(\(\delta, g\))-cages with \(g \geq 10\) are 4-connected \(\star\)

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Abstract

A regular graph \(G\) of degree \(\delta\) and girth \(g\) is said to be a \((\delta, g)\)-cage if it has the least number of vertices among all \(\delta\)-regular graphs with girth \(g\). A graph is called \(k\)-connected if the order of every cutset is at least \(k\). In this work, we prove that every \((\delta, g)\)-cage is 4-connected provided that either \(\delta = 4\), or \(\delta \geq 5\) and \(g \geq 10\). These results support the conjecture of Fu, Huang and Rodger that all \((\delta, g)\)-cages are \(\delta\)-connected.

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1. Introduction

Throughout this paper, all the graphs are simple, that is, without loops and multiple edges. Let \(G = (V, E)\) be a graph with the vertex set \(V = V(G)\) and the edge set \(E = E(G)\). For every \(v \in V\), \(N(v)\) denotes the neighbourhood of \(v\), that is, the set of all vertices adjacent to \(v\). If \(S \subset V\), then \(N(S) = \bigcup_{v \in S} N(v)\). If \(H\) is a subgraph of \(G\), then \(N_H(S) = N(S) \cap V(H)\). The subgraph of \(G\) induced by \(S\) is denoted \(G[S]\). For \(u, v \in V\), \(d(u, v) = d_G(u, v)\)

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denotes the distance between $u$ and $v$, that is, the length of a shortest $(u, v)$-path. For $S, W \subseteq V$, $d(S, W) = d_G(S, W) = \min\{d(s, w) : s \in S, w \in W\}$. The diameter $D = D(G)$ is the maximum distance over all pairs of vertices in $G$. A graph $G$ is called connected if every pair of vertices is joined by a path, that is, if $D < \infty$. If $S \subseteq V$ (resp. $S \subseteq E$) and $G - S$ is not connected, then $S$ is said to be a cutset (resp. edge-cut). Certainly, every connected graph different from a complete graph has a cutset. A (connected) component of a nonconnected graph $G$ is a maximal connected subgraph of $G$. A (noncomplete) connected graph is called $k$-connected if every cutset has cardinality at least $k$. The connectivity $\kappa$ of a (noncomplete) connected graph $G$ is defined as the maximum integer $k$ such that $G$ is $k$-connected. The minimum cutsets are those having cardinality $\kappa$. The connectivity $\kappa$ of a complete graph $K_{d+1}$ is defined as $\kappa(K_{d+1}) = d$.

The degree of a vertex $v$ is $\deg(v) = |N(v)|$, whereas the (minimum) degree $\delta$ of $G$ is the minimum degree over all vertices of $G$. A graph is maximally connected if $\kappa = \delta$. A graph is called regular if all its vertices have the same degree. The degree of a vertex $v$ in an induced subgraph $H$ of $G$ is $\deg_H(v) = |N(v) \cap V(H)| = |N_H(v)|$. The girth $g = g(G)$ is the length of a shortest cycle in $G$. A $(\delta, g)$-graph is a regular graph of degree $\delta$ and girth $g$. Let $f(\delta, g)$ denote the smallest integer $v$ such that there exists a $(\delta, g)$-graph having $v$ vertices. A $(\delta, g)$-cage is a $(\delta, g)$-graph with $f(\delta, g)$ vertices. These graphs have been intensely studied since introduced by Tutte in [11] (see [13] for a survey; see also [10]). Most of the work carried out so far has focused on the existence problem, whereas very little is known about structural properties. Recently, several authors have approached the problem of studying the connectivity of cages (see [3,5,6]). In the first paper on this issue (see [5]), Fu et al. proved that every $(\delta, g)$-cage is 2-connected. In addition, they conjectured that all $(\delta, g)$-cages are $\delta$-connected and proved this statement for $\delta = 3$. Subsequently, it has been proved that every $(\delta, g)$-cage with $\delta \geq 3$ is 3-connected (see [3,6]). More recently, $(4, g)$-cages have been seen to be 4-connected [14], and some of the authors have showed that every $(3, g)$-cage is quasi 4-connected [9]. As far as the edge-connectivity of cages is concerned, Wang et al. have proved in [12] that all $(\delta, g)$-cages with $g$ odd are $\delta$-edge-connected, an statement which Lin et al. [7] have showed to hold also for all value of $g$. Moreover, this result has been strengthened in [8] by the first two authors of this paper, who have proved that $(\delta, g)$-cages with $g$ odd are edge-superconnected.

This paper puts forward a further contribution towards the proof of the aforementioned conjecture, showing that every $(\delta, g)$-cage with $\delta \geq 4$ and $g \geq 10$ is 4-connected. Furthermore, this result is quite forwardly extended for $\delta = 4$ and $g \geq 3$, by presenting an independent proof of that in [14] of the fact that every $(4, g)$-cage is maximally connected. The statement for $\delta \geq 4$ and $g \geq 10$ has been proved taking into account the following known results.

**Theorem 1.1** (Erdös and Sachs [4], Fu et al. [5]). If $\delta \geq 3$ and $3 \leq g_1 < g_2$, then $f(\delta, g_1) < f(\delta, g_2)$.

**Theorem 1.2** (Jiang and Mubayi [6]). Let $S$ be a cutset of a $(\delta, g)$-cage with $\delta \geq 3$ and $g \geq 5$. Then, the diameter of $G[S]$ is at least $\lfloor g/2 \rfloor$. Furthermore, the inequality is strict if $d_{G[S]}(u, v)$ is maximized for exactly one pair of vertices.
The only \((\delta, g)\)-cages with \(g = 3\) and \(g = 4\) are \(K_{\delta+1}\) and \(K_{\delta, \delta}\) respectively. Certainly, the complete graph \(K_{\delta+1}\) is maximally connected. It is also clear that the complete bipartite graph \(K_{\delta, \delta}\) satisfies \(\kappa(K_{\delta, \delta}) = \delta\). For this reason, we henceforth assume \(g \geq 5\).

