are preserved in any extension.
- All the properties except (sCng) are preserved in any expansion\(^5\) and (sCng) is preserved in those expansions where the new connectives fulfill the symmetrized congruence property as well.
- If \(\Rightarrow\) is parameter-free, then all the properties are preserved in fragments containing the language of \(\Rightarrow\) (and that of \(E(p)\) in the case of (Alg)).

**Proof.** The only non-trivial part is to show the second part for parameterized sets. We present the proof for (T); the others are analogous. Let \(\langle L, \vdash_L \rangle\) be a logic satisfying (T) and \(\langle L', \vdash_{L'} \rangle\) an expansion. We know that for every \(\delta(p, q, \overline{r}) \in \Rightarrow\) we have: \(p \Rightarrow q, q \Rightarrow s \vdash_{L'} \delta(p, s, \overline{r})\). We can safely assume that none of the \(p, q, s\) occurs in \(\overline{r}\). For every \(\overline{r} \in Fm_{\Rightarrow}^{<\omega}\) we define an \(L'\)-substitution \(\sigma\) as: \(\sigma(p) = \varphi, \sigma(q) = \psi, \sigma(s) = \chi, \) and \(\sigma(\overline{r}) = \overline{r'}\). Thus we obtain: \(\sigma[p \Rightarrow q], \sigma[q \Rightarrow s] \vdash_{L'} \delta(\varphi, \chi, \overline{r'})\). To complete the proof just notice that \(\sigma[p \Rightarrow q] \subseteq \varphi \Rightarrow \psi \) and \(\sigma[q \Rightarrow s] \subseteq \psi \Rightarrow \chi\).

Notice that, unlike in the definition of (parameterized) equivalence sets, for weak (p-)implications we need to require the transitivity condition. Let us now clearly state the relation between equivalence and weak (p-)implication sets.

**Proposition 3.3.** Let \(L\) be a logic and \(\Rightarrow\) a weak \((p-)\)-implication in \(L\). Then, \(L\) is equivalential (protoalgebraic) with the (parameterized) equivalence set \(E(p, q, \overline{r}) = \Rightarrow(p, q, \overline{r}) \cup \Rightarrow(q, p, \overline{r})\).

\(^4\)We could also study the following symmetrized version of modus ponens (sMP): \(\varphi, \varphi \Rightarrow \psi, \psi \Rightarrow \varphi \vdash_L \psi\).

\(^5\)Many of the theorems we are going to prove would be also valid for this more general notion, but due to lack of any good motivating examples we are not going into this. The reader can easily recognize where we use full (MP) and which results would hold more generally (see e.g. the next proposition). Notice that if \(\Rightarrow\) satisfies (W), then (sMP) implies (MP). Indeed, from (W) we obtain \(\varphi, \varphi \Rightarrow \psi \vdash_L \psi \Rightarrow \varphi\) and hence by (sMP) we obtain \(\varphi, \varphi \Rightarrow \psi \vdash_{L'} \psi\).

\(^6\)Recall that an expansion of a logic can contain new connectives. Extensions are those expansions where the language remains the same.
On one hand, observe that if $L$ is an equivalential (protoalgebraic) logic then its (parameterized) equivalence set is a weak (p-)implication in the sense of the definition above. Recall that all (parameterized) equivalence sets in a given equivalential (protoalgebraic) logic are interderivable. On the other hand, in a given logic there could be different implications, take e.g. the classical logic: both implication and equivalence connectives are weak implications in the sense of the definition above, in fact, they are regular implications; but equivalence is not a Rasiowa implication.

**Proposition 3.4.** Each Rasiowa p-implication is a regular p-implication and each regular p-implication is an algebraic p-implication.

**Proof.** The first claim is trivial. To prove the second one let us by $\alpha$ denote a formula in one variable $p$ such that $\vdash_L \alpha$ (such formula clearly exists: take any formula from $p \Rightarrow p$ with all parameters substituted by $p$). Define $E(p) = \{p \approx \alpha\}$ and observe that (Reg) gives the first part of (Alg), whereas the second part follows from (MP).

We define several classes of logics according to the existence of several kinds of implications.

**Definition 3.5.** Let $L$ be a logic. We say that $L$ is a weakly/algebraically/regularly/Rasiowa-(p-)implicational logic if there is a (parameterized) set of formulae $\Rightarrow$ which is a weak/algebraic/regular/Rasiowa (p-)implication in $L$. We add the prefix ‘finitely’ if the set $\Rightarrow$ is finite and we use the adjective implicative instead of implicational if $\Rightarrow$ is a parameter-free singleton.

Rasiowa-implicative logics were already defined in 1974 by Rasiowa [26] and weakly implicative logics in 2006 by Cintula [6]. Now we study the relation between the classes of logics just defined and those in Leibniz hierarchy. First notice that the first two claims of the next proposition follow directly from Proposition 3.3, while the remaining ones are corollaries of the characterizations of the corresponding classes in Leibniz hierarchy presented in the preliminaries.

**Proposition 3.6.**

- Weakly p-implicational logics are exactly protoalgebraic logics.
- (Finitely) weakly implicational logics are exactly (finitely) equivalential logics.
- Algebraically p-implicational logics are exactly weakly algebraizable logics.
- Regularly p-implicational logics are exactly regularly weakly algebraizable logics.
- (Finitely) algebraically implicational logics are exactly (finitely) algebraizable logics.
- (Finitely) regularly implicational logics are exactly (finitely) regularly algebraizable logics.

**Proposition 3.7.** Let $L$ be a weakly implicative logic. If $L$ is regularly/algebraically p-implicational, then it is regularly/algebraically implicative.

**Proof.** As $L$ is regularly p-implicational we know that there is a parameterized equivalence set $E$ such that $\varphi, \psi \vdash_L E(\varphi, \psi, \tau)$. On the other hand, since $L$ is weakly implicative we know that $\{p \rightarrow q, q \rightarrow p\}$ is an equivalence set as well. As all (parameterized) equivalence sets are interderivable, the claim easily follows. The proof for the other case is analogous.
Thus, we have obtained a new classification of logics expanding the Leibniz hierarchy as drawn in Figure 2.

![Diagram of implicational logics hierarchy]

Figure 2: The hierarchy of implicational logics

We call it the *hierarchy of implicational logics*.

However, our intention is not to replace the traditional terminology. We only have provided a new systematic way to describe it: in one axis we go from p-implicational, implicational, finitely implicational to implicative; in the second one we use prefixes ‘weakly’, ‘algebraically’, ‘regularly’, or ‘Rasiowa-’. In the rest of the paper we will use the traditional names for particular old classes, and we will use the new systematic names only for new classes or when we need to formulate general theorems for more classes at once (see e.g. the previous or the next proposition).

**Proposition 3.8.** Let $L$ be a algebraically (regularly) p-implicational logic. Then, any weak p-implication is algebraic (regular).

Obviously the analogous statement is not true for Rasiowa p-implications.

**Example 3.9.** Let us consider the following examples showing separation of the classes in our hierarchy of implicational logics:

1. Consider first the equivalence fragment of the classical logic. This logic is clearly regularly implicative and we show that it is not Rasiowa-p-implicational. We know that this fragment is complete with respect to the two-valued matrix $M = \langle \langle \{0, 1\}, \leftrightarrow \rangle, \{1\} \rangle$, where $\leftrightarrow$ is the classical equivalence operation. Assume that there is a Rasiowa p-implication $\Rightarrow$. We

---

6The class of *finitely protoalgebraic logics* (finitely weakly p-implicational logics), i.e. logics with a finite parameterized equivalence set, has not been investigated so far as a part of Leibniz hierarchy. For this reason, and because they would make the diagram 3-dimensional, we will also disregard the classes of finitely algebraically/regularly/Rasiowa- p-implicational logics.
know that $x, x \Rightarrow y | \equiv_{M} y$. Take an evaluation $e$ such that $e(x) = 1$ and $e(y) = 0$. There has to be a formula $\chi(x, y, \overline{z}) \in \Rightarrow$ and a sequence of formulæ $\overline{\psi}$ (i.e. $\varphi = \chi(x, y, \overline{\psi}) \in x \Rightarrow y$) such that $e(\varphi) = 0$. Let us define a substitution $\sigma(v) = x$ if $e(v) = 1$ and $\sigma(v) = y$ otherwise. Define the formula $\hat{\varphi} = \chi(x, y, \overline{\sigma(\psi)})$. Observe that $\hat{\varphi} = \sigma \varphi$, it has just two variables $x$ and $y$, $\hat{\varphi} \in x \Rightarrow y$, and $e(\hat{\varphi}) = 0$. Let us write it as $\hat{\varphi}(1, 0) = 0$. Observe that from (R) we obtain $\hat{\varphi}(1, 1) = \hat{\varphi}(0, 0) = 1$ and from (W) we obtain that also $\hat{\varphi}(0, 1) = 1$. Thus we conclude that $\hat{\varphi}$ is the classical implication. As classical implication is not definable in the pure equivalence fragment, we have reached a contradiction and the proof is done.

2. Let UL be the Uninorm Logic studied in [24]. This logic is algebraically implicational but it is not regularly weakly algebraizable. Indeed, it has a binary primitive connective $\Rightarrow$ which fulfills the properties of algebraic implication; however, it is not regularly weakly algebraizable because $\text{MOD}^{*}(UL)$ is not unital (see Theorem 2.17). We can find many other examples with these features among well-known substructural logics: for instance, in [16], all the axiomatic extensions of FL which are not extensions of FLw.

3. Consider the degree-preserving three-valued Lukasiewicz logic $L_{3}^{\leq}$ defined in [3] by: $\Gamma \models_{L_{3}^{\leq}} \varphi$ iff for every evaluation $e$ on the three-element Lukasiewicz chain, $\min\{e(\gamma) | \gamma \in \Gamma\} \leq e(\varphi)$. This logic is weakly implicational with $E(p, q) = (p \leftrightarrow q)^{2}$ but it is not weakly algebraizable, as shown in [3]. A better known example of the same sort is the logic BCI which is clearly weakly implicational but known to be not weakly algebraizable.

4. Let $L$ be the logic in the language $\mathcal{L} = \{\rightarrow_{1}, \rightarrow_{2}\}$ with two binary connectives given by the matrix $A$ with a three element domain $\{0, a, 1\}$, the filter $\{1\}$ and the operations:

\[
\begin{array}{c|ccc}
\rightarrow_{1} & 0 & a & 1 \\
\hline
0 & 1 & 1 & 1 \\
\hline
a & 0 & 1 & 1 \\
1 & 1 & 0 & 1 \\
\end{array}
\quad
\begin{array}{c|ccc}
\rightarrow_{2} & 0 & a & 1 \\
\hline
0 & 1 & 1 & 1 \\
\hline
a & 0 & 1 & 1 \\
1 & 0 & 1 & 1 \\
\end{array}
\]

We show that $L$ is finitely Rasiowa-implicational but not weakly implicational. On one hand, it is easily checked that the finite set $p \Rightarrow q =\{p \rightarrow_{1} q, p \rightarrow_{2} q\}$ is a Rasiowa implication. On the other hand, assume in search of a contradiction that a formula $\delta(p, q)$ defines a weak implication. Observe that the element a never appears in the truth-table of a non-atomic formula. Due to (R) and (MP) the truth-table for $\delta$ should look like this:

\[
\begin{array}{c|ccc}
\delta & 0 & a & 1 \\
\hline
0 & 1 & ? & ? \\
\hline
a & ? & 1 & ? \\
1 & 0 & 0 & 1 \\
\end{array}
\]

Operations definable in $A$ are obtained as combinations of atoms by $\rightarrow_{1}$ and $\rightarrow_{2}$. One can prove that the truth-table of any binary operation has at most two 0s (it is routine to check it for definitions involving only two primitive operations, and then observe that because of the definitions of $\rightarrow_{1}$ and $\rightarrow_{2}$ any other combination will have the same property). Therefore, $\delta$ is not a weak implication, because it fails to satisfy (sCng): $\delta(0, a) = \delta(a, 0) = 1$, but also $\delta(a \rightarrow_{1} a, a \rightarrow_{1} 0) = 0$.

5. Dellunde’s logic presented in [9] is finitary and regularly algebraizable but it is not finitely equivalent. We improve the example by showing a Rasiowa-implicational but not finitely equivalent logic (although not finitary). Let $\langle\{\rightarrow_{i} | i \in \omega\}, \Rightarrow_{\mathcal{L}}\rangle$ be a logic, where all connectives are binary, given by the matrix $A$ with domain $\omega^{+}$, the filter $\{\omega\}$ and the operations $\rightarrow_{0}$ and $\rightarrow_{i}$ for $i > 0$: 

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The problem is in the last example: as far as we know, all known examples of left-to-right with one exception, the one from Rasiowa-implicational to Rasiowa-p-implicational logics. The problem is in the last example: as far as we know, all known examples of regularly weakly algebraizable logics which are not equivalential are unfortunately not Rasiowa-p-implicational.

We show the needed fact by the induction over the complexity of the formula, first notice that for non-atomic formulae \( \psi \) we have \( \psi^A(\omega, j) \in \{0, \omega\} \). We distinguish several cases:

- \( \varphi = p \rightarrow_i q \): \( \varphi^A(\omega, j) \neq \omega \) if \( j = i \).
- \( \varphi = \psi \rightarrow_i p \): clearly \( \varphi^A(\omega, j) = \omega \) for each \( j \in \omega \).
- Assume that \( \psi \neq q \) and \( \chi \) is not an atom:
  - \( \varphi = \psi \rightarrow_i q \): \( \varphi^A(\omega, j) \neq \omega \) only if \( j = i \).
  - \( \varphi = q \rightarrow_i \chi \): \( \varphi^A(\omega, j) \neq \omega \) only if \( j = i \).
  - \( \varphi = \psi \rightarrow_i \chi \) and \( i > 0 \): \( \varphi^A(\omega, j) = \omega \) for each \( j \in \omega \).
  - \( \varphi = \psi \rightarrow_0 \chi \): \( \varphi^A(\omega, j) \neq \omega \) only if \( \chi^A(\omega, j) = 0 \). From the induction assumption we know that \( \chi^A(\omega, j) \neq \omega \) for at most one \( j \in \omega \).

6. The logic of ortholattices (and, in general, any orthologic which is not orthomodular) is regularly weakly algebraizable but it is not equivalential (see [8, Chapter 4.7.]). Another example of such a situation is a subtractive logic defined by Agliano and Ursini (as proved in [8, Page 368]).

We need to show that all classes of logics in the implicational hierarchy depicted in Figure 2 are mutually different. The examples given above almost solve this problem: the first three examples show strictness of all right-to-left arrows; the next three ones show strictness of all left-to-right with one exception, the one from Rasiowa-implicational to Rasiowa-p-implicational logics. The problem is in the last example: as far as we know, all known examples of regularly weakly algebraizable logics which are not equivalential are unfortunately not Rasiowa-p-implicational.
Open Problem 3.10. Are the classes of Rasiowa-implicational and Rasiowa-p-implicational logics mutually different?

Another open problem is whether the relation of subsumption has the same nice intersection property that the original Leibniz hierarchy has, i.e. the intersection of any two classes is their infimum w.r.t. the subsumption order. Of course, everything works fine in the old part and due to Proposition 3.7 also in the weakly/algebraically/regularly implicative part. The rest is an open problem:

Open Problem 3.11. What are the intersections in the hierarchy involving Rasiowa classes?

3.2 Semantics of implications

The syntactical notion of weak p-implication that we have introduced has a natural semantical counterpart: a preorder in the models that becomes an order in reduced models. Let us formalize this notion.

Definition 3.12. Let \( L \) be a logic, \( \Rightarrow \) a weak p-implication, and \( A = \langle A, D \rangle \) an L-matrix. We define a binary relation \( \leq^\Rightarrow_A \) on \( A \) as: \( a \leq^\Rightarrow_A b \) iff \( a \Rightarrow^A b \subseteq D \).

Proposition 3.13. Let \( L \) be a logic, \( \Rightarrow \) a weak p-implication, and \( A = \langle A, D \rangle \in \text{MOD}(L) \). Then:

\begin{itemize}
  \item \( \leq^\Rightarrow_A \) is a preorder.
  \item \( \leq^\Rightarrow_A \) is an order if, and only if, \( A \) is reduced.
  \item The symmetrization of \( \leq^\Rightarrow_A \) is the Leibniz congruence of \( A \), i.e. \( \Omega_A(D) = \leq^\Rightarrow_A \cap (\leq^\Rightarrow_A)^{-1} \).
  \item \( D \) is an up-set w.r.t. \( \leq^\Rightarrow_A \), i.e. if \( a \in D \) and \( a \leq^\Rightarrow_A b \), then \( b \in D \).
\end{itemize}

Proof. All the properties are easily checked.

If there is a weak p-implication \( \Rightarrow \) in \( L \), we can say by virtue of Theorem 2.5 that \( L \) is complete with respect to the class of ordered matrices. Also for an \( L \)-matrix \( A \) and a weak p-implication \( \Rightarrow \) we call \( \leq^\Rightarrow_A \) the matrix (pre)order of \( A \). The following theorem shows an interesting link between reduced models and regularity of implication.

Theorem 3.14. Let \( L \) be a logic and \( \Rightarrow \) a weak p-implication. Then:

\begin{itemize}
  \item \( \Rightarrow \) is a regular p-implication if, and only if, for each \( A = \langle A, D \rangle \in \text{MOD}^\ast(L) \) there is an element \( a \in A \) such that \( D = \{a\} \).
  \item \( \Rightarrow \) is a Rasiowa p-implication if, and only if, for each \( A = \langle A, D \rangle \in \text{MOD}^\ast(L) \) there is an element \( a \in A \) such that \( D = \{a\} \) and \( a \) is the maximum of \( \leq^\Rightarrow_A \).
\end{itemize}

Proof. The first claim follows from Theorem 2.17. For the second one assume that \( \Rightarrow \) is a Rasiowa p-implication, i.e. \( q \vdash_L p \Rightarrow q \) and take any \( A = \langle A, D \rangle \in \text{MOD}^\ast(L) \). By the first claim, \( D = \{a\} \) for some \( a \in A \). Let \( b \) be an arbitrary element of \( A \) and \( e \) and \( A \)-evaluation such that \( e(q) = a \) and \( e(p) = b \). Then, \( b \Rightarrow^A a = \{a\} \) and hence \( b \leq^\Rightarrow_A a \). Conversely, if all reduced matrices have a singleton filter whose element is the maximum, then it is clear that \( q \vdash_L p \Rightarrow q \).
Given a logic \( L \) with a weak p-implication \( \Rightarrow \) (i.e. a protoalgebraic logic) and \( A = \langle A, D \rangle \in \text{MOD}^\ast(L) \), we denote by \([D, A]\) the set of all filters from \( \mathcal{F}_{\text{fil}}(A) \) that contain \( D \). Recall that \( \mathcal{F}_{\text{fil}}(A) \) is complete lattice (hence bounded) where the meet is the set intersection, the bottom is the intersection of all filters and the top is the set \( A \), and thus \([D, A]\) is a complete sublattice. It is easy to show that if \( L \) is weakly algebraizable, then we actually have \( \mathcal{F}_{\text{fil}}(A) = [D, A] \). Let us denote by \( \text{Up}^\Rightarrow(A) \) the complete bounded lattice of \( \leq^\Rightarrow_A \)-up-sets.

**Proposition 3.15.** Let \( L \) a logic with a weak p-implication \( \Rightarrow \) and \( A = \langle A, D \rangle \in \text{MOD}^\ast(L) \). Then, \([D, A]\) forms a sublattice of \( \text{Up}^\Rightarrow(A) \).

**Proof.** Take any \( D' \in [D, A] \) and \( A' = \langle A, D' \rangle \). Assume that \( a \in D' \) and \( a \leq^\Rightarrow_A b \). Then \( a \Rightarrow A b \subseteq D \) and since \( D \subseteq D' \) we obtain \( a \leq^\Rightarrow_{A'} b \). As we know that \( D' \) is up-set w.r.t. \( \leq^\Rightarrow_{A'} \), we obtain \( b \in D' \), and hence \( D' \in \text{Up}^\Rightarrow(A) \).

**Definition 3.16.** Let \( L \) be a logic, \( \Rightarrow \) a weak p-implication, and \( A = \langle A, F \rangle \in \text{MOD}(L) \). Then, \( F \) is called \( \Rightarrow \)-linear if \( \leq^\Rightarrow_A \) is a total preorder, i.e. for every \( a, b \in A \), \( a \Rightarrow A b \subseteq F \) or \( b \Rightarrow A a \subseteq F \). Furthermore, we say that \( A \) is a linearly ordered model (or just linear model) with respect to \( \Rightarrow \) if \( \leq^\Rightarrow_A \) is a linear order (equivalently: \( F \) is \( \Rightarrow \)-linear and \( A \) is reduced). We denote the class of all linear models with respect to \( \Rightarrow \) as \( \text{MOD}_\text{Lin}(L) \).

Observe that the class of linear models is not intrinsically defined for a given logic for it depends on the chosen implication. However, we will see later that in a reasonably wide class of logics all semilinear implications define the same linear models. But even in a general case we can make an interesting observation about the linear models. Observe that if \( \leq^\Rightarrow_A \) is a linear order, then \( \text{Up}^\Rightarrow(A) \) is linearly ordered by inclusion, and hence by Proposition 3.15 we easily obtain:

**Corollary 3.17.** Let \( L \) be a logic with a weak p-implication \( \Rightarrow \), and \( A = \langle A, D \rangle \in \text{MOD}_\text{Lin}(L) \). Then, \([D, A]\) is linearly ordered by inclusion.

**Theorem 3.18.** Let \( L \) be a protoalgebraic logic. Then, for any weak p-implication \( \Rightarrow \), \( \text{MOD}_\text{Lin}(L) \subseteq \text{MOD}^\ast(L)_{\text{RFSI}} \).

**Proof.** Given any linear model \( \langle A, D \rangle \), we know that the filters in \([D, A]\) are linearly ordered and so \( D \) is clearly finitely meet-irreducible. Thus, by Proposition 2.8 the model is relatively finitely subdirectly irreducible.

Another interesting question is under which condition the \( \Rightarrow \)-linear theories form a basis of the closure system \( \text{Th}(L) \). We formulate this question in an equivalent way (due to Proposition 2.3) as a form of extension property: (LEP). Our characterization is based on a generalization of the so-called ‘Prelinearity property’ (see [6]). However, we prefer the new name ‘Semilinearity Property’ following the tradition in Universal Algebra of calling a class of algebras ‘semiX’ whenever its subdirectly irreducible members have the property \( X \) (see Theorem 3.23).

**Definition 3.19.** Let \( L = (\mathcal{L}, \vdash_L) \) be a logic, \( \Rightarrow \) a parameterized set of formulae, and \( A \) an \( \mathcal{L} \)-algebra. We say that \( L \) has the

- Linear Extension Property (LEP) with respect\(^7\) to \( \Rightarrow \) if for every theory \( T \in \text{Th}(L) \) and every formula \( \varphi \in \text{Fm}_{\mathcal{L}} \setminus T \), there is a \( \Rightarrow \)-linear theory \( T' \supseteq T \) such that \( \varphi \notin T' \).

\(^7\)We omit the references to the set \( \Rightarrow \) when it is clear from the context.
• Semilinearity Property (SLP) with respect to \( \Rightarrow \) if the following meta rule is valid:

\[
\Gamma, \varphi \Rightarrow \psi \vdash_L \chi \quad \Gamma, \psi \Rightarrow \varphi \vdash_L \chi \\
\Gamma \vdash_L \chi
\]

The proof of the next proposition relating (LEP) and (SLP) can be obtained as an generalization of [6, Lemmata 10 and 11].

**Proposition 3.20.** Let \( L \) be a logic, \( \Rightarrow \) be a parameterized set of formulae. Then we have:

• If \( L \) has the (LEP), then it has the (SLP).

• If \( L \) is finitary, then \( L \) has the (LEP) iff it has the (SLP).

The next theorem shows that in finitary logics the notion of (LEP) can be also meaningfully defined for other than Lindenbaum matrices.\(^8\)

**Theorem 3.21.** Let \( L = \langle L, \vdash_L \rangle \) be a finitary logic with (LEP) and \( A \in \text{ALG}^*(L) \). Then, linear filters form a basis of \( \mathcal{F}i_L(A) \).

**Proof.** We show that for each \( F \in \mathcal{F}i_L(A) \) and \( t \in A \setminus F \) there is an \( \Rightarrow \)-linear filter \( F' \supseteq F \) such that \( t \notin F' \). We distinguish two cases based on the cardinality of \( A \).

1) **First assume** \( A \) is at most countable set. We can assume that the set of propositional atoms contains (or is equal to, in the infinite case) the set \( \{v_a \mid a \in A\} \). Let us by \( \varphi(v_{a_1}, \ldots, v_{a_n}) \) denote that a formula \( \varphi \) has variables \( v_{a_1}, \ldots, v_{a_n} \), and by \( \varphi^A(a_1, \ldots, a_n) \) the value of \( \varphi \) in \( A \) in evaluation \( e(v_{a_i}) = a_i \). Finally by \( E \) we denote the symmetrization of \( \Rightarrow \) (a parameterized) equivalence set of \( L \). Consider a theory \( T \) axiomatized by a set of formulae

\[
\Gamma = \{v_a \mid a \in F\} \cup \bigcup_{\varphi(v_{a_1}, \ldots, v_{a_n}) \in \text{Fm}_L} E(\varphi(v_{a_1}, \ldots, v_{a_n}), v_{\varphi^A(a_1, \ldots, a_n)}).
\]

Clearly \( v_t \notin T \) (because for the \( A \)-evaluation \( e(v_a) = a \) we obtain \( e(\Gamma) \subseteq F \) and \( e(v_t) \notin F \)). Now we use (LEP) to obtain a linear theory \( T' \supseteq T \) such that \( v_t \notin T' \). Consider a subset of \( A \) defined as \( F' = \{a \mid v_a \in T'\} \). Clearly \( F' \supseteq F \) and \( t \notin F' \), what remains to be shown is that \( F' \in \mathcal{F}i_L(A) \) and it is linear.

Linearity is simpler: if we show \( v_a \Rightarrow v_b \subseteq T' \) implies \( a \Rightarrow \chi b \subseteq F' \) the proof is done by the linearity of \( T' \). To show \( a \Rightarrow \chi b \subseteq F' \) we need to have \( \chi^A(a, b, \overline{v}) \in F' \) for any \( \chi \in \Rightarrow \) and any vector \( \overline{v} \) of values assigned to the parameters of \( \chi \). From our assumption we know that \( \chi(v_{a_1}, v_{a_2}, \overline{v}) \in T' \) and from the construction of \( T' \) we have \( E(\chi(v_{a_1}, v_{a_2}, \overline{v}), v_{\chi^A(a_1, a_2, \overline{v})}) \subseteq T' \) and so by (MP) we obtain \( v_{\chi^A(a_1, a_2, \overline{v})} \in T' \) and so \( \chi^A(a_1, a_2, \overline{v}) \in F' \).

To show that \( F' \in \mathcal{F}i_L(A) \) we first prove the following chain of equivalences for any \( A \)-evaluation \( e \) and any formula \( \psi(p_1, \ldots, p_m) \): \( e(\psi) \in F' \) iff \( v_{\psi(e(p_1), \ldots, e(p_m))} \in T' \) iff \( \psi(v_{e(p_1)}, \ldots, v_{e(p_m)}) \in T' \) iff \( \sigma \psi \in T' \) for the substitution \( \sigma p = e(p) \). The first equivalence is the definition of \( F' \); the second and the last ones are simple, and the third one follows from the fact \( E(\psi(v_{e(p_1)}, \ldots, v_{e(p_m)}), v_{\psi^A(e(p_1), \ldots, e(p_m))}) \subseteq T' \) and (MP). Assume that \( S \vdash_L \varphi \) and

\(^8\)This theorem can be seen as one of the so-called transfer theorems in the theory of g-matrices (see [13]), since it transfers a property (the (LEP) in this case) from Lindenbaum basic full g-models to other basic full g-models.
for some $A$-evaluation $e$ holds $e[S] \subseteq F'$. Using our equivalences we obtain $\sigma[S] \subseteq T'$ thus also $\sigma \varphi \in T'$ (because $\sigma[S] \vdash_{L} \sigma \varphi$). Thus finally $e(\varphi) \in F'$.

2) Secondly assume that $A$ is uncountable. We introduce a new set of propositional variables $VAR = \{v_{a} \mid a \in A\}$ that we can safely assume that contains the original propositional variables. We define a new logic $L'$ in the language $L'$ which has the same connectives as $L$ and atoms $VAR$ (let us, in this proof, assume that the set of atoms is part of the notion of propositional language) in the following way: $T \vdash_{L'} \varphi$ iff there is finite subset $T' \subseteq T$ and $L'$-substitution $\sigma$ such that $\sigma[T'] \cup \{\sigma \varphi\} \subseteq \text{Fm}_{L}$ and $\sigma[T'] \vdash_{L} \sigma \varphi$. From [8] (remark after Exercise 0.3.3.) we know that $L'$ is a finitary logic and it is a conservative extension of $L$.