2. Almost every cage is 4-connected

Let \(G\) be a \((\delta, g)\)-cage with \(\delta \geq 4\) and \(g \geq 5\). Observe that, for every pair of vertices \(u,v\) such that \(d(u, v) \leq \lfloor (g - 1)/2 \rfloor\), there is only one shortest \((u, v)\)-path. Throughout this work, if \(\rho\) is a path or a cycle in a graph \(G\), \(||\rho||\) will denote its length.

Consider the set \(\mathcal{F}\) of all cutsets of \(G\) having cardinality 3 (recall that \(G\) is 3-connected, see [3,6]). The main goal of this section is to show that \(\mathcal{F} = \emptyset\), i.e., that \(G\) is 4-connected. To this end, suppose on the contrary that \(\mathcal{F} \neq \emptyset\), that is, \(\kappa = \kappa(G) = 3\). For every \(F \in \mathcal{F}\), let \(C_F\) denote a smallest component of \(G - F\), that is, such that \(|V(C_F)| \leq |V(G)|\) for every component \(C\) of \(G - F\). Let \(S = \{x, y, z\}\) denote any cutset of \(G\) satisfying:

\[
|V(C_S)| \leq |V(C_F)|, \quad \text{for every } F \in \mathcal{F}. \tag{1}
\]

In the rest of this work, we use the following notation: \(C_1 = C_S, C_2 = G - (S \cup V(C_1))\), \(X = N_{C_1}(x), Y = N_{C_1}(y),\) and \(Z = N_{C_1}(z)\). Notice that \(C_2\) is not necessarily connected, and \(|V(C_1)| \geq 2\) since \(\delta \geq 4\) and \(\kappa = 3\). Additionally, the minimality of \(C_1\) allows us to write:

\[
|V(G)| = |V(C_1)| + |S| + |V(C_2)| \geq 2|V(C_1)| + 3 \tag{2}
\]

**Lemma 2.1.** If \(S = \{x, y, z\}\) is a minimum cutset satisfying (1), then \(|X|, |Y|, |Z| \geq 2\). Moreover, if at least two of the sets \(X, Y, Z\) have cardinality \(\delta - 1\), then \(|V(G)| \geq 2|V(C_1)| + 5\).

**Proof.** To prove the first assertion, suppose, for instance, \(X = \{x_1\}\). Then, the set \(F = \{x_1, y, z\}\) is clearly a cutset of cardinality 3 satisfying \(|V(C_F)| < |V(C_1)|\), contradicting the definition of \(S\).

To prove the second claim, suppose for example, that \(|X| = |Y| = \delta - 1\). Observe that, in this case, \(C_2\) must be connected because \(\kappa = 3\). If \(xy, yz\) are two edges with \(a, b \in V(C_2)\), then \(Q = \{a, b, z\}\) is a cutset and hence, \(a\) and \(b\) must be different. Taking \(C'_2 = C_2 - \{a, b\}\), we can write \(|V(G)| = |V(C_1)| + |S \cup Q| + |V(C'_2)| = |V(C_1)| + |V(C'_2)| + 5\). But \(|V(C'_2)| \geq |V(C_1)|\), since \(|V(C_1)|\) is minimal, and thus \(|V(G)| \geq 2|V(C_1)| + 5\). \(\square\)

Under the assumption that \(G\) is a \((\delta, g)\)-cage with \(\delta \geq 4\), \(g \geq 5\), and connectivity \(\kappa = 3\), a certain minimum cutset \(S = \{x, y, z\}\) satisfying (1) is assumed to be arbitrarily chosen, whence we can write \(V(G) = V(C_1) \cup S \cup V(C_2)\); and \(N_{C_1}(x) = X, N_{C_1}(y) = Y,\) and \(N_{C_1}(z) = Z\), such that \(|X|, |Y|, |Z| \geq 2\). In this context, we introduce \(L\) and \(M\) as:

\[
L = \min\{d_{C_1}(X,Y), d_{C_1}(X,Z), d_{C_1}(Y,Z)\}
\]

\[
M = \max\{d_{C_1}(X,Y), d_{C_1}(X,Z), d_{C_1}(Y,Z)\}
\]

**Lemma 2.2.** Let \(G\) be a \((\delta, g)\)-cage with \(\delta \geq 4\), \(g \geq 10\), and \(\kappa = 3\). Let \(S = \{x, y, z\}\) be a cutset satisfying (1). Assume that \(d_{C_1}(X,Y) = d_{C_1}(a_0, a_L) = L \leq \lfloor (g - 3)/2 \rfloor\), for some
For every true only if holds.

Let $G$ be the only $(a_0, a_L)$-path of length $L$ in $C_1$, and $H = C_1 - \Pi$. Then, $X - a_0, V(\Pi), N_H(a_0), \ldots, N_H(a_L)$, $Y - a_L$, and $Z$ are pairwise disjoint sets.

**Proof.** Firstly, $X - a_0, V(\Pi), N_H(a_0), \ldots, N_H(a_L)$ and $Y - a_L$ are pairwise disjoint; otherwise, there would exist a cycle $C$ in $G[V(C_1) \cup \{x, y\}]$ of length at most $L + 4$ (see Fig. 1). Therefore, $|C| \leq L + 4 \leq \lfloor (g + 5)/2 \rfloor < g$, for $g \geq 6$, yielding a contradiction.

Next, take a vertex $z_i \in Z$. If $z_i \in X - a_0$, then $L = 0, a_0 = a_L$ and hence the path $z_i x a_0 y$ is a cutset of length 4, contradicting Theorem 1.2 as $\lfloor g/2 \rfloor \geq 5$. The same contradiction is obtained if we suppose that $z_i \in Y - a_L$.

Finally, suppose that $z_i \in N_{C_1}(V(\Pi))$. Let $P$ be the subgraph induced by the cutset $V(\Pi) \cup \{x, y, z_i, z\}$. Certainly, $|g/2| \leq D(P) \leq L + 3$ (the first inequality is a consequence of Theorem 1.2). But this means that $L \geq 2$ because $g \geq 10$. Hence, by the minimality of $L$, $z_i \notin \{a_0, a_L\} \cup N_{C_1}(a_0) \cup N_{C_1}(a_L)$. This fact allows us to assure that $D(P) \leq L + 2$ and thus, $|g/2| \leq L + 2$. In consequence, we obtain that $L \geq 3$ as $g \geq 10$. On the other hand, again by the minimality of $L, L \leq d_{C_1}(z_i, \{a_0, a_L\}) \leq 1 + \lfloor L/2 \rfloor$. As the inequality $L \leq 1 + \lfloor L/2 \rfloor$ is true only if $L \leq 2$, we have got the desired contradiction. \[ \square \]

At this point, we need to prove a useful technical lemma.