Notice that $\Rightarrow$ is a weak p-implication in $L'$; we show that $L'$ has (SLP): assume that $T, \varphi \Rightarrow \psi \vdash_{L'} \chi$ and $T, \psi \Rightarrow \varphi \vdash_{L'} \chi$. As $L'$ is a finitary logic we know that there is a finite $T' \subseteq T$ such that $T', \varphi \Rightarrow \psi \vdash_{L'} \chi$ and $T', \psi \Rightarrow \varphi \vdash_{L'} \chi$. Obviously, there is an $L'$-substitution $\sigma$ such that $\sigma[T'] \cup \{\sigma \varphi, \sigma \psi, \sigma \chi\} \subseteq \text{Fm}_{L}$. Clearly also $\sigma[T'], \sigma \varphi \Rightarrow \sigma \psi \vdash_{L'} \sigma \chi$ and $\sigma[T'], \sigma \psi \Rightarrow \sigma \varphi \vdash_{L'} \sigma \chi$. Using the fact that $L'$ extends $L$ conservatively we obtain $\sigma[T'], \sigma \varphi \Rightarrow \sigma \psi \vdash_{L} \sigma \chi$ and $\sigma[T'], \sigma \psi \Rightarrow \sigma \varphi \vdash_{L} \sigma \chi$. Using (SLP) of $L$ we know that $\sigma[T'] \vdash_{L} \sigma \chi$. The definition of $L'$ gives us $T' \vdash_{L'} \chi$ and thus we obtain (SLP) in $L'$.

By the previous proposition (notice that the cardinality of the set of atoms does not play a role in its proof) $L'$ has also (LEP). Knowing this we can repeat the constructions from the first part of this proof; we construct $T$ (we obtain $v_{i} \notin T$ from an obvious observation $A \in \text{MOD}^{*}(L')$), $T'$ (because $L'$ has (LEP)), and $F'$. The rest of the proof is fully analogous. $\square$

3.3 Semilinear implications

Now we introduce the central concept of semilinear implication which, among others, provides another characterization of (LEP) and allows us to characterize implications where the converse inclusion of Theorem 3.18 holds.

**Definition 3.22.** Let $L$ be a logic and $\Rightarrow$ a weak p-implication. We say that $\Rightarrow$ is a weak semilinear p-implication if $\vdash_{L} = \vdash_{\text{MOD}^{*}(L)}$.

**Theorem 3.23 (Characterization of semilinear implications).** Let $L$ be a logic and $\Rightarrow$ a weak p-implication. Then, the following are equivalent:

1. $\Rightarrow$ is semilinear in $L$,
2. $L$ has the (LEP) w.r.t. $\Rightarrow$.

Furthermore, if $L$ is finitary the list of equivalences can be expanded with:

3. $L$ has the (SLP) w.r.t. $\Rightarrow$,
4. $\text{MOD}^{*}(L)_{RSI} \subseteq \text{MOD}^{0}_{\Rightarrow}(L)$.

Moreover, if $\Rightarrow$ is finite we can add:

5. $\text{MOD}^{*}(L)_{RFSI} \subseteq \text{MOD}^{0}_{\Rightarrow}(L)$.

**Proof.** The equivalence of 1. and 2. is proved by generalizing the proof of [6, Theorem 1]; 2. and 3. are equivalent due to Proposition 3.20; 4. implies 1. is an easy consequence of Corollary 2.7; we prove that 2. implies 4. Assume that $A = \langle A, F \rangle \notin \text{MOD}^{0}_{\Rightarrow}(L)$. If $A \notin \text{MOD}^{*}(L)$, then $A \notin \text{MOD}^{*}(L)_{RSI}$. Assume that $A \in \text{MOD}^{*}(L)$. By (LEP) and the Theorem 3.21 we obtain
that $F$ is the intersection of a family of $\Rightarrow$-linear filters and, since $F$ is not $\Rightarrow$-linear, it follows that $F$ is meet-reducible. By Proposition 2.8 we obtain $A \not\in \text{MOD}^*(L)_{\text{RFSI}}$.

Clearly 5. implies 4., thus to complete the proof we just show that 1. implies 5.: from the assumptions we know that $L$ is a finitary protoalgebraic logic complete with respect to the class $\text{MOD}^\ell_\Rightarrow(L)$, which clearly contains the trivial matrix. Thus, from Theorem 2.12, we obtain $\text{MOD}^*(L)_{\text{RFSI}} \subseteq H_S \text{SP}_U(\text{MOD}^\ell_{\Rightarrow}(L))$. By [8, Theorem 0.6.1] we know that $\text{SP}_U(\text{MOD}^\ell_{\Rightarrow}(L)) \subseteq \text{MOD}(L)$. It suffices to show that for each matrix in $\text{SP}_U(\text{MOD}^\ell_{\Rightarrow}(L))$ its matrix preorder is a linear order (since all models in $\text{MOD}^\ell_{\Rightarrow}(L)$ are reduced and $H_S$ only yields isomorphic copies when applied to reduced matrices). Consider the matrices as first-order structures with one unary predicate. The matrix preorder can be formally defined as: $a \leq^\L A b = \forall \exists \forall^A D(\chi(a, b, \exists \forall))$. Then the fact that $\leq^\L A$ is a linear ordering is expressible in the first-order language and hence preserved under $P_U$. Finally, the preservation under $S$ is obvious.

The implication ‘1. implies 3.’ of the theorem above holds even without the additional assumption of finitarity of the logic (by Proposition 3.20). Also clearly the implication ‘5. implies 1.’ holds even without the additional assumption of finiteness of the implications.

The previous theorem has several interesting and important corollaries. Using Theorem 3.18 we obtain that, at least in a reasonably wide class of logics, being the class of linear models with respect to any finite semilinear implication is an intrinsic property of a logic, for it corresponds to the class of relatively finitely subdirectly irreducible matrices.

**Corollary 3.24.** Let $L$ be a finitary protoalgebraic logic. Then, for any finite weak semilinear $p$-implication $\Rightarrow$, it holds: $\text{MOD}^*(L)_{\text{RFSI}} = \text{MOD}^\ell_{\Rightarrow}(L)$.

The eyesore restriction to finite implications in this corollary (and in the last part of the previous theorem) can be removed in a certain wide class of logics. To formalize this statement we need to introduce the notion of disjunction (see the next two sections and Corollary 5.6 in particular).

The second corollary uses the trivial observation that $\varphi, \psi \vdash_L \psi \Rightarrow \varphi$ and for regular implications also $\varphi, \varphi \vdash \psi \Rightarrow \psi \Rightarrow \varphi$. Thus, if $\Rightarrow$ is semilinear, we can use (SLP) to obtain that $\Rightarrow$ is a Rasiowa implication.

**Corollary 3.25.** Let $L$ be a logic. Then each regular semilinear $p$-implication is a Rasiowa $p$-implication.

Another interesting corollary is obtained by a simple observation that (LEP) of a logic is preserved in all its axiomatic extensions.

**Corollary 3.26.** Let $L$ be a logic and $\Rightarrow$ a weak semilinear $p$-implication. Then $\Rightarrow$ is semilinear in all axiomatic extensions of $L$.

This corollary will be particularly useful for showing that some logic has no semilinear implication: all we have to do is to find an axiomatic extension with this property.

It is quite easy to show that an implication in some logic is not semilinear, consider e.g. the normal implication of the intuitionistic logic. The well-know fact that the linear Heyting algebras do not generate the variety of Heyting algebras does the job. The next example uses again our characterization theorem (together with Corollary 3.17) to show much more: there is no weak semilinear $p$-implication definable in the intuitionistic logic, i.e. not only the
standard nice Rasiowa implication given by a single formula is not semilinear but even using an infinite set with parameters we could never obtain an implication whose linearly ordered Heyting algebras would generate the variety of Heyting algebras.

**Example 3.27.** There is no weak semilinear p-implication definable in intuitionistic logic.

**Proof.** We provide two alternative proofs of this fact. First a very simple ad hoc one, and then a more sophisticated proof using the machinery introduced in the present paper which has the advantage of providing a general method to show undefinability of weak semilinear p-implications in many logics.

1. Assume that \( \Rightarrow \) is a weak semilinear p-implication in intuitionistic logic and we show that \( p \Rightarrow q \vdash_{IPC} p \rightarrow q \) (\( \rightarrow \) is the usual implication of intuitionistic logic), which entails that \( \rightarrow \) is a semilinear implication—a contradiction. One direction is simple: from \( p, p \Rightarrow q \vdash_{IPC} q \Rightarrow p \). The reverse direction: using the first direction we obtain \( q \Rightarrow p, p \rightarrow q \vdash_{IPC} q \Rightarrow p \). Since, trivially, \( q \Rightarrow p, p \rightarrow q \vdash_{IPC} q \rightarrow p \) and all equivalence sets are interderivable (Proposition 2.11), we obtain \( q \Rightarrow p, p \rightarrow q \vdash_{IPC} q \Rightarrow p \). Now, using the trivial entailment \( p \Rightarrow q, p \rightarrow q \vdash_{IPC} p \Rightarrow q \) and (SLP), we conclude that \( p \Rightarrow q \vdash_{IPC} p \Rightarrow q \).

2. Consider any presentation of the intuitionistic propositional calculus, IPC, in the language \( L = \{ \land, \lor, \rightarrow, \bot, \top \} \). It is well known that it is a regularly algebraizable logic whose equivalent quasivariety semantics is \( \text{ALG}^*(IPC) = \text{HA} \), i.e. the variety of Heyting algebras, with \( E(p,q) = \{ p \rightarrow q, q \rightarrow p \} \) and \( E(p) = \{ p \approx \top \} \). On one hand, the regularity implies that the class of reduced models is unital, hence the filters of these models are just singletons of the form \( \{ \top_A \} \). On the other hand, the bijective correspondence between filters and congruences in any Heyting algebra given by the Leibniz operator gives that \( \{ \top_A \} \) is meet-irreducible in \( \text{F}_{IPC}(A) \) if, and only if, the identity relation is meet-irreducible in \( \text{Co}(A) \), i.e. \( A \) is subdirectly irreducible. Thus, \( \text{MOD}^*(IPC)_{RSI} = \{ \langle A, \{ \top^A \} \rangle \mid A \in \text{HASI} \} \). Assume now, in search of a contradiction, that \( \Rightarrow(p, q, \neg p) \subseteq \text{Fm}_C \) is a weak p-implication in IPC. By Theorem 3.23, we have \( \{ \langle A, \{ \top^A \} \rangle \mid A \in \text{HASI} \} \subseteq \text{MOD}^*_\Rightarrow(IPC) \). Now, it is sufficient to consider a subdirectly irreducible Heyting algebra where the natural lattice order is not linear (it is well-known that these algebras exist) and it will have two incomparable filters (IPC-filters are known to be the same as lattice filters over the Heyting algebra). Then, Corollary 3.17 gives the contradiction.

Combining this example and the previous corollary we obtain:

**Proposition 3.28.** Let \( L \) be the logic of any quasivariety of pointed residuated lattices\(^9\) containing the variety of Heyting algebras. Then, there is no weak semilinear p-implication definable in \( L \).

Let us give a list of some well-known logics falling under the scope of the previous proposition: Full Lambek logic (possibly extended with exchange/weakening/contraction), multiplicative-additive fragment of (Affine) Intuitionistic Linear logic, Relevance logic \( R \). On

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\(^9\)See e.g. the monograph [16].
the other hand, observe that the second proof in Example 3.27 can be used in many other protoalgebraic finitary logics $L$: all one needs is to find a relatively subdirectly irreducible member of $\text{MOD}^\ast(L)$ with two incomparable logical filters. For instance, consider now the variety $V$ of pointed residuated lattices generated by the symmetric rotation (see this construction e.g. in [17, 22]) of all Heyting algebras. Clearly, its corresponding logic has an involutive negation, that is, it proves $\neg\neg\varphi \rightarrow \varphi$. Reasoning exactly in the same way as before, we can show that no weak semilinear p-implication is definable in this logic and thus the same holds for the logic of any quasivariety of pointed residuated lattices containing $V$. In particular, this shows that no weak semilinear p-implication is definable in Girard’s Linear logic (precisely, in its reduct in the language of pointed residuated lattices).

At the end of this subsection we present another corollary of Theorem 3.23 that shows that semilinearity of implications is preserved under intersections of logics and discuss some of its consequences.

**Corollary 3.29.** Let $\Rightarrow$ be a parameterized set of formulae, $\mathcal{I}$ be a family of logics in the same language and $L$ its intersection. If $\Rightarrow$ is a weak semilinear p-implication in every logic of $\mathcal{I}$, then so it is in $\hat{L}$.

**Proof.** We show that $\hat{L}$ has the (LEP). Let $T$ be an $\hat{L}$-theory and $\varphi \notin T$, i.e. $T \not\models _{\hat{L}} \varphi$. Thus there has to be a logic $L \in \mathcal{I}$ such that $T \not\models _L \varphi$, i.e. $\varphi \notin \bar{T}$ where $\bar{T}$ is the $L$-theory generated by $T$. Thus by the (LEP) of $L$ there is a linear $L$-theory $T' \supseteq \bar{T} \supseteq T$ and $\varphi \notin T'$. As clearly $T'$ is also an $\hat{L}$-theory the proof is done.

The following theorem states that each logic with an implication can be extended to the weakest logic where that implication is semilinear. The proof is a trivial consequence of the previous corollary because any weakly p-implicational logic has at least one extension where its weak p-implication is semilinear, namely the inconsistent logic.

**Theorem 3.30.** Let $L$ be a logic and $\Rightarrow$ a weak p-implication. Then, there is the weakest logic extending $L$ where $\Rightarrow$ is semilinear (the intersection of all its extensions where $\Rightarrow$ is semilinear). Let us denote this logic as $L_{\Rightarrow}^\ell$.

In Section 5 we will show how to axiomatize $L_{\Rightarrow}^\ell$. However, to determine a complete semantics is simple, as described in the following straightforward proposition.

**Proposition 3.31.** Let $L$ be a logic and $\Rightarrow$ a weak p-implication. Then $L \models _{L_{\Rightarrow}^\ell} \Rightarrow = \models _{\text{MOD}^\ell_{\Rightarrow}(L)}$ and $\text{MOD}^\ell_{\Rightarrow}(L_{\Rightarrow}^\ell) = \text{MOD}^\ell_{\Rightarrow}(L)$.

Moreover, taking $L_{\Rightarrow}^\ell$ preserves finitarity:

**Proposition 3.32.** Let $L$ be a finitary logic and $\Rightarrow$ a weak p-implication. Then $L_{\Rightarrow}^\ell$ is finitary.

**Proof.** Recall that the finitary companion of a logic $S$, denoted as $\mathcal{FC}(S)$, is defined as: $\Gamma \vdash _{\mathcal{FC}(S)} \varphi$ iff there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash _S \varphi$. Observe that $\mathcal{FC}(S)$ is the largest finitary logic contained in $S$. Thus, since $L$ is finitary, we know that $L \subseteq \mathcal{FC}(L_{\Rightarrow}^\ell) \subseteq L_{\Rightarrow}^\ell$. If we show that $\Rightarrow$ is semilinear in $\mathcal{FC}(L_{\Rightarrow}^\ell)$ we obtain $\mathcal{FC}(L_{\Rightarrow}^\ell) = L_{\Rightarrow}^\ell$ and hence $L_{\Rightarrow}^\ell$ is finitary. Actually, one can easily show in general that if $\Rightarrow$ is semilinear in a logic $S$ then so it is in $\mathcal{FC}(S)$ just by checking that satisfaction of (SLP) preserved.

Finally, observe that if $L$ is finitary and $\Rightarrow$ a weak p-implication, then $L_{\Rightarrow}^\ell$ is the intersection of all the finitary extensions of $L$ where $\Rightarrow$ is semilinear.
4 Disjunctions

The goals of this section are to provide an abstract analysis of generalized disjunction connectives, present several possible notions of disjunction, and prepare the ground for the next section, where we study the interplay of disjunctions and implications. As we will see, disjunctions can be used to provide a powerful characterization of semilinear implications. Moreover, they are crucial for the notion of first-order predicate fuzzy logics (see e.g. [19] for some particular examples; a more general treatment is the subject for a planned further paper).

This section is based on the results from Section 2.5.1 of [8] and the paper [31]; we combine them into a uniform framework, generalize some of them (more details about the relation to these two works are in the text), and prove several new results (e.g. Proposition 4.5, Lemma 4.8 and Theorem 4.12).

4.1 A hierarchy of disjunctions

As in the case of implication, given a parameterized set of formulae \( \nabla(p, q, \rightarrow) \) we define \( \varphi \nabla \psi \) as \( \bigcup \{ \nabla(\varphi, \psi, \rightarrow) \mid \rightarrow \in \text{Fm}_L^{\leq \omega} \} \). Given sets \( \Sigma_1, \Sigma_2 \subseteq \text{Fm}_L \), \( \Sigma_1 \nabla \Sigma_2 \) denotes the set \( \bigcup \{ \varphi \nabla \psi \mid \varphi \in \Sigma_1, \psi \in \Sigma_2 \} \). When there are no parameters in the set \( \nabla(p, q) \) and it is unitary, we write \( \varphi \lor \psi \) instead of \( \varphi \nabla \psi \).

We start with a useful convention:

**Convention 4.1.** A parameterized set of formulae \( \nabla(p, q, \rightarrow) \) is called a \( p \)-protodisjunction in \( L \) whenever it satisfies:

\[
\text{(PD)} \quad \varphi \vdash_L \varphi \nabla \psi \quad \text{and} \quad \psi \vdash_L \varphi \nabla \psi
\]

If \( \nabla \) has no parameters we drop the prefix ‘\( p \)-’.

This convention does not define an interesting notion on its own because, actually, any (set of) theorem(s) in two variables of a given logic would be a protodisjunction in this logic; we only introduce it as a useful means to shorten the formulation of many upcoming definitions and results. In contrast, adding (a variant of) the property of proof by cases of classical disjunction results in a more interesting notion of disjunction.

**Definition 4.2.** Let \( L \) be a logic and \( \nabla \) a \( (p-) \)protodisjunction in \( L \) whenever it satisfies:

\[
\text{(wPCP)} \quad \text{If } \varphi \vdash_L \chi \text{ and } \psi \vdash_L \chi, \text{ then } \varphi \nabla \psi \vdash_L \chi.
\]

\( \nabla \) is called a \( (p-) \)disjunction whenever it satisfies Proof by Cases Property:

\[
\text{(PCP)} \quad \text{If } \Gamma, \varphi \vdash_L \chi \text{ and } \Gamma, \psi \vdash_L \chi, \text{ then } \Gamma, \varphi \nabla \psi \vdash_L \chi.
\]

Observe that \( \nabla \) remains (weak) disjunction in those fragments of a given logic containing the all connectives used in \( \nabla \). In Subsection 4.3 we will give sufficient and necessary conditions for the preservation of proof by cases in expansions (notice that (PD) is preserved in all expansions). Interestingly enough, all weak \( (p-) \)disjunctions in a given logic are mutually interderivable as we can easily prove:

**Lemma 4.3.** Let \( L \) be a propositional logic and \( \nabla, \nabla' \) parameterized sets of formulae. Assume that \( \nabla \) is a weak \( p \)-disjunction in \( L \). Then: \( \nabla' \) is a weak \( p \)-disjunction in \( L \) iff \( \varphi \nabla \psi \vdash_L \varphi \nabla \psi \).
Now we study relation of proof by cases and other properties a disjunction is expected to satisfy: commutativity, idempotency and associativity (which, however, are typically also satisfied by conjunction\textsuperscript{10} connectives). The following lemma is straightforward:

**Lemma 4.4.** Let $L$ be a logic and $\triangledown$ a p-protodisjunction. If $\triangledown$ satisfies (wPCP), then it also satisfies conditions:

(C) $\varphi \triangledown \psi \vdash L \psi \triangledown \varphi$

(I) $\varphi \triangledown \varphi \vdash L \varphi$

(A) $\varphi \triangledown (\psi \triangledown \chi) \vdash L (\varphi \triangledown \psi) \triangledown \chi$

We can show that the converse direction of the above lemma is not valid and also that properties (wPCP) and (PCP) are indeed different:

**Proposition 4.5.** There is a logic with a protodisjunction satisfying condition (C), (I), (A) which is not a weak disjunction. Moreover, there is a logic with a weak disjunction which is not a disjunction.\textsuperscript{11}

**Proof.** Define $L$ as the multiplicative and additive fragment of the linear logic (MALL) extended by the axiom $(\varphi \rightarrow \psi) \lor (\psi \rightarrow \varphi)$. Let $\lor$ be the additive disjunction and let $t$ be the 0-ary connective corresponding to the neutral element of the multiplicative conjunction. Notice that $\lor$ clearly satisfies all required conditions. Assume that $\lor$ has (wPCP) and we show that then $L$ proves the formula $\chi$ defined as $(\varphi \lor \psi) \land t \rightarrow (\varphi \land t) \lor (\psi \land t)$ which will be a contradiction with \cite[Example 3.2]{21}.

Obviously $\varphi \rightarrow \psi \vdash L \varphi \lor \psi \rightarrow \psi$ and so $\varphi \rightarrow \psi \vdash L (\varphi \lor \psi) \land t \rightarrow \psi \land t$. Thus finally $\varphi \rightarrow \psi \vdash L \chi$. Analogously we could prove $\psi \rightarrow \varphi \vdash L \chi$. Using (wPCP) and the fact that $\vdash L (\varphi \rightarrow \psi) \lor (\psi \rightarrow \varphi)$ the proof is done.

To prove the second claim consider the Gentzen calculus $G_L$ (introduced and studied in \cite{27}) in the language $L = \{\land, \lor\}$ whose rules are:

\[
\begin{align*}
\Gamma \vdash \varphi & \quad \Gamma \vdash \psi \quad \varphi \lor \psi \quad \psi \lor \chi & \quad \psi \lor \chi \quad \varphi \land \psi \quad \varphi \land \chi & \quad \varphi \land \chi \quad \Gamma \vdash \varphi \lor \psi \quad \Gamma \vdash \varphi \lor \psi
\end{align*}
\]

Let $L$ be the sentential logic defined from $G_L$ in the following way: given $\Gamma \cup \{\varphi\} \subseteq \text{Fm}_L$, $\Gamma \vdash_L \varphi$ iff there is a finite $\Delta \subseteq \Gamma$ such that the sequent $\Delta \vdash \varphi$ is derivable in $G_L$. It is clear that $\lor$ enjoys the (wPCP). However, it has not the (PCP) as implicitly proved in \cite[Chapter 5]{13}.

The notions of disjunction and p-disjunction are not new. They have already been considered in the framework of Abstract Algebraic Logic in several works (see e.g. \cite{8, 11, 13, 15, 29, 30}), with a different (but equivalent) definition. Indeed, according to these works, given a propositional logic $L = \langle \mathcal{L}, \vdash_L \rangle$ and its associated closure operator $C$, a connective $\lor$ (primitive or definable by a single formula) is a disjunction if, and only if, it satisfies the Property of Disjunction (PDI),\textsuperscript{12} i.e. for every set of formulae $\Gamma \cup \{\varphi, \psi\} \subseteq \text{Fm}_L$ it holds that

\textsuperscript{10}Notice the properties introduced so far are typically satisfied only by disjunction connectives.

\textsuperscript{11}We could also show the independence of the conditions (C), (I), (A) of protodisjunctions by several artificial examples. We leave it as an exercise for a reader and just mention a natural example: any substructural non-contractive involutive logic (e.g. linear logic or Lukasiewicz infinite-valued logic) has the multiplicative disjunction $\oplus$ which satisfies conditions (PD), (C), and (A) but not (I).

\textsuperscript{12}This property should not be confused with the so-called disjunction property, which typically holds in intuitionistic logic: If $\vdash_L \varphi \lor \psi$, then $\vdash_L \varphi$ or $\vdash_L \psi$.\footnote{Notice the properties introduced so far are typically satisfied only by disjunction connectives.}

\footnote{We could also show the independence of the conditions (C), (I), (A) of protodisjunctions by several artificial examples. We leave it as an exercise for a reader and just mention a natural example: any substructural non-contractive involutive logic (e.g. linear logic or Lukasiewicz infinite-valued logic) has the multiplicative disjunction $\oplus$ which satisfies conditions (PD), (C), and (A) but not (I).}

\footnote{This property should not be confused with the so-called disjunction property, which typically holds in intuitionistic logic: If $\vdash_L \varphi \lor \psi$, then $\vdash_L \varphi$ or $\vdash_L \psi$.}
\( C(\Gamma, \varphi \lor \psi) = C(\Gamma, \varphi) \cap C(\Gamma, \psi) \). \( \lor \) is a weak disjunction if, and only if, it satisfies the weak Property of Disjunction (wPDI), i.e. the same equation with \( \Gamma = \emptyset \). (PDI) and (wPDI) are Tarski-style conditions, i.e. conditions imposed on a consequence operator that involve only one connective.

In [31] Wang and Cintula consider logics with a connective \( \lor \) (primitive or definable by a single formula) called disjunction, which corresponds to our protodisjunction satisfying conditions (C), (I), (A), and study under which conditions this connective satisfies (PCP) as well. Based on some of his previous works, in [8] Czelakowski provides a more general approach by allowing a parameterized set of formulae instead of a single formula \( p \lor q \), which gives rise to the notions of p-disjunction and weak p-disjunction, in our terminology. To preserve most of the existing terminology and for the sake uniformity with the previous section we propose the following definition:

**Definition 4.6.** We call a logic (weakly) (p-)disjunctional if it has a (weak) (p-)disjunction. Furthermore, we call a logic (weakly) disjunctive if it has a (weak) disjunction given by a single parameter-free formula.

Lemma 4.3 shows us the notions of p-disjunction and weak p-disjunction are intrinsic for a given the logic; namely: a logic is (weakly) (p-)disjunctional iff it has some (weak) (p-)disjunction, and it does not matter which one we choose since all of them are interderivable. Let us now consider the question whether the above defined classes of logics are really distinct.

**Proposition 4.7.** There is a weakly disjunctive logic which is not p-disjunctional. Furthermore, there is a disjunctive logic which is not weakly disjunctive.

**Proof.** Consider again the second logic appearing in the proof of Proposition 4.5. As we argued, this logic has a weak disjunction \( \lor \) which is not a p-disjunction. Therefore, \( L \) is weakly disjunctive. If it would be p-disjunctional with some p-disjunction \( \triangledown \) then, according to Lemma 4.3, \( \triangledown \) would be mutually interderivable with \( \lor \), so \( \lor \) would be a p-disjunction – a contradiction. For the proof of the second claim see Theorem 5.24 and Example 5.25. \( \square \)

It follows that the classes of disjunctive, weakly disjunctive, disjunctional and weakly disjunctional logics are mutually different. The open problem is the relation of these classes and those given by parameterized (weakly) disjunctions.

We close this subsection with a lemma showing a strengthening of (PCP) which will be very useful later on.

**Lemma 4.8.** Let \( L \) be a logic, \( \triangledown \) a p-disjunction, \( \Gamma \cup \{ \chi \} \) a set of formulae, and \( A, B \) finite sets of formulae. Then the following metarule holds:

\[
\frac{\Gamma, A \vdash_L \chi \quad \Gamma, B \vdash_L \chi}{\Gamma, A \triangledown B \vdash_L \chi}
\]

Furthermore, if \( L \) is finitary we can drop the condition that \( A \) and \( B \) are finite.

**Proof.** The proof of the second claim is a straightforward consequence of finitarity and the first claim which we prove by using induction. Call a pair \( \Gamma, A \vdash_L \chi \) and \( \Gamma, B \vdash_L \chi \) a situation; define the complexity of a situation as a pair \( \langle n, m \rangle \) where \( n \) and \( m \) are respectively the cardinals of \( A \setminus B \) and \( B \setminus A \). We show by the induction over \( k = n + m \) that in each situation we obtain \( \Gamma, A \triangledown B \vdash_L \chi \).
First assume $k \leq 2$. If $n = 0$, i.e. $A \subseteq B$, we obtain $A \uparrow B \subseteq A \uparrow B$ and since $\Gamma, A \uparrow B \vdash L \chi$, the proof is done. The proof for $m = 0$ is the same. If $n = m = 1$ we use (PCP). The induction step: consider a situation with complexity $(n, m)$, where $n + m > 2$. We can assume without loss of generality that $n \geq 2$, take a formula $\varphi \in A \setminus B$ and define $A' = A \setminus \{\varphi\}$. We know that $\Gamma, A', \varphi \vdash L \chi$ and $\Gamma, B \vdash L \chi$. Thus we also know that $\Gamma, A', \varphi \vdash L \chi$ and $\Gamma, A', B \vdash L \chi$; notice that the complexity of this situation is $(1, m')$, where $n' \leq n - 1$ and $m' \leq m$, and so by the induction assumption we obtain $\Gamma, \varphi \uparrow B, A' \uparrow B \vdash L \chi$ (which is exactly what we wanted).

4.2 (PCP) and prime filters

In this subsection we study semantical characterizations of p-disjunctions in an analogous way to what we have done for semilinear implications in Subsections 3.2 and 3.3, i.e. in terms of a convenient notion of filter and its corresponding extension principle. In this analogy, (PCP) will play the role of (SLP), prime filters will play the role of linear filters, and (PEP) that of (LEP). Moreover, we will have again a strong link between this notion of filter and relatively finitely subdirectly irreducible reduced models of the logic.