**Lemma 2.3.** Let $G$ be a $\delta$-regular graph with $\delta \geq 3$ and girth $g \geq 5$. Let $H$ be a subgraph of $G$ of minimum degree $\delta - 1$ and let $\Omega = \{v \in V(H) : \deg_H(v) = \delta - 1\}$. Suppose that $\Omega$ can be partitioned into two sets $\Omega_1$ and $\Omega_2$ with $|\Omega_2| = m \geq 2$, in such a way that $d_H(u_i, u_j) \geq \lfloor (g - 1)/2 \rfloor$ for every pair of different vertices $u_i, u_j \in \Omega_1$; and $\Omega_2 = \{z_1, z_2, \ldots, z_m\} \subset N(z)$ for some vertex $z \notin V(H)$. Then, $f(\delta, g) \leq 2|V(H)|$ if any of the following conditions holds:

1. $|\Omega_1| < m$;
2. For every $\{u_1, u_2, \ldots, u_m\} \subset \Omega_1 : 2m + 2\sum_{i=1}^m d_H(u_i, z_i) \geq g$.

**Proof.** Take a copy $H'$ of $H$ and consider the one-to-one map $\varphi$ between $H$ and $H'$ such that: $\varphi(z_i) = z_{i+1}'$ for every $i \in \{1, 2, \ldots, m - 1\}$, $\varphi(z_m) = z_1'$, and $\varphi(v) = v'$ whenever $v \in V(H) \setminus \Omega_2$.

![Fig. 1. Detail of a $(\delta, g)$-cage when $\kappa = 3, L \leq \lfloor (g - 3)/2 \rfloor$.](image-url)
Let $G^*$ be the graph such that $V(G^*) = V(H) \cup V(H')$, and $E(G^*) = E(H) \cup E(H') \cup E^+$, where $E^+ = \{w \varphi(w) : w \in \Omega\} = E_1^+ \cup E_2^+$, with $E_1^+ = \{uu' : u \in \Omega_1\}$ and $E_2^+ = \{z_1z_2', z_2z_3', \ldots, zmz_1'\}$. Note that $G^*$ is a $\delta$-regular graph satisfying $|V(G^*)| = 2|V(H)|$.

Hence, from Theorem 1.1 it suffices to show that $g(G^*) \geq g(G) = g$ to get the desired result. To this end, consider a cycle $\mathcal{C}$ in $G^*$, such that $E(\mathcal{C}) \cap E^+ \neq \emptyset$. Observe that, since $E^+$ is an edge-cut, $\mathcal{C}$ must contain an even number, say $2r$, of edges in $E^+$. At this point, consider the nonconnected graph $\mathcal{C} - (E(\mathcal{C}) \cap E^+)$, which is the disjoint union of $2r$ paths, $r$ of them contained in $H$ (resp. in $H'$). Any of these paths in $H$ (resp. $H'$) is called an $\eta$-path if its endvertices are both either in $\Omega_1$ (resp. $\Omega_1'$), or in $\Omega_2$ (resp. $\Omega_2'$); otherwise, it will be said to be a $\mu$-path. Furthermore, observe that $d_H(z_i, z_j) \geq g - 2 > \lfloor (g - 1)/2 \rfloor$ for every two vertices $z_i, z_j \in \Omega_2$, because $z \notin V(H)$.

Suppose firstly that $\mathcal{C}$ contains an $\eta$-path $\rho_1$ in $H$. Observe that, in this case, it must contain at least one more $\eta$-path $\rho_2$ either in H or in $H'$. This means that: $\|\mathcal{C}\| \geq \|\rho_1\| + \|\rho_2\| - 2 \geq 2\lfloor (g - 1)/2 \rfloor + 2 \geq g$, and we are done.

Assume next that $\mathcal{C}$ is a cycle without $\eta$-paths (thus $|E_1^+| = |E_2^+| = r$) and $r < m$. Therefore, there exists an $h \in \{1, 2, \ldots, m\}$ such that $z_{h-1}z_h' \in E(\mathcal{C})$ and $zhz_{h+1} \notin E(\mathcal{C})$ (where $h - 1, h + 1$ are taken modulo $m$). Then, if $\rho_1$ is a $(z_h, u_x')$-path in $\mathcal{C}$, then it must contain a $(u_x, z_k)$-path satisfying $k \neq h$. Hence,

$$
\|\mathcal{C}\| = 1 + d_H(z_h', u_x') + 1 + d_H(u_x, z_k) = 2 + d_H(z_h, u_x) + d_H(u_x, z_k) \\
\geq 2 + d_H(z_h, z_k) \geq g.
$$

So, if condition 1 is satisfied, then the proof is ended. Suppose next that $|\Omega_1| \geq m$, $\mathcal{C}$ is a cycle without $\eta$-paths, and $r = m$. This means that $\mathcal{C}$ contains the $2m$ edges in $E^+ = E_1^+ \cup E_2^+$; it also contains $m$ $\mu$-paths in $H$, say $\{\rho_i\}_{i=1}^m$, where $\rho_i$ is a $(u_i, z_i)$-path; and $m$ $\mu$-paths in $H'$, $\{\rho_i'\}_{i=1}^m$, where $\rho_i'$ is a $(u_i', z_i')$-path (see Fig. 2). Therefore,

$$
\|\mathcal{C}\| = 2m + \sum_{i=1}^m \|\rho_i\| + \sum_{i=1}^m \|\rho_i'\| \\
\geq 2m + \sum_{i=1}^m d_H(u_i, z_i) + \sum_{i=1}^m d_H'(u_i', z_i') \\
= 2m + 2 \sum_{i=1}^m d_H(u_i, z_i) \geq g. \quad \square
$$

![Fig. 2. Cycle in $G^*$ without $\eta$-paths with $r = m = 3$.](image-url)
Next, as a consequence of Lemma 2.2 and Lemma 2.3, a new partial result providing bounds for \( L \) and \( M \) is exhibited.