**Definition 4.9.** Let $L = \langle L, \vdash \rangle$ be a logic, $\uparrow$ a parameterized set of formulae, $A$ an $L$-algebra, and $F \in \mathcal{F}_{iL}(A)$ a filter. Then, $F$ is called $\uparrow$-prime if for every $a, b \in A$, $a \uparrow Ab \subseteq F$ iff $a \in F$ or $b \in F$.

Notice that when $\uparrow$ defines a disjunction connective $\lor$, the previous definition gives just the usual notion of prime filter. The prime extension property (PEP) with respect to a set $\uparrow$ is defined as the (LEP) by substituting the notion of $\rightarrow$-linear filter for that of $\uparrow$-prime filter. Therefore, again due to Proposition 2.3, (PEP) is equivalent to the fact that $\uparrow$-prime filters form a basis of the closure system $Th(L)$. (PEP) is a sufficient condition for the (PCP), and it even characterizes (PCP) in finitary logics (the proof is an easy generalization of Proposition 4 and Theorem 1 of [31] and the definition of p-disjunction):

**Proposition 4.10.** Let $L = \langle L, \vdash \rangle$ be a logic and $\uparrow$ a p-protodisjunction. If $L$ has the (PEP), then it has (PCP). Furthermore, if $L$ is finitary the following are equivalent:

1. $L$ has the (PEP),
2. $L$ has the (PCP),
3. $\uparrow$ is p-disjunction.

We can obtain the following result completely analogous to Theorem 3.21.

**Theorem 4.11.** Let $L = \langle L, \vdash \rangle$ be a finitary logic with (PEP) and $A \in \text{ALG}^*(L)$. Then prime filters form a basis of $\mathcal{F}_{iL}(A)$.

**Proof.** In [8, Corollary 2.5.4] the author proves\textsuperscript{13} that if a finitary logic $L$ has the (PCP), then every pair of $L$-algebra $A$ and the set of its $L$-filters $\mathcal{F}_{iL}(A)$ enjoy a the following semantical

\textsuperscript{13}As noted by Font and Jansana in [13], the result can be generalized to the broader class the so-called full g-models.
Theorem 4.14 (P). The distributivity of the lattice of filters.

If the disjunction is parameter-free, all equivalences hold after removing the surjectivity of the substitutions.

Theorem 4.13. Let \( L = \langle \mathcal{L}, \models_L \rangle \) be a finitary logic and \( \nabla \) a protodisjunction. Then the following are equivalent:

1. \( \nabla \) is a p-disjunction.
2. For every \( \langle A, F \rangle \in \text{MOD}^*(L) \), \( \langle A, F \rangle \in \text{MOD}^*(L)\text{-RFSI} \) iff \( F \) is \( \nabla \)-prime.
3. For every \( \Gamma \cup \{ \varphi \} \subseteq \text{Fm}_L \), \( \Gamma \models_L \varphi \) if, and only if, \( \Gamma \models_{\{\langle A,F \rangle \in \text{MOD}^*(L) | F \text{ is } \nabla \text{-prime}\}} \varphi \).

Proof. 1. \( \rightarrow \) 2.: Assume that \( \nabla \) is a p-disjunction and take any \( \langle A, F \rangle \in \text{MOD}^*(L) \). We know from [8, Proposition 2.5.7] that finitely meet-irreducible filters coincide with \( \nabla \)-prime filters. This, together with Proposition 2.8, gives 2. The implication 2. \( \rightarrow \) 3. follows directly from Corollary 2.7. 3. \( \rightarrow \) 1.: Assume that \( L \) satisfies 3. and we show that it has (PCP). Assume that \( \Gamma, \varphi \nabla \psi \not\models_L \chi \), then there is an \( \langle A, F \rangle \in \text{MOD}^*(L) \) where \( F \) is \( \nabla \)-prime and an \( A \)-evaluation \( e \) such that \( e(\Gamma, \varphi \nabla \psi) \subseteq F \) and \( e(\chi) \notin F \). Since \( F \) is \( \nabla \)-prime we know that \( e(\varphi) \in F \) or \( e(\psi) \in F \) and so \( \Gamma, \varphi \not\models_L \chi \) or \( \Gamma, \psi \not\models_L \chi \).

We close this subsection with two results by Czelakowski expressed in our terminology. First, he shows that p-disjunctonal logics are a special kind of logics with distributive set of theories.

Theorem 4.13 ([8]). Let \( L = \langle \mathcal{L}, \models_L \rangle \) be a finitary logic. The following are equivalent:

1. \( L \) is p-disjunctonal.
2. The lattice defined over \( \text{Th}(L) \) is distributive, and for every \( \varphi, \psi \in \text{Fm}_L \) and every surjective substitution \( \sigma \), \( C(\sigma(\varphi)) \cap C(\sigma(\psi)) = C(\sigma(\varphi \cap C(\psi))) \).
3. The lattice defined over \( \text{Th}(L) \) is distributive, and for some \( \varphi \neq \psi \in \text{Fm}_L \), \( C(\sigma(\varphi)) \cap C(\sigma(\psi)) = C(\sigma(\varphi \cap C(\psi))) \) for every surjective substitution \( \sigma \).
4. The lattice defined over \( \text{Th}(L) \) is distributive and for every surjective substitution \( \sigma \) and every \( \nabla \)-prime \( T \in \text{Th}(L) \), \( \sigma^{-1}[T] \) is also \( \nabla \)-prime.

If the disjunction is parameter-free, all equivalences hold after removing the surjectivity of the substitutions.

Second, he proves that in protoalgebraic logics, being p-disjunctonal is equivalent to distributivity of the lattice of filters.

Theorem 4.14 ([8]). Let \( L \) be a finitary protoalgebraic logic. The following are equivalent:

1. \( L \) is p-disjunctonal.
2. For every \( L \)-algebra \( A \), the lattice defined over \( \mathcal{F}i_L(A) \) is distributive.
4.3 Syntactic characterization of (PCP)

This subsection in based on [31]; however we need to change the basic notion due to the possible presence of parameters in $\nabla$. Let us fix a parameterized set of formulae $\nabla(p,q,\vec{r})$ and a logic $L = (\mathcal{L}, \vdash_L)$ throughout this subsection.

**Definition 4.15.** Let $R = \Gamma \vdash \varphi$ be an $\mathcal{L}$-consecution. Then by $R^\nabla$ we denote the set of consecutions $\{ \Gamma \nabla \chi \vdash \delta \mid \chi \in \text{Fm}_\mathcal{L} \text{ and } \delta \in \varphi \nabla \chi \}$.

In [31] the authors used a different notion, which can be generalized to non-singleton $\nabla$ as $R^* = \{ \Gamma \nabla x \vdash \delta \mid \delta \in \varphi \nabla x \}$, where $x$ is a propositional variable not appearing in any formula of $R$. Notice that for a parameter-free $\nabla$ we have: $R^\nabla \subseteq L$ iff $R^* \subseteq L$, and the absence of parameters is crucial for this fact. Another problem could arise if $R$ would contain all propositional variables, and hence we could not pick any $x$ for the definition of $R^*$ (without some syntactic manipulation of the set of variables); this is in fact a subtle mistake in [31] where the authors assume that such $x$ always exists. The proofs of the following lemma and proposition can be taken directly from [31, Section 4] if we read them for $R^\nabla$ instead of $R^*$. Nevertheless, to be on the safe side and since the proofs are short, we will present them anyway.

**Lemma 4.16.** Let $R$ be a consecution and $\nabla$ a p-protodisjunction in $L$.

1. If $\nabla$ satisfies (I), then $R^\nabla \subseteq L$ implies $R \in L$.

2. If $\nabla$ satisfies (A), then $R^\nabla \subseteq L$ implies $(R^\nabla)^\nabla \subseteq L$.

3. If $\nabla$ is a p-disjunction and $R$ is finitary, then $R^\nabla \subseteq L$ iff $R \in L$.

4. If $\nabla$ satisfies (C) and (I), and $R^\nabla \subseteq L$ for each $R \in L$, then $\nabla$ is a p-disjunction.

**Proof.**

1. By (PD) we obtain $\Gamma \vdash_L \Gamma \nabla \varphi$. From $R^\nabla \subseteq L$ we know $\Gamma \nabla \varphi \vdash_L \varphi \nabla \varphi$ and so $\Gamma \vdash_L \varphi \nabla \varphi$. The rest is done by rule (I).

2. From $R^\nabla \subseteq L$ we know $\Gamma \nabla (\psi_1 \nabla \psi_2) \vdash_L \varphi \nabla (\psi_1 \nabla \psi_2)$; repeated use of (A) completes the proof.

3. One direction is part 1. The converse one: from $\Gamma \vdash_L \varphi$ we obtain $\Gamma \vdash_L \varphi \nabla \psi$. Observe that for arbitrary $\Gamma'$ and $\chi \in \Gamma'$ such that $\Gamma' \vdash_L \varphi \nabla \psi$ we can use the trivial fact that $\Gamma' \setminus \{ \chi \}, \psi \vdash_L \varphi \nabla \psi$ and obtain $\Gamma' \setminus \{ \chi \}, \chi \nabla \psi \vdash_L \varphi \nabla \psi$ by (PCP). If $\Gamma'$ is finite, then after we apply this observation enough number of times we obtain $\Gamma' \nabla \psi \vdash_L \varphi \nabla \psi$. Setting $\Gamma' = \Gamma$ completes the proof.

4. Assume that $\Gamma, \varphi \vdash_L \chi$ and $\Gamma, \psi \vdash_L \chi$. Using our assumption we obtain that $\Gamma \nabla \psi, \varphi \nabla \psi \vdash_L \chi \nabla \psi$ and $\Gamma \nabla \chi, \psi \nabla \chi \nabla \chi \nabla \chi$. Using (C) and (I) we obtain $\Gamma \nabla \psi, \Gamma \nabla \chi, \varphi \nabla \psi \vdash_L \chi$. As clearly $\Gamma \vdash_L \Gamma \nabla \psi$ and $\Gamma \vdash_L \Gamma \nabla \chi$ the proof is done.

\[\Box\]
There is an interesting open problem: does part 3. hold for all (not necessarily finitary) consecutions $R$? Observe that part 4. is a nice sufficient condition for a logic to have (PCP) and it is in fact a corollary of [8, Theorem 2.5.3.], which we can strengthen in the following proposition. Observe that we do not assume that $L$ is finitary.

**Proposition 4.17.** Let $L$ be a logic with a presentation $\mathcal{A}S$ and $\nabla$ a $p$-protodisjunction satisfying (C) and (I). If $R^\nabla \subseteq L$ for each $R \in \mathcal{A}S$, then $\nabla$ is a $p$-disjunction.

**Proof.** We show that $R^\nabla \subseteq L$ for each $R \in L$ and use the last part of the previous lemma. Assume that $\Gamma \vdash_L \varphi$ and we show $\Gamma \nabla \psi \vdash_L \delta \nabla \psi$ for each $\delta$ appearing in the proof of $\varphi$ from $\Gamma$. If $\delta \in \Gamma$ or $\delta$ is an axiom, the proof is trivial. Now assume that $R = \Gamma' \triangleright \delta \in \mathcal{A}S$ is the inference rule we use to obtain $\delta$. From the induction assumption we have $\Gamma \nabla \psi \vdash_L \Gamma' \nabla \psi$. As we know that $R^\nabla \subseteq L$ the proof is done. □

Putting it all together we can provide a nice characterization of the (PCP) for finitary logics. The proofs of the next theorem and the subsequent corollary are straightforward.

**Theorem 4.18.** Let $L$ be a finitary logic with a presentation $\mathcal{A}S$ and $\nabla$ a $p$-protodisjunction satisfying (C) and (I). Then, the following are equivalent:

1. $\nabla$ is a $p$-disjunction,
2. $R^\nabla \subseteq L$ for each finitary $R \in L$,
3. $R^\nabla \subseteq L$ for each $R \in L$,
4. $R^\nabla \subseteq L$ for each $R \in \mathcal{A}S$.

This theorem has an interesting corollary, answering the question whether $\nabla$ remains a disjunction in an expanded logic.

**Corollary 4.19.** Let $L_1$ be a finitary logic, $\nabla$ a $p$-disjunction in $L_1$, and $L_2$ an expansion of $L_1$ by a set of consecutions $C$. Then:

- If $R^\nabla \subseteq L_2$ for each $R \in C$, then $\nabla$ is a $p$-disjunction in $L_2$.
- If all the consecutions from $C$ are finitary, then $R^\nabla \subseteq L_2$ for each $R \in C$ iff $\nabla$ is a $p$-disjunction in $L_2$.

- If all the consecutions from $C$ are axioms, then $\nabla$ is a $p$-disjunction.

At the end of this subsection we show one more application of our main theorem. Observe that if we take an intersection of logics where $\nabla$ is a $p$-disjunction, $\nabla$ will remain a $p$-disjunction in the intersection. Also observe that any $\nabla$ is a $p$-disjunction in the inconsistent logic. Thus, the following definition is sound:

**Definition 4.20.** Let $L$ be a logic and $\nabla$ a parameterized set of formulae. We denote by $L^\nabla$ the least logic extending $L$ where $\nabla$ is a $p$-disjunction.

As in the case of $L^\ell_\omega$ (see the end of Subsection 3.3), we can easily determine a complete semantics $L^\nabla$ and show that taking $L^\nabla$ preserves finitarity. Moreover now we can also present a simple axiomatization of $L^\nabla$. 30
Proposition 4.21. Let $L$ be a logic and $\nabla$ a protodisjunction. Then,

$$\vdash_L \nabla = \models \{ (A,F) \in \text{MOD}^*(L) | F \text{ is } \nabla \text{-prime} \}.$$ 

Proposition 4.22. Let $L$ be a finitary logic and $\nabla$ a protodisjunction. Then $L^\nabla$ is finitary.

Theorem 4.23. Let $L$ be a finitary logic with a presentation $\mathcal{AS}$ and $\nabla$ a protodisjunction satisfying (C), (I), and (A). Then $L^\nabla$ is axiomatized by $\mathcal{AS} \cup \bigcup \{ R^\nabla \mid R \in \mathcal{AS} \}$.

Proof. Let $\hat{L}$ denote the logic axiomatized by $\mathcal{AS} \cup \bigcup \{ R^\nabla \mid R \in \mathcal{AS} \}$. Clearly, for each $R$ from that axiomatic system $R^\nabla \subseteq \hat{L}$ (due to Lemma 4.16 part 2.), hence we can use Proposition 4.17 to obtain that $\hat{L}$ has (PCP).

Let $L'$ be any logic with (PCP) extending $L$. Notice that for any $R \in \mathcal{AS}$ we have both $R \in L'$ and $R^\nabla \subseteq L'$ (due to Lemma 4.16 part 3.). Thus clearly $\hat{L} \subseteq L'$.

Observe that we could relax some of the conditions of $\nabla$ if we would add them together with their $\nabla$ variants to obtain the axiomatization of $L^\nabla$.

5 Interplay of disjunction and implication

In this section we consider the relationships between the several kinds of disjunctions and implications we have defined and their corresponding properties.

5.1 (LEP), (PEP), and semilinearity

First, we introduce two natural syntactical conditions that will play an important role in the interplay of disjunctions and implications: a version of modus ponens with disjunction (DMP), and a natural generalization of the prelinearity axiom used in fuzzy logics (P).

Proposition 5.1. Let $L$ be a logic. If $\nabla$ is a protodisjunction, then for any weak p-implication $\Rightarrow$ it holds:

\begin{align*}
\text{(DMP)} & \quad \varphi \Rightarrow \psi, \varphi \nabla \psi \vdash_L \psi \quad \text{and} \quad \varphi \Rightarrow \psi, \psi \nabla \varphi \vdash_L \psi \\
\text{If } \Rightarrow & \text{ is a weak p-implication with (SLP), then for any protodisjunction it holds:} \\
\text{(P)} & \quad \vdash_L (\varphi \Rightarrow \psi) \nabla (\psi \Rightarrow \varphi)
\end{align*}

Proof. The first part: clearly $\varphi, \varphi \Rightarrow \psi \vdash_L \psi$ and $\psi, \varphi \Rightarrow \psi \vdash_L \psi$. (PCP) completes the proof. The second part: clearly $\varphi \Rightarrow \psi \vdash_L (\varphi \Rightarrow \psi) \nabla (\psi \Rightarrow \varphi)$ and $\psi \Rightarrow \varphi \vdash_L (\varphi \Rightarrow \psi) \nabla (\psi \Rightarrow \varphi)$. (SLP) completes the proof.

Lemma 5.2. Let $L$ be a logic, $\nabla$ a protodisjunction, and $\Rightarrow$ a weak p-implication, $\mathcal{A} \in \text{ALG}^*(L)$.

- If $L$ fulfills (DMP), then each $\Rightarrow$-linear filter is $\nabla$-prime.
- If $L$ fulfills (P), then each $\nabla$-prime filter is $\Rightarrow$-linear.
Proof. The first claim: assume that $F$ is $\Rightarrow$-linear and $a \nabla A b \subseteq F$. We know that $a \Rightarrow^A b \subseteq F$ or $b \Rightarrow^A a \subseteq F$. Thus from (DMP) we obtain that $b \in F$ or $a \in F$.

The second claim: assume that $F$ in not $\Rightarrow$-linear, i.e. there are elements $a, b, x, y$ such that $x \in a \Rightarrow^A b$, $y \in a \Rightarrow^A b$, and $x, y \not\in F$. Because $x \nabla A y \subseteq (a \Rightarrow^A b) \nabla (b \Rightarrow^A a)$ and $L$ satisfies (P) we obtain $x \nabla A y \subseteq F$, i.e. $F$ is not $\nabla$-prime.

\[\square\]

**Theorem 5.3.** Let $L$ be a logic, $\nabla$ a p-protodisjunction, and $\Rightarrow$ a weak p-implication.

- If $L$ fulfills (DMP), we have:
  1. each $\Rightarrow$-linear theory is $\nabla$-prime,
  2. if $\Rightarrow$ has the (LEP), then $\nabla$ has the (PEP),
  3. if $\Rightarrow$ has the (SLP), then $\nabla$ has the (PCP).

- If $L$ fulfills (P), we have:
  4. each $\nabla$-prime theory is $\Rightarrow$-linear,
  5. if $\nabla$ has the (PEP), then $\Rightarrow$ has the (LEP).

- If $L$ fulfills (P) and either it is finitary or $\Rightarrow$ is finite and parameter-free, we have:
  6. if $\nabla$ has the (PCP), then $\Rightarrow$ has the (SLP).

Proof. Parts 1. and 4. are special cases of the previous lemma, and parts 2. and 5. are their straightforward corollaries. We prove the remaining two cases.

3. Let $\Gamma$ be a set of formulae and $\Gamma, \varphi \vdash_L \chi$ and $\Gamma, \psi \vdash_L \chi$. Using (DMP) we know that $\Gamma, \varphi \nabla \psi, \varphi \Rightarrow \psi \vdash_L \chi$ and $\Gamma, \varphi \nabla \psi, \psi \Rightarrow \varphi \vdash_L \chi$. (SLP) completes the proof.

6. From $\Gamma, \varphi \Rightarrow \psi \vdash_L \chi$ and $\Gamma, \psi \Rightarrow \varphi \vdash_L \chi$ we obtain $\Gamma, (\varphi \Rightarrow \psi) \nabla (\psi \Rightarrow \varphi) \vdash_L \chi$ (using Lemma 4.8). Knowing that $L$ satisfies (P) we obtain $\Gamma \vdash_L \chi$.

\[\square\]

This theorem together with known relations of the properties (SLP), (PCP), (PEP), (LEP) and semilinearity of an implication (Theorem 3.23 and Proposition 4.10) allows us to formulate a number of corollaries about their mutual relationship. We choose those we consider the most important ones:

**Corollary 5.4.** Let $L$ be a finitary logic, $\nabla$ a p-protodisjunction, and $\Rightarrow$ a weak p-implication. The following are equivalent:

1. $L$ satisfies (DMP) and $\Rightarrow$ is semilinear.
2. $L$ satisfies (DMP) and $\Rightarrow$ has the (SLP).
3. $L$ satisfies (DMP) and $\Rightarrow$ has the (LEP).
4. $L$ satisfies (P) and $\nabla$ has the (PEP).
5. $L$ satisfies (P) and $\nabla$ has the (PCP).
Corollary 5.5. Let $L$ be a finitary logic, $\triangledown$ a p-disjunction, and $\Rightarrow$ a weak p-implication. Then, $\Rightarrow$ is semilinear iff $L$ satisfies $(P)$.

As an application of Theorem 5.3 and its corollaries we can strengthen two important results from the previous section in the presence of a (definable) p-disjunction. First, we can remove the precondition of finiteness of implication in Part 5. of Theorem 3.23.

Corollary 5.6. Let $L$ be a finitary logic, $\triangledown$ a p-protodisjunction, and $\Rightarrow$ a weak p-implication. If $L$ satisfies $(DMP)$, then the following are equivalent:

1. $\Rightarrow$ is semilinear in $L$,
2. $\text{MOD}^*(L)_{\text{RFSI}} \subseteq \text{MOD}^\ell(L)$.

Furthermore, in any finitary p-disjunctional protoalgebraic logic holds: $\text{MOD}^*(L)_{\text{RFSI}} = \text{MOD}^\ell(L)$ for any semilinear p-implication $\Rightarrow$.

Proof. 2. $\Rightarrow$ 1. follows from the remark after Theorem 3.23. The converse direction: since $L$ satisfies $(DMP)$ and it is semilinear, it satisfies $(P)$ and (PCP) (due to the previous corollary). Thus we can use (the first part of) Corollary 4.12 to obtain that $(A, F) \in \text{MOD}^*(L)_{\text{RFSI}}$ iff $F$ is $\triangledown$-prime. As we have both (DMP) and (P) we know from Lemma 5.2 that linear and prime filters coincide over any algebra $A$ and so the proof of the first claim is done. Notice that in fact we have proved more, we showed that $\text{MOD}^*(L)_{\text{RFSI}} = \text{MOD}^\ell(L)$. This, together with the fact that p-disjunctional logic satisfies $(DMP)$ for any weak p-implication, gives us the proof of the second claim.

Second, we can use Corollary 4.19 to generalize Corollary 3.26 to show in which expansions of a logic an implication remains semilinear.

Corollary 5.7. Let $L_1$ be a finitary logic, $L_2$ an expansion of $L_1$ by a set of consecutions $C$, and $\triangledown$ a p-protodisjunction and $\Rightarrow$ a weak p-implication in both $L_1$ and $L_2$. Let further $L_1$ satisfy $(DMP)$ and $\Rightarrow$ be semilinear in $L_1$. Then:

- If $R^\triangledown \subseteq L_2$ for each $R \in C$, then $\Rightarrow$ is semilinear in $L_2$.
- If $L_2$ is an axiomatic expansion of $L_1$, then $\Rightarrow$ is semilinear in $L_2$.

Finally, we can use this result to provide an axiomatization of the least extension of a given logic $L$ where $\Rightarrow$ is semilinear. Recall that we denote this logic by $L^\ell_{\Rightarrow}$. Also recall that by $L^\triangledown$ we denote the weakest logic extending $L$ where $\triangledown$ is a p-disjunction (see Theorem 4.23 for an axiomatization of this logic).

Theorem 5.8. Let $L$ be a finitary logic, $\Rightarrow$ a weak p-implication, and $\triangledown$ a p-protodisjunction satisfying $(C)$, $(I)$, and $(A)$. Further assume that $L$ satisfies $(DMP)$. Then $L^\ell_{\Rightarrow}$ is the extension of $L^\triangledown$ by $(P)$.

Proof. Let us denote the extension of $L^\triangledown$ by $(P)$ as $\hat{L}$. Because $\hat{L}$ is an axiomatic extension of $L^\triangledown$, $\triangledown$ is a p-disjunction in $\hat{L}$ as well (by Corollary 4.19). Thus $\Rightarrow$ is a weak semilinear p-implication in $\hat{L}$ (by Corollary 5.4).

Let $L'$ be any extension of $L$ where $\Rightarrow$ is semilinear. Clearly $L'$ satisfies $(DMP)$ as well. Thus it satisfies $(P)$ and $\triangledown$ is a p-disjunction there (by Corollary 5.4) and so $\hat{L} \subseteq L'$. 

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If \( \nabla \) is a \( p \)-disjunction, \( L \) satisfies (DMP) (by Proposition 5.1). Thus, we can obtain an interesting corollary:

**Corollary 5.9.** Let \( L \) be a finitary logic, \( \nabla \) a \( p \)-disjunction, and \( \Rightarrow \) a weak \( p \)-implication. Then \( L_{\Rightarrow} \) is the extension of \( L \) by (P).

### 5.2 Dense semantics

We have introduced semilinear implications as those which induce completeness of the logic with respect to all linearly ordered matrices. However, there is a strong tradition in Fuzzy Logic which considers completeness theorems of logics with respect to particular kinds of linearly ordered matrices, mainly those defined over the real unit interval and, to a lesser extent, those defined over the rational unit interval (see e.g. [18, 7]). Some works even consider that completeness with respect to matrices over the real numbers is the constitutive property of fuzzy logics (see e.g. [24]). In this subsection we propose a way to study completeness with respect to densely ordered matrices (a common feature of both real and rational ones) by means of a special kind of filters and some meta-rules, analogously to previous sections.

**Definition 5.10.** Let \( L = \langle \mathcal{L}, \vdash \rangle \) be a logic, \( \Rightarrow \) a parameterized set of formulae, \( A \) an \( \mathcal{L} \)-algebra, and \( F \in \mathcal{F}_{\mathcal{L}}(A) \) a filter. \( F \) is called a \( \Rightarrow \)-dense filter if \( F \) is \( \Rightarrow \)-linear and for every \( a, b \in A \) such that \( 14 \ a \nless_{\Rightarrow} b \) there is \( z \in A \) such that \( a \nless_{\Rightarrow} z \) and \( z \nless_{\Rightarrow} b \).

A matrix \( A = \langle A, F \rangle \) is called a dense linear matrix w.r.t. \( \Rightarrow \) if it is reduced and \( F \) is \( \Rightarrow \)-dense (equivalently: if \( \leq_{A} \) is a dense order). The set of all dense linear \( L \)-matrices is denoted as \( \text{MOD}^F_{\Rightarrow}(L) \).

The notion of dense extension property (DEP) with respect to a set \( \Rightarrow \) is defined analogously as (LEP) and (PEP) but with some non-trivial changes. Recall that in the previous two cases we started with the extension properties and then managed to characterize them (in finitary logics) by some suitable meta-rule (viz. (SLP) and (PCP)). In this case we start with the meta-rule (DP) as it was already introduced in the literature [28] in a much more specific context and later generalized to a wide class of fuzzy logics in [24]. Recently this rule was studied in [5] in a very general context of hypersequent calculi; however, the level of generality of this study is clearly incomparable with ours. Our goal is to provide a corresponding extension principle (which was not explicitly formulated in the literature yet). The problem is that (DP) is not structural because it refers to an unused propositional variable. That is why we formulate (DEP) only in Lindenbaum matrices, and not for theories but only for some particular sets of formulae.

However, we will see that also in this case we obtain a nice interplay between a filter extension principle, a completeness property, and a logical meta-rule.

**Definition 5.11.** Let \( L \) be a logic and \( \Rightarrow, \nabla \) be parameterized sets of formulae. We say that \( L \) has the Density Property (DP) with respect to \( \Rightarrow \) and \( \nabla \) if for any set of formulae \( \Gamma \cup \{ \varphi, \psi, \chi \} \) we obtain \( \Gamma \vdash_L (\varphi \Rightarrow \psi) \nabla \chi \) whenever \( \Gamma \vdash_L (\varphi \Rightarrow p) \nabla (p \Rightarrow \psi) \nabla \chi \) for some variable \( p \) not occurring in \( \Gamma \cup \{ \varphi, \psi, \chi \} \).

**Definition 5.12.** Let \( L \) be a logic and \( \Rightarrow \) a parameterized set of formulae. We say that \( L \) has the dense extension property (DEP) with respect to \( \Rightarrow \) if every set of formulae \( \Gamma \) such that \( \Gamma \vdash_L \varphi \) and there are infinitely many variables not occurring in \( \Gamma \) can be extended into a \( \Rightarrow \)-dense theory \( T \supseteq \Gamma \) such that \( T \not\vdash_L \varphi \).

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14 By \( a \nless_{A} b \) we understand \( a \leq_{A} b \) and \( b \nless_{A} a \)
Lemma 5.13. Let $L$ be a logic, $\Rightarrow$ be a parameterized set of formulae, $A \in \text{MOD}^\delta_\Rightarrow(L)$, $T$ a theory, and $\varphi$ a formula. If $T \not\models_A \varphi$, then there is a countable submatrix $A'$ of $A$ such that $A' \in \text{MOD}^\delta_\Rightarrow(L)$ and $T \not\models_{A'} \varphi$.