Lemma 2.4. If \( G \) is a \((\delta, g)\)-cage with \( \delta \geq 4 \), \( g \geq 10 \), and \( k = 3 \) then,

\[
[(g - 5)/2] \leq L \leq M \leq [(g - 3)/2].
\]

Proof. To prove the first inequality, assume, for example, that \( d_{\mathcal{C}_1}(X, Y) = d_{\mathcal{C}_1}(a_0, a_L) = L \leq [(g - 7)/2] \), where \( a_0 \in X \) and \( a_L \in Y \). Let \( \Pi = a_0 \cdots a_L \) denote the only path of length \( L \) in \( \mathcal{C}_1 \) joining \( a_0 \) to \( a_L \) (\( \Pi \) consists of a single vertex, \( a_0 = a_L \), when \( L = 0 \) and consider the subgraph \( H = \mathcal{C}_1 \cap \Pi \). Lemma 2.2 allows us to state that the sets \( F = \bigcup_{i=0}^{F} N_H(a_i) \), \( X^* = X - a_0 \), \( Y^* = Y - a_L \), and \( Z \) are pairwise disjoint, since we are assuming \( L \leq [(g - 7)/2] \) (see Fig. 1).

Define \( \Omega = F \cup X^* \cup Y^* \cup Z \), \( \Omega \subset V(H) \). Notice that \( \deg_H(v) = \delta \) for all \( v \in V(H) \setminus \Omega \), and \( \deg_H(w) = \delta - 1 \) for all \( w \in \Omega \). Let us define \( \Omega_1 = F \cup X^* \cup Y^* \), \( \Omega_2 = Z \). At this point, take two vertices \( u, v \in \Omega_1 \) and consider the shortest \((u, v)\)-path \( \rho \) in \( G[\Omega_1 \cup V(H) \cup \{x, y\}] \). The length \( \|\rho\| \) of this path is at most \( L + 4 \), being not greater than \( L + 3 \) when \( \{u, v\} \not\subset X^* \cup Y^* \).

Hence,

\[
d_H(u, v) \geq g - \|\rho\| \geq g - (L + 4) \geq (g - 1)/2.
\]

Next, take two vertices \( z_i, z_j \in \Omega_2 \). Observe that: \( d_H(u, z_i) + d_H(v, z_j) \geq g - 2 - \|\rho\| \). It is also clear that \( d_H(u, z_i) + d_H(v, z_j) \geq 2L \), whenever \( \{u, v\} \subset X^* \cup Y^* \). In consequence,

\[
d_H(u, z_i) + d_H(v, z_j) \geq \\
\begin{cases}
\begin{aligned}
&\geq (g - 2 - (L + 3) \geq (g - 3)/2) & \text{if } \{u, v\} \not\subset X^* \cup Y^*; \\
&\geq (g - 2 - (L + 4) \geq (g - 3)/2) & \text{if } \{u, v\} \subset X^* \cup Y^*, \ L \leq [(g - 9)/2]; \\
&\geq 2L & \text{if } \{u, v\} \subset X^* \cup Y^*, \ L = [(g - 7)/2].
\end{aligned}
\end{cases}
\]

So, we have proved that for any \( u, v \in \Omega_1 \) and \( z_i, z_j \in \Omega_2 \),

\[
4 + 2(d_H(u, z_i) + d_H(v, z_j)) \geq 4 + 2((g - 3)/2) \geq g.
\]

Set \( m = |\Omega_2| \geq 2 \); if \( |\Omega_1| \geq m \) and \( \{u_1, u_2, \ldots, u_m\} \) are \( m \) different vertices in \( \Omega_1 \), then the left-hand term of the above inequality is lesser than or equal to \( 2m + 2 \sum_{k=1}^{m} d_H(u_k, z_k) \).

Therefore, the subgraph \( H \) satisfies either condition 1 or condition 2 of Lemma 2.3, whence \( f(\delta, g) \leq 2|V(H)| - |V(G)| = f(\delta, g) \), which is impossible. Hence, \( [(g - 5)/2] \leq L \).

Now, we prove that \( M \leq [(g - 3)/2] \). On the contrary we have that \( M = d_{\mathcal{C}_1}(X, Y) \geq (g - 1)/2 \). Notice that the sets \( X, Y \) and \( Z \) are pairwise disjoint since \( L \leq [(g - 5)/2] \geq 3 \), and define \( H = \mathcal{C}_1 \cap (X \cup Y \cup Z) \subset V(H) \). \( \Omega_1 = X \cup Y, \ \Omega_2 = Z \). Then, for all \( v \in V(H) \setminus \Omega \), \( \deg_H(v) = \delta \), and for every \( w \in \Omega \), \( \deg_H(w) = \delta - 1 \). Observe that \( d_H(u, v) \geq (g - 1)/2 \) for every two different vertices \( u, v \in \Omega_1 \). Moreover, for every \( z_i, z_j \in \Omega_2 \), \( d_H(u, z_i) + d_H(v, z_j) \geq 2L \geq 2[(g - 5)/2] \geq g - 5 \). So, the subgraph \( H \) satisfies the conditions of Lemma 2.3, and we again obtain the contradiction \( f(\delta, g) \leq 2|V(H)| - |V(G)| = f(\delta, g) \).

\[\square\]

All of these previous lemmas enable us to derive a number of structural properties, which are put forward next.
Lemma 2.5. Let $G$ be a $(\delta, g)$-cage with $\delta \geq 4$, $g \geq 10$, and $\kappa = 3$. Let $d_x = |X|$, $d_y = |Y|$ and $d_z = |Z|$. Then the following assertions hold:

1. If $g$ is even, then $3 \leq L = M = g/2 - 2$, and $2 \leq d_x, d_y, d_z \leq \delta - 2$.
2. If $g$ is odd, then $3 \leq \lfloor (g-5)/2 \rfloor \leq L \leq M \leq \lfloor (g-3)/2 \rfloor$. Moreover, at most one of the sets $X, Y, Z$ has $\delta - 1$ vertices; if for instance, $d_x = \delta - 1$, then $d_{C_1}(X, Y) = d_{C_1}(X, Z) = L = \lfloor (g-5)/2 \rfloor$.