Proof. Clearly $A$ is non-trivial and so it is infinite (because its matrix order $\leq$ is dense). Let $e$ be the evaluation witnessing $T \not\models_A \varphi$; we define $K$ as the subset of $A$ containing valuations assigned to all subformulae of formulae from $T \cup \{\varphi\}$ by $e$. Clearly $K$ is countable. We define an sequences $K_i$ of countable subset of $A$ and $A_i$ of submatrices of $A$ as: $K_1 = K$ and for $i > 0$:

- $A_i$ is the submatrix $A$ generated by $K_i$. (Clearly each $A_i$ is countable.)
- $K_{i+1}$ is any countable dense subset of $A$ containing $A_i$.

Clearly $A_i$ is a directed family a matrices and so their union $A'$ is a reduced $L$-matrix (see [8, Theorem 0.7.2]). Obviously $A'$ is countable submatrix $A'$ of $A$ such that $A' \in \text{MOD}^\delta_\Rightarrow(L)$ (because of the construction) and $T \not\models_{A'} \varphi$ (the evaluation $e$ does the job because $K \subseteq A'$).

Theorem 5.14. Let $L$ be a logic and $\Rightarrow$ be a parameterized set of formulae. Then, $\models_L = \models_{\text{MOD}^\delta_\Rightarrow(L)}$ iff $L$ has the (DEP).

Proof. Right-to-left: we repeat the usual completeness proof via constructing appropriate Lindenbaum-Tarski matrix with an interesting twist to overcome the restrictions of (DEP).

Let us consider a theory $T$ and a formula $\varphi$ such that $\varphi \notin T$. Let us enumerate the propositional variables and define substitutions $\sigma$ and $\sigma'$ by setting: $\sigma(v_i) = v_{2i}; \sigma'(v_{2i}) = v_i$, $\sigma'(v_{2i+1}) = v_i$ for each $i \geq 0$. Observe that $\sigma'(\sigma\psi) = \psi$ for any formula $\psi$. Thus also $\sigma[T] \not\models_L \sigma\varphi$. Notice that there are infinitely many variables not occurring in $\sigma[T]$ we can use (DEP) to obtain a $\Rightarrow$-dense theory $T'$ such that $T' \supseteq \sigma[T]$ and $T' \not\models_L \sigma\varphi$. Take the matrix $A = \langle \text{Fm}_L/\Omega(T'), T'/\Omega(T') \rangle$, observe that $A \in \text{MOD}^\delta_\Rightarrow(L)$, and consider the $A$-evaluation $e(\psi) = \psi/\Omega(T')$. We know that $e[T'] \subseteq T'/\Omega(T')$ and $e(\sigma\varphi) \notin T'/\Omega(T')$.

Let us now consider the $A$-evaluation $e'(\psi) = e(\sigma\psi)$ and observe that $e'(\varphi) = e(\sigma\varphi) \notin T'/\Omega(T')$. As $\sigma[T] \subseteq T'$, we obtain that $e'[T] = e[\sigma[T]] \subseteq e[T'] \subseteq T'/\Omega(T')$. Thus, we obtain $T \models_{A'} \varphi$.

Left-to-right: consider a set of $L$-formulae $\Gamma$ with infinitely many unused variables and a formula $\Gamma \models_L \delta$. $L$ can use our assumption to obtain a dense linear $L$-matrix $A = \langle A, F \rangle$ and an $A$-evaluation $e$ such that $e[\Gamma] \subseteq F$ and $e(\delta) \notin F$. Without a loss of generality we can assume that $A$ is countable (due to previous lemma). Let us consider a variable $v_a$ not occurring in $\Gamma \cup \{\delta\}$ for any $a \in A$ (such variables exist). Further consider an $A$-evaluation $e'$ defined as $e'(p) = e(p)$ for variables in $\Gamma \cup \{\delta\}$ and $e'(v_a) = a$ for $a \in A$.

Consider the set of formulae $T = \{ \varphi \mid e'(\varphi) \in F \}$. Clearly $T$ is a theory, $T \supseteq \Gamma$, and $\delta \notin T$; it remains to be shown that $T$ is $\Rightarrow$-dense in $L$. Linearity is simple (for each $\varphi$ and $\psi$, clearly $e(\varphi) \Rightarrow_A e(\psi) \subseteq F$ or $e(\psi) \Rightarrow_A e(\varphi) \subseteq F$). Observe that $\varphi \not\models_{\text{Fm},T} \psi$ if $e'(\varphi) <_{\text{Fm},T} e'(\psi)$. As $A$ is dense there is $a \in A$ such that $e'(\varphi) <_{\text{A}} a = e'(v_a) <_{\text{A}} e'(\psi)$. Thus $\varphi \not\models_{\text{Fm},T} v_a$ and $v_a \not\models_{\text{Fm},T} \psi$.

Using Theorem 3.23 we obtain:

Corollary 5.15. Let $L$ be a logic. Then, (DEP) implies (LEP).
**Lemma 5.16.** Let $L$ be a logic satisfying (DMP). Then, (DEP) implies (DP).

**Proof.** Assume that $\Gamma \not\vdash_L (\varphi \Rightarrow \psi) \nabla \chi$ and $\Gamma \vdash_L (\varphi \Rightarrow p) \nabla (p \Rightarrow \psi) \nabla \chi$ for some variable $p$ not occurring in $\Gamma, \varphi, \psi, \chi$. From the first assumption we know that there are formulae $\nu \in (\varphi \Rightarrow \psi)$ and $\delta \in (\psi) \nabla \chi)$ such that $\Gamma \not\vdash_L \delta$. Thus there is a dense linear $L$-matrix $A = \langle A, F \rangle$ and an $A$-evaluation $e$ such that $e[\Gamma] \subseteq F$ and $e(\delta) \notin F$. Clearly $e(\varphi) \not\leq e(\psi)$ (otherwise $e(\nu) \in F$ and so $e(\delta) \in F$) and so $e(\varphi) < e(\psi)$ (because $A$ is linear). As $A$ is dense there is an element $e(\varphi) < a < e(\psi)$. Take evaluation $e'(v) = a$ for $v = p$ and $e(v)$ otherwise (clearly $e'[\Gamma] \subseteq F$). There has to be elements $\nu_1 \in (\varphi \Rightarrow p), \nu_2 \in (p \Rightarrow \varphi)$ such that $e'(\nu_1) \notin F$. Notice that also $e'(\chi) \notin F$ (otherwise $e(\delta) \in F$). Using Theorem 5.3 we know that $F$ is also $\nabla$-prime filter and so $e'(\nu_1) \nabla e'(\nu_2) \nabla e'(\chi) \notin F$. Thus $\Gamma \not\vdash_L (\varphi \Rightarrow p) \nabla (p \Rightarrow \psi) \nabla \chi$, a contradiction. 

In the absence of parameters we can prove the equivalence of (DEP) with (SLP) and (DP):

**Lemma 5.17.** Let $L$ be a finitary logic satisfying (DMP), $\nabla$ a protodisjunction, and $\Rightarrow$ a weak implication. Then, $L$ has (DEP) iff it has (SLP) and (DP).

**Proof.** One direction follows from the previous lemma and corollary. To prove the second direction we introduce two sequences of sets of formulae $\Gamma_i$ and $A_i$ such that $\Gamma_i \vdash_L A_i$. Clearly, $\Gamma_0 = \Gamma$ and $A_0 = \{ \varphi \}$. Let us enumerate all pairs of formulae and proceed by induction:

- If $\Gamma, \varphi_i \Rightarrow \psi_i \not\vdash_L A_i$, then we define $\Gamma_{i+1} = \Gamma_i \cup (\varphi_i \Rightarrow \psi_i)$ and $A_{i+1} = A_i$. Clearly, $\Gamma_{i+1} \not\vdash_L A_{i+1}$.

- If $\Gamma, \varphi_i \Rightarrow \psi_i \vdash_L A_i$, then we define $\Gamma_{i+1} = \Gamma_i \cup (\psi_i \Rightarrow \varphi_i)$ and $A_{i+1} = A_i \nabla (\varphi_i \Rightarrow p) \nabla (p \Rightarrow \psi_i)$ for some variable $p$ not occurring in $\Gamma_i \cup A_i \cup \{ \varphi_i, \psi_i \}$ (since there are infinitely many variables not occurring in $\Gamma$, we can find in each step some unused one; notice that his would be no longer true if either $\nabla$ or $\Rightarrow$ would be parameterized). Assume, by the way of contradiction, that $\Gamma_{i+1} \vdash_L A_{i+1}$. As clearly also $\Gamma_{i+1} \nabla (\varphi_i \Rightarrow \psi_i)$ we can use (SLP) to obtain $\Gamma_{i+1} \vdash_L A_{i+1}$. Thus we also have $\Gamma_i \vdash_L A_i \nabla (\varphi_i \Rightarrow \psi_i)$ using (DP). Finally observe that from $\Gamma_i, A_i \vdash_L A_i$ and $\Gamma_i, \varphi_i \Rightarrow \psi_i \vdash_L A_i$ we can obtain $\Gamma_i, A_i \nabla (\varphi_i \Rightarrow \psi_i) \vdash_L A_i$ via Lemma 4.8. Putting this together we obtain $\Gamma_i \vdash_L A_i$—a contradiction.

Define $T$ as the $L$-theory generated by the union of all $\Gamma_i$’s. First observe that $T \not\vdash_L A_i$ for each $i$ (otherwise by finitarity there would be some $j$ such that $\Gamma_j \vdash_L A_i$ and so clearly $\Gamma_{\max\{i,j\}} \vdash_L A_{\max\{i,j\}}$—a contradiction).

Thus $T \not\vdash_L \varphi_i$ and from the construction it follows that $T$ is $\Rightarrow$-linear. Now assume that $T \not\vdash_L \varphi_i \Rightarrow \psi_i$, then we had to proceed via the second case in the construction (otherwise $T \vdash_L \varphi_i \Rightarrow \psi_i$) thus also $T \not\vdash_L \varphi_i \Rightarrow p$ and $T \not\vdash_L p \Rightarrow \psi_i$ (because otherwise $T \vdash_L A_{i+1}$).

All these results together give the main theorem of this subsection: the syntactical characterizations of completeness with respect to dense linear models.

**Theorem 5.18.** Let $L$ be a finitary logic, $\nabla$ a protodisjunction, and $\Rightarrow$ a weak implication. If $L$ satisfies (DMP), then the following are equivalent:

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1. \( \vdash_{L} \models_{\text{MOD}_{\delta}(L)} \).
2. \( L \) has the (DEP).
3. \( L \) has the (SLP) and the (DP).

Analogously to the case of \( L_{\ell} \Rightarrow \) and \( L_{\nabla} \), we can also consider the weakest extension of a logic enjoying completeness with respect to dense linear models.

**Lemma 5.19.** Let \( \Rightarrow \) be a parameterized set of formulae, \( I \) be a family of logics in the same language and \( L \) its intersection. If every logic of \( I \) has the (DEP), then so has \( \hat{L} \).

**Proof.** Let \( \Gamma \) be a set of formulae with infinitely many variables not occurring in it and \( \varphi \) a formula such that \( \Gamma \not\vdash_{\hat{L}} \varphi \). Thus there has to be a logic \( L \in I \) such that \( \Gamma \not\vdash_{L} \varphi \). Thus by the (DEP) of \( L \) there is an \( \Rightarrow \)-dense \( L \)-theory \( T \supseteq \Gamma \) and \( \varphi \not\in T \). As clearly \( T \) is also an \( \hat{L} \)-theory the proof is done.

This, together with the fact that any logic with a weak p-implication \( \Rightarrow \) has at least one extension which is complete w.r.t. its \( \Rightarrow \)-dense linear models (namely the inconsistent logic), gives the following result:

**Theorem 5.20.** Let \( L \) be a logic and \( \Rightarrow \) a weak p-implication. Then, there is the weakest logic extending \( L \) which is complete w.r.t. its \( \Rightarrow \)-dense linear models. Let us denote this logic as \( L_{\delta}^{\Rightarrow} \).

The proofs of the following two results run parallel to those of their analogues in previous sections.

**Proposition 5.21.** Let \( L \) be a logic and \( \Rightarrow \) a weak p-implication. Then \( \vdash_{L_{\delta}^{\Rightarrow}} = \models_{\text{MOD}_{\delta}(L)} \) and \( \text{MOD}_{\delta}(L_{_{\delta}^{\Rightarrow}}) = \text{MOD}_{\delta}(L) \).

**Proposition 5.22.** Let \( L \) be a finitary logic and \( \Rightarrow \) a weak p-implication. Then \( L_{\delta}^{\Rightarrow} \) is finitary.

**Theorem 5.23.** Let \( L \) be a finitary logic satisfying (DMP), \( \nabla \) a protodisjunction, and \( \Rightarrow \) a weak implication. Then, \( L_{\delta}^{\Rightarrow} \) is equal to the intersection of all its finitary extensions satisfying (DP) iff this intersection satisfies (SLP).

**Proof.** Let us denote that intersection as \( \hat{L} \). One direction is a simple consequence of Lemma 5.15. The converse direction: by Lemma 5.16 we know that each finitary extension of \( L \) with (DEP) has also (DP). Thus \( \hat{L} \subseteq L_{\delta}^{\Rightarrow} \). We prove the converse subsethood: since clearly \( \hat{L} \) is finitary, it satisfies (DMP), (SLP), and (DP), and hence it also has (DEP) (Lemma 5.17); thus \( \hat{L} \supseteq L_{\delta}^{\Rightarrow} \).

Although from our point of view this is just a slight reformulation, it simplifies and gives a new insight into an approach used in the fuzzy literature to prove dense completeness. In particular, in [24] the authors first describe the intersection of all finitary extensions of a given logic satisfying (DP) in a syntactic way;\(^{15}\) then they show that their syntactically described logic is complete w.r.t. dense linear models of the original logic. In the course of this proof they easily show that their extended logic has (PCP); which as we know implies (SLP) that would allow to skip the next two pages of their proof. Finally, by proof-theoretic means, they show the eliminability of the new rule to conclude that the original logic is indeed complete w.r.t. its dense linear models.

\(^{15}\)In a sense they add (DP) as a new rule, which of course cannot be done in the usual way for it is not a rule but a metarule (see [24, Definition 20]).
5.3 Some additional topics

In this final subsection we deal with two small topics. First we demonstrate how we can use a (strong enough) semilinear implication to produce a p-disjunction. Second we study a stronger notion of disjunction: a disjunction whose interpretation in reduced matrices is the supremum w.r.t. the matrix order.

**Theorem 5.24.** Let $L$ be a logic with a weak semilinear (p-)implication $\Rightarrow$. Let us define $\nabla = (p \Rightarrow q) \cup (q \Rightarrow p)$. If $\varphi \vdash_L (\varphi \Rightarrow \psi) \Rightarrow \psi$ and $\psi \vdash_L (\varphi \Rightarrow \psi) \Rightarrow \psi$, then $\nabla$ is a (p-)disjunction.