Proof. As a direct consequence of Lemma 2.4, it is clear that $3 \leq L = M = g/2 - 2$ whenever $g$ is even, and $3 \leq \lfloor (g-5)/2 \rfloor \leq L \leq M \leq \lfloor (g-3)/2 \rfloor$ if $g$ is odd.

To prove the other assertions, assume firstly that at least two of the sets $X, Y, Z$ have cardinality $\delta - 1$, for instance, $d_y = d_x = \delta - 1$. Let $H = G[V(C_1) \cup \{x, y\}]$, and define $\Omega_1 = \{x, y\}$, $\Omega_2 = Z$. Since the subgraph $H$ satisfies the conditions of Lemma 2.3, and recalling Lemma 2.1, we obtain $f(\delta, g) \leq |V(H)| = 2|V(C_1)| + 2 < |V(G)| = f(\delta, g)$, a contradiction. In consequence, we have shown that at least two of the sets $X, Y, Z$, have cardinality $\leq \delta - 2$.

Lastly, suppose that $d_x = \delta - 1$ and $d_{C_1}(X, Y) = \lfloor (g-3)/2 \rfloor$, and take $H = G[V(C_1) \cup x]$. Observe that $H$ satisfies the conditions of Lemma 2.3, since $d_H(x, y') = 1 + d_{C_1}(X, y') \geq 1 + \lfloor (g-3)/2 \rfloor = \lfloor (g-1)/2 \rfloor$, for every $y' \in Y$. So, we obtain $f(\delta, g) \leq |V(H)| = 2|V(C_1)| + 1 < |V(G)| = f(\delta, g)$, because of Lemma 2.1, a contradiction. Therefore, when $d_x = \delta - 1$, we must assume $d_{C_1}(X, Y) = d_{C_1}(X, Z) = L = \lfloor (g-3)/2 \rfloor - 1$; in other words, $g$ odd and $L = \lfloor (g-5)/2 \rfloor$. □

Now, we are ready to prove the main result of this work, in which we use the following notation:

\[ g = 2\ell + 1, \quad \text{if } g \text{ is odd}; \]
\[ g = 2\ell + 2, \quad \text{if } g \text{ is even}. \]

In other words, $\ell = \lfloor (g-1)/2 \rfloor$.

Theorem 2.1. Every $(\delta, g)$-cage with $\delta \geq 4$ and $g \geq 10$ is 4-connected.

Proof. Let $G$ be a $(\delta, g)$-cage with $\delta \geq 4$, $g \geq 10$, and $\kappa = 3$. Consider a cutset $S$ satisfying (1). According to Lemma 2.5, we can assume without loss of generality, $2 \leq d_x \leq \delta - 1$, $2 \leq d_y, d_z \leq \delta - 2$, $\ell - 2 = \lfloor (g-5)/2 \rfloor \leq d_{C_1}(X, Y) = d_{C_1}(a_0, a_L) = L \leq \lfloor (g-3)/2 \rfloor = \ell - 1$, and $L \geq 3$.

Let $II = a_0a_1 \cdots a_L$ denote the only $(a_0, a_L)$-path of length $L$ in $C_1$, and consider the subgraph $H = C_1 - II$. From Lemma 2.2, the sets $X - a_0, N_H(a_0), \ldots, N_H(a_L)$, $Y - a_L$ and $Z$, are pairwise disjoint (see Fig. 1). Observe that $|X - a_0| = d_x - 1 \leq \delta - 2$, $|Y - a_L| = d_y - 1 \leq \delta - 3$, $|Z| = d_z \leq \delta - 2$ and $|N_H(a_j)| = \delta - 2$, for every $j = 0, 1, \ldots, L$. Next, consider the set:

\[ \Omega = (X - a_0) \cup (Y - a_L) \cup Z \cup \left( \bigcup_{i=0}^{L} N_H(a_i) \right) \]

Certainly, $\deg_H(v) = \delta$ for all $v \in V(H) \setminus \Omega$, and $\deg_H(w) = \delta - 1$ for all $w \in \Omega$. 

Notice that, for every pair of different vertices \( p, q \in V(G) \), there exists at most one vertex \( q_i \in N_G(q) \) such that \( d_{G_q}(p, q_i) \leq \ell - 1 \), since \( g \geq 2\ell + 1 \). This fact allows us to label the vertices of \( \Omega \) in a suitable way by carrying out the following steps.

1. \( Y^* = Y - a_L = \{y_1, y_2, \ldots, y_{d_i-1}\} \), \( Z = \{z_1, z_2, \ldots, z_{d_f}\} \), and \( N_H(a_L) = \{a_{i1}, a_{i2}, \ldots, a_{i\delta-2}\} \), \( 2 \leq i \leq L - 1 \), are arbitrarily labelled.
2. \( N_H(a_0) = \{a_{01}, a_{02}, \ldots, a_{0\delta-2}\} \), so that \( d_H(y_i, a_{0i}) \geq \ell, 1 \leq i \leq d_y - 1 \).
3. \( N_H(a_1) = \{a_{11}, a_{12}, \ldots, a_{1\delta-2}\} \), where \( d_H(z_i, a_{1j}) \geq \ell \) if \( i \neq j \), \( 1 \leq i \leq d_z, 1 \leq j \leq \delta - 2 \).
4. If there exists a vertex \( r^* \in N_H(a_L) \) such that \( d_H(r^*, Z) \leq \ell - 2 \), then it is unique, and we set \( a_{L\delta-2} = r^* \). Otherwise, \( a_{L\delta-2} \) is any vertex in \( N_H(a_L) \).
5. \( X^* = X - a_0 = \{x_1, x_2, \ldots, x_{d_i-1}\} \), \( N_H(a_L) - a_{L\delta-2} = \{a_{L1}, a_{L2}, \ldots, a_{L\delta-3}\} \), so that \( d_H(x_i, a_{Lj}) \geq \ell \) if \( i \neq j \), \( 1 \leq i \leq d_x - 1, 1 \leq j \leq \delta - 3 \).