**Proof.** Clearly $\nabla$ is a (p-)protodisjunction. We show that $\nabla$ satisfies the (PCP). Assume that $\Gamma, \varphi \vdash_L \chi$ and $\Gamma, \psi \vdash_L \chi$. Thus clearly $\Gamma, \varphi \Rightarrow \psi, \psi \nabla \psi \vdash_L \chi$; analogously for $\psi \Rightarrow \varphi$. (SLP) completes the proof.

The hypotheses of the previous theorem are satisfied by a wide range of logics; for instance all finitary logics with a Rasiowa semilinear implication given by a binary connective $\rightarrow$ such that $p \vdash (p \rightarrow q) \rightarrow q$, and hence they are disjunctional logics. Nevertheless, they are not disjunctive in general as the following example shows.

**Example 5.25.** Let $G$ be Gödel-Dummett logic and $G \rightarrow$ its purely implicational fragment. Then, $G \rightarrow$ is a disjunctional logic which is not weakly disjunctive.

**Proof.** It is clear that $G \rightarrow$ is a disjunctional logic falling under the scope of Theorem 5.24. Assume that $\varphi(p, q)$ is a weak disjunction. As a consequence of the completeness theorem for $G$ proved in [10], we know that $G \rightarrow$ is complete with respect to the matrix $A$ whose universe is the real unit interval $[0, 1]$, the filter is $\{1\}$ and the only operation is:

$$a \rightarrow^A b = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{otherwise}. \end{cases}$$

By Lemma 4.3, the formula $\varphi(p, q)$ and the set $\{(p \rightarrow q) \rightarrow q, (q \rightarrow p) \rightarrow p\}$ are equivalent with respect to the semantics given by $A$. It implies, using the Deduction Theorem, that $\varphi(p, q)$ is interpreted in $A$ as the function maximum. So, in particular, for every $a, b \in [0, 1]$ we have $\varphi^A(a, b) = \max\{a, b\}$. We show by an infinite descent argument that this is impossible. Since $\rightarrow$ is the only connective in the language, the formula must be $\varphi(p, q) = \alpha(p, q) \rightarrow \beta(p, q)$. Take any $a, b \in [0, 1]$. If $a \leq b$, $\varphi^A(a, b) = \alpha(a, b) \rightarrow^A \beta(a, b) = b$, which implies $\beta^A(a, b) = b$. Analogously, if $a > b$, we have $\beta^A(a, b) = a$. Thus, $\beta(p, q)$ would be a strictly shorter formula with the same property – a contradiction.

\[\diamond\]

At the end this section we present a stronger notion of disjunction, a disjunction whose interpretation in reduced matrices is the supremum w.r.t. the matrix order. Of course, this idea makes sense only if $\nabla$ is a parameter-free singleton; let us use $\lor$ instead of $\nabla$ in this case.

**Definition 5.26.** Let $L$ be a logic, $\Rightarrow$ a weak p-implication, and $\varphi$ a formula in two variables. We say that $\lor$ is a lattice protodisjunction for $\Rightarrow$ if:

\[
\begin{align*}
(\lor_1) & \quad \Gamma, \varphi \Rightarrow \varphi \lor \psi \\
(\lor_2) & \quad \Gamma, \psi \Rightarrow \varphi \lor \psi \\
(\lor_3) & \quad \varphi \Rightarrow \chi, \psi \Rightarrow \chi, \Gamma \vdash_L \varphi \lor \psi \Rightarrow \chi
\end{align*}
\]
We say that $\lor$ is a lattice disjunction for $\Rightarrow$ (resp. lattice weak disjunction for $\Rightarrow$) if it also has the (PCP) (resp. the (wPCP)).

Notice that if a logic $L$ has a lattice p-protodisjunction for $\Rightarrow$, then $\leq \Rightarrow A$ is a lattice order for every $A \in \text{MOD}^+(L)$. By combining ($\lor$1), ($\lor$2), ($\lor$3), (R) and (T) we can easily show that any lattice protodisjunction for $\Rightarrow$ satisfies the properties which entail that it is in fact a $\{(C), (I), (A)\}$-protodisjunction.

$$
\begin{align*}
(\text{iC}) & \vdash^L \varphi \lor \psi \Rightarrow \psi \lor \varphi \\
(\text{iI}) & \vdash^L \varphi \lor \varphi \Rightarrow \varphi \\
(\text{iA}) & \vdash^L (\varphi \lor \psi) \lor \chi \Rightarrow \varphi \lor (\psi \lor \chi) \text{ and } \vdash^L \varphi \lor (\psi \lor \chi) \Rightarrow (\varphi \lor \psi) \lor \chi
\end{align*}
$$

This special kind of disjunctions does not collapse with the general notion in weakly p-implicational logics. For instance, consider the expansion of BL with Baaz’s $\triangle$ projection, $\text{BL}_{\triangle}$ (see e.g. [18]). In this logic we have a disjunction which is not a lattice disjunction. Indeed, if $\lor$ is the disjunction defined by $(((p \rightarrow q) \rightarrow q) \land ((q \rightarrow p) \rightarrow p)$, then the formula $\triangle p \lor \triangle q$ gives a disjunction which is not a lattice disjunction for $\rightarrow$. In contrast, $\lor$ is a lattice disjunction for $\rightarrow$.

6 The hierarchy of implicational semilinear logics

According to [2] fuzzy logics are the logics of chains in the sense that they enjoy a complete semantics based on linearly ordered algebras. Such a claim must be read as a methodological statement, pointing at a roughly defined class of logics, rather than a precise mathematical description of what fuzzy logics are, since there could be many different ways in which a logic might enjoy a complete semantics based on chains. Nevertheless, the framework we have developed in the present paper provides in a natural way a particular technical notion corresponding to this intuition. We define implicational semilinear logics as those possessing some weak semilinear p-implication. Obviously, they are fuzzy logics in the sense of [2] for they happen to be complete w.r.t. the class of models where the weak semilinear p-implication induces a linear order. Notice that we choose the term ‘semilinear’ over ‘fuzzy’ in spite of the fact that a first step towards the general definition we are offering here had been done by the first author in [6], when he defined the class of weakly implicitive fuzzy logics (in our new framework: logics with a weak semilinear implication given a single binary connective). We have realized that the attempt of [6] of using the term ‘fuzzy’ to formally define a class of logics was rather futile because such word is heavily charged with many conflicting potential meanings which are hard to be put away for many. Therefore, we have opted now for the new neutral name ‘semilinear’ (which, on the other hand, has the advantage of describing an equivalent algebraic property in the finitary case) although our intention remains the same: to capture the class of fuzzy logics among the protoalgebraic ones (originally, in weakly implicative ones only). Nevertheless, by doing so we do not yet expect to capture in a mathematical definition the whole intuitive notion of arbitrary fuzzy logic, for there could be (and some works in the literature suggest that this is the case; see e.g. [3] or some recent work on modal fuzzy logics) several other ways in which a logic might have a complete semantics somehow based on chains. But at least we aim to clearly define a broad class of logics which can arguably be regarded as fuzzy logics and containing almost all the prominent examples known so far. To this end, analogously as in Definition 3.5 we define classes of implicational semilinear logics based on the form of semilinear implication they possess.
Definition 6.1. Let $L$ be a logic. We say that $L$ is a weakly/algebraically/Rasiowa- (p-)implicational semilinear logic if there is a (parameterized) set of formulae $\Rightarrow$ such that it is a weak/algebraic/Rasiowa semilinear (p-)implication in $L$. We add the prefix ‘finitely’ if the set $\Rightarrow$ is finite and we use the adjective implicative instead of implicative if $\Rightarrow$ is a parameter-free singleton.

Notice that if we would have defined the class of regularly (p-)implicational (implicative) semilinear logics, by the Corollary 3.25 we would obtain that each regularly p-implicational semilinear logic is a Rasiowa-p-implicational semilinear logic (and analogously for the other three Rasiowa- classes in the hierarchy of implicational logics).

In accordance with Proposition 3.6 and our decision of preserving traditional terminology as much as possible, we will use the name ‘protoalgebraic semilinear logics’ instead of ‘weakly p-implicational semilinear logics’, ‘(finitely) equivalential semilinear logics’ instead of ‘(finitely) weakly implicational semilinear logics’, and ‘(finitely) algebraizable semilinear logics’ instead of ‘(finitely) algebraically implicational semilinear logics’. However, in the light of the previous observation, we will have no ‘regularly (finitely/weakly) algebraizable semilinear logics’ and we will use ‘(finitely) Rasiowa-(p-)implicational semilinear logics’ instead. See all the classes and their inclusions in Figure 3.

Figure 3: The hierarchy of implicational semilinear logics

Proposition 6.2. Let $X$ be one of the following expressions: ‘protoalgebraic’, ‘equivalential’, ‘finitely equivalential’, or ‘weakly implicational’. Then, a logic is algebraically/Rasiowa-$X$ semilinear logic iff it is simultaneously an $X$ semilinear logic and an algebraically/Rasiowa-$(p)$-implicational logic.

Proof. Let a logic $L$ be $X$ semilinear logic, then it possesses a semilinear implication $\Rightarrow$ of a proper form (parameterized set, any parameter-free set, finite parameter-free set or a singleton). Let $E$ be the (parametrized) equivalence set induced by this implication by symmetrization. Further assume that $L$ is a Rasiowa-p-implicational logic, i.e. there is a parameterized equivalence set $E'$ such that $\varphi, \psi \vdash_L E'(\varphi, \psi, \top)$. As all parameterized equivalence
sets are interderivable, \( \Rightarrow \) is a regular semilinear p-implication and so by Corollary 3.25 it is a Rasiowa p-implication. Thus the claim easily follows. The proof for \( L \) being algebraically p-implicational is analogous.

Roughly speaking, this proposition says that to locate a logic in the hierarchy of implicational semilinear logics it is enough to place it in one of the semilinear classes on the left-down branch (i.e. those given by \( \mathcal{X} \)) and in one of the classes on the right-down branch of the general (not semilinear) diagram (i.e. weakly algebraizable or regularly weakly algebraizable). Notice that no semilinearity is required in the second step. This proposition has an interesting corollary and raises an open problem.

**Corollary 6.3.** The intersection of any two classes of the hierarchy of implicational semilinear logics is their infimum w.r.t. the subsumption order.

The proof is simple if we notice that it is sufficient to show it for the first class being one of the equivalential, finitely equivalential, or weakly implicative semilinear logics and the second one of the weakly algebraizable or Rasiowa-implicational semilinear logics, and this follows from the previous proposition. The open problem is a kind-of dual of the proposition, where we switch the right-down and the left-down branches.

**Open Problem 6.4.** Let \( \mathcal{X} \) be either ‘protoalgebraic’, ‘weakly algebraizable’ or ‘Rasiowa-p-implicational’. Is a logic (finitely) \( \mathcal{X} \) implicational/implicative semilinear logic iff it is simultaneously an \( \mathcal{X} \) semilinear logic and a (finitely) implicational/implicative logic?

If we inspect the logics in Example 3.9 showing separations of the classes in the hierarchy of implicational logics, we notice that 2, 3, 4 and 5 have, in fact, semilinear implications. Thus, they also show the separation of the corresponding classes of implicational semilinear logics. Only three differences remain to be seen:

**Open Problem 6.5.** Rasiowa-implicational semilinear logics \( \neq \) Rasiowa-p-implicational semilinear logics, algebraizable semilinear logics \( \neq \) weakly algebraizable semilinear logics, and equivalential semilinear logics \( \neq \) protoalgebraic semilinear logics.

Recall that Corollary 3.24 together with Corollary 3.17 give us a nice way to show that a finitary weakly algebraizable logic is not semilinear: all we need to do is to find some relatively subdirectly irreducible reduced model and show that it has two incomparable filters. From Proposition 3.28 we conclude that many important logics (those which can be axiomatically extended to intuitionistic logic) are not protoalgebraic semilinear logics and hence do not belong to the hierarchy of implicational semilinear logics. As the intuitionistic logic is Rasiowa-implicative we can conclude:

**Proposition 6.6.** Let \( \mathcal{X} \) be any class in the hierarchy of implicational logics. Then, there is an \( \mathcal{X} \) logic which is not an \( \mathcal{X} \) semilinear logic.

Finally, it will be interesting for the reader coming from the Fuzzy Logic world to locate some well known families of fuzzy logics in the hierarchy. The three main logics based on continuous t-norms (Lukasiewicz logic, Gödel-Dummett logic and Product logic) as well as the logic of all continuous t-norms BL have a primitive implication connective \( \rightarrow \) which is known to be a Rasiowa semilinear implication in our terminology. So, they are Rasiowa-implicative semilinear logics. The same can be said in general as regards to left-continuous t-norm-based
logics such as MTL and its t-norm based axiomatic extensions, and even for all axiomatic
extensions of MTL (even those which are not complete with respect to a semantics of t-norms)
because all of them are complete with respect to a subvariety of MTL-algebras generated by
its linearly ordered members. Two incomparable superclasses of this one have been considered
in the literature. On one hand, we have the so-called core fuzzy logics introduced in [19] as
finitary logics expanding MTL or MTL△, satisfying (sCng) for →, and one of the following
forms of Deduction Theorem: (i) T, ϕ ⊢ MTL△ ψ iff T ⊢ MTL△ Δϕ → ψ, for expansions of
MTL△, or (ii) T, ϕ ⊢ MTL ψ iff there is n ∈ N such that T ⊢ MTL ϕn → ψ, for expansions
of MTL. On the other hand, we can consider the class of all semilinear finitary extensions
of MTL. Their equivalent quasiverse semantics are the subquasivarieties of MTL-algebras
generated by chains. Since there are such kind of quasivarieties that are not varieties, we have
that this class is strictly bigger than that of axiomatic extensions of MTL. Both incomparable
classes are included in the class of semilinear extensions of MTL, and finally this one is
included in the Rasiowa-implicative semilinear logics. In the recent paper [24] the fuzzy
logic UL based on uninorms instead of t-norms has been studied. It is an algebraizable logic
without weakening, so it belongs to the class of algebraically implicative semilinear logics.
We can consider the same structure of classes as above without weakening by replacing MTL
for UL. See the resulting hierarchy of classes of semilinear logics in Figure 4. We realize
that all of them lie on the top of our classification, above Rasiowa-implicative or algebraically
implicative semilinear logics. But if, by means of our definition of semilinear implication
presented in this paper, we have succeeded in capturing an interesting way by which a logic
can be fuzzy this means that fuzzy logics are a much wider class than those studied so far.
Thus, future research in the field will probably bring new significant examples of fuzzy logics
throughout the whole hierarchy of implicational semilinear logics.

![Hierarchy of Classes](image-url)

Figure 4: Prominent classes of fuzzy logics on the top of the hierarchy of implicational semi-
linear logics. All of them are mutually different.

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