Let \( A = \lceil (L - 2)/2 \rceil \geq 1 \). At this point, we make the following remarks for every two vertices \( b, c \in \Omega \):

(i) Suppose first \( b = x_i \) and \( c = y_j \). Let us see that \( d_H(x_i, y_j) \geq \ell - 1 \). Otherwise \( L = \ell - 2 = d_H(x_i, y_j) \), because \( \ell - 2 \leq L = d_H(X, Y) \leq d_H(x_i, y_j) \leq \ell - 2 \). In this case, the \( (a_0, a_{i-L}) \)-path in \( C_1 \), the path \( a_Ly_jy_i \) in \( G \) of length two, the shortest \( (x_i, y_j) \)-path in \( H \), and the path \( x_i x_0 a_0 \) in \( G \) of length two, form a cycle of length \( L + 4 + d_H(x_i, y_j) = 2\ell < g \), a contradiction.

(ii) If \( b, c \notin (X^* \cup Y^* \cup Z) \), then \( d_H(b, c) \geq g - (L + 2) \geq \ell \).

(iii) If either \( b, c \notin (X^* \cup Z) \), or \( b, c \notin (Y^* \cup Z) \), then \( d_H(b, c) \geq g - (L + 3) \geq \ell - 1 \).

(iv) Suppose next \( b = z_i \). If \( c \in (X^* \cup Y^*) \), then certainly \( d_H(z_i, c) \geq L > A \). If \( c = a_{kj} \in N_H(a_k) \) for some \( k \), then the shortest \( (z_i, a_{kj}) \)-path in \( H \), along with the edge \( a_{kj}a_k \) and the \( (a_k, a_{00}) \)-subpath of \( \Pi \), form a \( (z_i, a_{00}) \)-path in \( C_1 \). By the minimality of \( L \), we get \( d_H(z_i, a_{kj}) \geq L - 1 - k \). Analogously, the shortest \( (z_i, a_{kj}) \)-path in \( H \), along with the edge \( a_{kj}a_k \) and the \( (a_k, a_{L2}) \)-subpath of \( \Pi \), form a \( (z_i, a_{L2}) \)-path in \( C_1 \). Again, by the minimality of \( L \), \( d_H(z_i, a_{kj}) \geq k - 1 \). So,

\[
d_H(z_i, a_{kj}) \geq \max\{L - 1 - k, k - 1\} \geq A,
\]

because either \( k \leq \lfloor L/2 \rfloor \) or \( L - k \leq \lfloor L/2 \rfloor \). Furthermore, notice that \( d_H(z_i, a_{kj}) = A \) only if \( k = \lfloor L/2 \rfloor \) or \( k = \lceil L/2 \rceil \).

(v) All of the previous remarks allow us to derive that there exists at most one path of length exactly \( A \) in \( H \) joining vertices in \( \Omega \), since otherwise (see remark (iv)) there would be a cycle of length at most either \( 2A + 5 \) for \( L \) odd, or \( 2A + 4 \) for \( L \) even. But since

\[
2A + 5 = 2\left\lceil \frac{L - 2}{2} \right\rceil + 5 \leq L + 4 \leq \ell + 3 < 2\ell + 1 \leq g,
\]

we obtain a contradiction.

Therefore, we can state for every two vertices \( b, c \in \Omega \):

\[
d_H(b, c) \geq \begin{cases} \frac{A}{\ell - 1} & \text{if } \{b, c\} \cap Z \neq \emptyset, \\ \ell - 1 & \text{otherwise.} \end{cases}
\]

(3)
At this point, we introduce the one-to-one map $\sigma : \Omega \to \Omega$, defined as follows:

\[
\begin{align*}
\text{For every } i, &\quad 1 \leq i \leq d_x - 1: \quad \begin{cases} 
\sigma(x_i) = d_0i, \\
\sigma(a_0i) = x_i.
\end{cases} \\
\text{For every } i, &\quad 1 \leq i \leq d_y - 1: \quad \begin{cases} 
\sigma(y_i) = d_li, \\
\sigma(a_li) = y_i.
\end{cases} \\
\text{For every } i, &\quad 1 \leq i \leq d_z: \quad \begin{cases} 
\sigma(z_i) = a_{li}, \\
\sigma(a_{li}) = z_{i+1} \quad \text{(sum for indices modulo } d_z),
\end{cases}
\end{align*}
\]

Otherwise : $\sigma(p) = p$.

Take a copy $H'$ of $H$. Let $G^*$ be the graph such that $V(G^*) = V(H) \cup V(H')$, and $E(G^*) = E(H) \cup E(H') \cup E^+$, where $E^+ = \{w\sigma(w)' : w \in \Omega\}$. Note that $G^*$ is a $\delta$-regular graph satisfying $|V(G^*)| = 2|V(H)| < |V(G)|$. Reasoning in a similar way as in Lemma 2.3, we can also show that $g(G^*) \geq g(G) = g$, which contradicts Theorem 1.1. Therefore, every $(\delta, g)$-cage with $g \geq 10$ must be 4-connected.

To prove that $g(G^*) \geq g(G) = g$, consider a cycle $\mathcal{C}$ in $G^*$ such that $E(\mathcal{C}) \cap E^+ \neq \emptyset$. Certainly, the cycle $\mathcal{C}$ contains an even number, say $m = 2r$, of edges in the edge-cut $E^+$. It is also clear that $\mathcal{C}$ must contain $r$ paths in $H$ having their two endvertices in $\Omega$, and $r$ more paths through $H'$ with both endvertices in $\Omega'$, all these paths being pairwise disjoint. To approach the mentioned proof, we must again recall Lemma 2.5, which assures that $L = \ell - 1$ if $g$ is even and $\ell - 2 \leq L \leq \ell - 1$ if $g$ is odd. We can henceforth assume that none of those $r$ paths through $H$ (resp. $H'$) has $b, c \in N(v), v \notin V(H) \cup V(H')$, as endvertices; otherwise, $\|\mathcal{C}\| > 2 + d_H(b, c) > 2 + (g - 2) = g$ and we are done. From now on, we denote by $\Omega_2 = Z \cup N_H(a_1)$, and $\Omega_1 = \Omega \setminus \Omega_2$. Let us distinguish three cases:

Case 1: Suppose $m \geq 6$. Notice that $\mathcal{C}$ contains at most two paths of length $A$, one in $H$ and the other one in $H'$, because of remark (v). Then,

\[
\|\mathcal{C}\| \geq 6 + 2A + 4(A + 1) = 10 + 6[(L - 2)/2] \geq 10 + 3(L - 2) \geq 4 + 3(\ell - 2) \geq 3\ell - 2 \geq g,
\]

since $g \geq 10$.

Case 2: Assume $m = 4$. Consider $|E(\mathcal{C}) \cap E^+_2|$, where $E^+_2 = \{w\sigma(w)' : w \in \Omega_2\}$.

- If $|E(\mathcal{C}) \cap E^+_2| \leq 2$, then $\mathcal{C}$ contains at most one path in $H$ (resp. $H'$) with some endvertex in $Z$ (resp. $Z'$). Hence, by (3):

\[
\|\mathcal{C}\| \geq 4 + 2A + (\ell - 1) \geq 2\ell + 2 + 2A > 2\ell + 2 \geq g.
\]

- Suppose $|E(\mathcal{C}) \cap E^+_2| = 3$. In this case, the cycle $\mathcal{C}$ must be as illustrated in Fig. 3 (taking $w \in \Omega_1, \sigma(w)' \in \Omega_1'$); notice that $d_H(z_i, a_{1j}) \geq L - 2$, by remark (iv). Hence

\[
\|\mathcal{C}\| \geq 4 + 2(L - 2) + A + (\ell - 1) = 2L + A + \ell - 1
\]

\[
\geq \begin{cases} 
3(\ell - 1) + A \geq 2\ell + 2 = g & \text{if } g \text{ is even}; \\
3\ell - 5 + A \geq 2\ell + 1 = g & \text{if } g \text{ is odd}.
\end{cases}
\]

- Assume $|E(\mathcal{C}) \cap E^+_2| = 4$. Then, the cycle $\mathcal{C}$ must be as illustrated in Fig. 3 (taking $w = a_{1k}$), and hence:

\[
\|\mathcal{C}\| > 4 + d_H(z_i, a_{1j}) + d_H(z_k, a_{1k}) \geq 4 + (g - 4) \geq g.
\]
since the shortest \((z_i, a_{1j})\)-path in \(H\), the shortest \((z_h, a_{1k})\)-path in \(H\), and the paths of
length two \(z_izhz\) and \(a_{1j}a_{1k}\), form a cycle in \(G\).

Case 3: Lastly, suppose \(m=2\). Let \(\eta\) (resp. \(\eta')\) denote the \((b, c)\)-path (resp. \(((\sigma(b))', (\sigma(c))')\)-path) in \(\mathcal{C} \cap H\) (resp. \(\mathcal{C} \cap H'\)). Then, \(\|\mathcal{C}\| = 2 + \|\eta\| + \|\eta'\|\). Suppose first \(E(\mathcal{C}) \cap E_2^+ = \emptyset\),
where \(E_2^+ = \{w\sigma(w)': w \in \Omega_2\}\).

- If \(b = x_i\):
  - If \(c = y_j\), then \(\|\eta\| \geq d_H(x_i, y_j) \geq \ell - 1\), by remark (i); and \(\|\eta'\| \geq d_H'(a'_{0i}, a'_{Lj}) \geq g - (L + 2)\), by remark (ii); therefore \(\|\mathcal{C}\| \geq 2 + (\ell - 1) + g - (L + 2) = g + (\ell - 1) - L \geq g\).
  - If \(c = a_{0j}\), then taking into account the structure illustrated in Fig. 1, \(\|\mathcal{C}\| \geq d_H(x_i, a_{0j}) \geq g - 3\). Therefore, since \(\|\eta'\| \neq 0\), we have \(\|\mathcal{C}\| \geq 2 + (g - 3) + 1 = g\).
  - If \(c = a_{kj}\), \(k \notin \{0, 1\}\), then

\[
\begin{align*}
\|\eta\| & \geq d_H(x_i, a_{kj}) \\
& \geq \begin{cases} 
  g - (L + 3) & \text{(remark (iii))}, \\
  \ell & \text{if } k = L, \text{ and } j \neq i \text{ (Point 5 of labelling)},
\end{cases}
\end{align*}
\]

\[
\|\eta'\| \geq \begin{cases} 
  \ell & \text{if either } k \neq L, \text{ or } k = L \text{ and } j \neq d_j \text{ (remark (ii))} \\
  d_H'(a'_{0i}, a'_{kj}) & \text{otherwise.}
\end{cases}
\]

Therefore, in any case \(\|\mathcal{C}\| \geq 2 + g - (L + 3) + \ell = g + (\ell - 1) - L \geq g\).

- If \(b = y_i\):
  - If \(c = a_{Lj}\), then \(\|\eta\| \geq d_H(y_i, a_{Lj}) \geq g - 3\) (see Fig. 1). Therefore, since \(\|\eta'\| \neq 0\), we have \(\|\mathcal{C}\| \geq 2 + (g - 3) + 1 = g\).
To end the proof of Case 3, assume next that
\[ b(134) \]
If
\[ b \in (X^* \cup Y^*) \]
Consider the case
\[ c(134) \]
Finally, if
\[ c(134) \]

Therefore, in any case \( \|E\| \geq 2 + g - (L + 3) + \ell = g + (\ell - 1) - L \geq g \).

To end the proof of Case 3, assume next that \( b(\sigma(b))' \in E(\mathcal{G}) \cap E_2^+ \), where \( b = z_j \), \( (\sigma(b))' = a'_i \). As for the edge \( c(\sigma(c))' \), we must distinguish the following cases:

- If \( c = x_j \), then taking into account the structure illustrated in Fig. 1 we get:
  \[ \|E\| \geq 2 + d_H(z_i, x_j) + d_H(a_{1i}, a_{0j}) \geq 2 + L + (g - 3) = g + (L - 1) > g. \]

- If \( c = a_{0j} \) and \( j \leq d_x - 1 \), then:
  \[ \|E\| \geq 2 + d_H(z_i, a_{0j}) + d_H(a_{1i}, x_j) \geq 2 + (L - 2) > g. \]

When \( j \geq d_x \), then:
\[ \|E\| \geq 2 + d_H(z_i, a_{0j}) + d_H(a_{1i}, a_{0j}) \geq 2 + (L - 1) + (g - 3) = g + (L - 2) > g. \]

- If \( c = a_{1j} \) with \( i \in \{j, j + 1\} \), then:
  \[ \|E\| \geq 2 + d_H(z_i, a_{1j}) + d_H(a_{1i}, z_{j+1}) \geq 2 + (g - 3) = g. \]

So, assume that \( i \notin \{j, j + 1\} \). If \( j \leq d_x \), then by the labelling of vertices in \( \Omega \) point 3, \( \|E\| \geq 2 + d_H(z_i, a_{1j}) + d_H(a_{1i}, z_{j+1}) \geq 2 + \ell + \ell \geq g \). Finally, if \( j \geq d_x + 1 \), then
\[ \|E\| \geq 2 + d_H(z_i, a_{1j}) + d_H(a_{1i}, a_{1j}) \geq 2 + d_H(z_i, a_{1j}) + (g - 3) > g. \]

- Consider the case \( c = a_{kj} \), with \( 2 \leq k \leq L - 1 \). By applying remark (iv), we get:
  \[ \|E\| \geq 2 + d_H(z_i, a_{kj}) + d_H(a_{1i}, a_{kj}) \geq 2 + (k - 1) + (g - (k - 1) - 2) = g. \]

- If \( c = a_{Lj} \) and \( 1 \leq j \leq d_y - 1 \), then by the labelling of vertices in \( \Omega \) point 4, \( a_{Lj} \neq a_{L \delta - 2} \), and hence:
  \[ \|E\| \geq 2 + d_H(z_i, a_{Lj}) + d_H(a_{1i}, y_j) \geq 2 + (\ell - 1) + g - (L + 2) = g + (\ell - 1 - L) \geq g. \]
When \( j \geq d_y \), then \( \sigma(a_Lj) = a_Lj \), and hence:

\[
\|C\| \geq 2 + d_H(z_i, a_Lj) + d_H(a_{1i}, a_Lj) \\
\geq 2 + (L - 1) + g - (L + 1) \geq g.
\]

- If \( c = y_j \), then:

\[
\|C\| \geq 2 + d_H(z_i, y_j) + d_H(a_{1i}, a_Lj) \geq 2 + L + g - (L + 1) > g.
\]

Finally, assume that \( E(\mathcal{C}) \cap E^+ = \{b(\sigma(b))', c(\sigma(c))'\} \), with \( b(\sigma(b))' \in E_2^+, b = a_{1i}, (\sigma(b))' = a'_{1i} \). As the case \( c = z_j \) has been studied above, and since \( d_H(a_{1i}, c) \geq \ell \) for every \( c \notin Z \) by remark (ii), it follows:

\[
\|C\| \geq 2 + d_H(a_{1i}, c) + d_H(a_{1i}, \sigma(c)) \geq 2 + 2\ell \geq g.
\]

From Theorem 2.1, we know that every \((4, g)\)-cage with \( g \geq 10 \) is maximally connected, and the same fact holds for \( g \in \{3, 4\} \) because the \((4, 3)\)-cage is the complete graph \( K_5 \), and the \((4, 4)\)-cage is the complete bipartite graph \( K_{4,4} \). We can extend this result for \( g \in \{5, 6, 7, 8, 9\} \), with the help of the following theorem.

**Theorem 2.2 (Balbuena et al.\[1\]).** Let \( G \) be a connected graph with minimum degree \( \delta \) and girth \( g \), where either \( g = 2\ell + 1 \) or \( g = 2\ell + 2 \). Then, \( G \) is maximally connected if

\[
|V(G)| \leq p(\delta, \ell) = 2(1 + \delta + \delta(\delta - 1) + \cdots + \delta(\delta - 1)^{\ell-1}) - \delta.
\]

**Corollary 2.1.** Every \((4, g)\)-cage is maximally connected.

**Proof.** Let us denote by \( b(4, g) \) the smallest order (known thus far) of a 4-regular graph with girth \( g \), and let \( p(4, \ell) = 2(1 + 4 + 4 \cdot 3 + \cdots + 4 \cdot 3^{\ell-1}) - 4 \) the quantity introduced in Theorem 2.2 for \( \delta = 4 \). For \( g \in \{5, 6, 7, 8, 9\} \), we obtain the results shown in the following table:

<table>
<thead>
<tr>
<th>( g )</th>
<th>( \ell )</th>
<th>( b(4, g) )</th>
<th>( p(4, \ell) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>2</td>
<td>19</td>
<td>30</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>26</td>
<td>30</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>67</td>
<td>102</td>
</tr>
<tr>
<td>8</td>
<td>3</td>
<td>80</td>
<td>102</td>
</tr>
<tr>
<td>9</td>
<td>4</td>
<td>275</td>
<td>318</td>
</tr>
</tbody>
</table>

where the values of \( b(4, g) \) can be found in \[10\]. Therefore, Theorem 2.2 allows us to state that \((4, g)\)-cages with \( g \in \{5, 6, 7, 8, 9\} \) are maximally connected, since \( f(4, g) \leq b(4, g) \leq p(4, \ell) \). Notice that this result is widely known for the \((4, 6)\)-cage and the \((4, 8)\)-cage (respectively, the generalized polygons \( P_3 \) and \( Q_3 \), see \[2\]).

**3. Open questions**

The techniques used in this work have not allowed us to prove that every \((\delta, g)\)-cage is 4-connected for \( \delta \geq 5 \) and \( 5 \leq g \leq 9 \). For instance, for \( g = 5 \), Lemma 2.2 is far from being
true. So, a different approach seems to be necessary. Other suitable steps towards proving that every \((\delta, g)\)-cage is \(\delta\)-connected, might be by proving the following statements:

I. Every \((4, g)\)-cage is quasi 5-connected.
II. Every \((\delta, g)\)-cage is \(\delta\)-connected for some particular range of its girth.

**Conjecture.** Every \((\delta, g)\)-cage is quasi \((\delta + 1)\)-connected.

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**References**