

# Time-Varying Risk Premia in Emerging Markets: Explanation by a Multi-Factor Affine Term Structure Model

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## Abstract

From the empirical viewpoint, the Expectation Hypothesis Theory (EHT) of the term structure of interest rates has been extensively tested and rejected for US. term structure data. Dai and Singleton (2002) show that under the settings of Affine term structure models it is possible that one matches both the historical term structure dynamics and capture an important stylized fact that have contradicted the EHT: Time-varying risk premia.

In Emerging Markets, economic conditions tend to be much less stable than in developed markets. For this reason, if risk premia is dynamic in such markets, intuition would suggest that it is more volatile than in developed markets, implying a stronger statistical rejection of the EHT. In this paper, we verify the robustness of Dai and Singleton's results under these more extreme market conditions. We estimate an arbitrage free Affine Gaussian model for the term structure of Swaps in the Brazilian market. We propose an extensive empirical analysis which consists on: defining the optimal number of factors to be used in the model, estimating the model, giving interpretation to the state variables in terms of risk factors, and studying the model implied risk premia. In the end, we propose an application for risk management of interest rates futures portfolios.

Keywords: Interest Rates, Affine Models, Expectation Hypothesis Theory, Market Prices of Risk, Risk Premia.

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# 1 Introduction

The Expectation Hypothesis Theory states in its strong version that forward rates are unbiased predictors of expected future short term interest rates. In its weaker form, it allows forward rates to be biased predictors of expected short term rates, but with the bias being a deterministic function of the maturities, not depending on time. Many empirical studies (among them Fama (1984), Fama and Bliss (1987), Campbell and Shiller (1991) and Campbell (1995)) present empirical evidences against both forms of the EHT. If the EHT was to hold, coefficients of regressions of forward rates (or returns) on the slope of the term structure should be unitary. On the reality, in these empirical studies these coefficients usually appear negative and increasing with maturity.

As stated in Backus et al. (2001) evidence against the EHT translates directly into evidence of time-varying risk premia for zero coupon bonds. Time-varying risk premia is captured by Dai and Singleton's (2002) proposal which consists of using multi-factor Affine term structure models to capture the failure of the EHT on the US market. They estimate and compare different models for the term structure of the US swaps market using maximum likelihood estimation. They conclude that a simple 3 factor Gaussian model is successful in fitting the historical dynamics of the term structure in particular capturing variation of the risk premia along time. A key fact contributing for their success is their flexible parameterization of the market prices of interest rates risk which is compatible with time-varying (possible changing signs) risk premia.<sup>1</sup>

Economic conditions in Emerging Markets are much more unpredictable than in developed countries. For instance, uncertainty related to the payments of the public debt is a source of credit risk. This risk contributes to an increase in the volatility of interest rates (see Favero and Giavazzi (2002)). As a direct consequence, investors seek for high risk premia, usually increasing with maturity due to the potencial variability of interest rates along time. We conjecture that these conditions contribute to a stronger rejection of the EHT, stronger in the sense that risk premia have high variations along time.

In this paper we test if Dai and Singleton's (2002) setting works well in a particular Emerging Market. We perform this test using data on important Brazilian fixed income instruments, namely the Brazilian swaps. After verifying the empirical rejection of the EHT by the data, we present an econometric model for the dynamics of the Brazilian swap term structure that encompasses the following interesting characteristics: it is an arbitrage free model, it fits well the available data capturing the failure of the EHT, it provides the possibility of dynamic hedging portfolios composed of swaps and/or DIs, and it can be used for risk management purposes. The model comes from the family of Affine models first described in Duffie and Kan (1996), recently generalized in Duffie et al. (2003), and classified and empirically tested in Dai and Singleton (2000) and Dai and Singleton (2002). Briefly explaining, the model assumes the yield of the swap can be expressed as a linear combination of variables which appear in the state space vector. The dynamics for the evolution of these variables satisfy certain restrictions which guarantee that arbitrages are ruled out. More details are presented along the paper.

The paper is organized as follows. In Section 2, we describe the general structure of a  $A_m(N)$  model as in Dai and Singleton (2000). In Section 3, we estimate the model using historical data on swap yields for different maturities. There, we briefly explain the Brazilian swap market, apply principal component analysis to identify the ideal number of factors for the model, estimate the model, and provide interpretations for the state space vector and for the model implied risk pre-

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<sup>1</sup>Backus et al. (2001) propose a similar analysis, but much more restrictive using forward rate regressions. They only test one specific Affine multifactor model.

mia. In Section 4, we present an application on risk management of a portfolio of DIs. Section 5, concludes. The Appendix presents all the necessary steps for the implementation of the maximum likelihood estimator adopted.

## 2 Affine Term Structure Models and the $A_m(N)$ Representation

### 2.1 The Model

Consider a complete probability space  $(\Omega, \mathcal{F}, P)$  and a  $N$ -dimensional state space process  $Y = \{Y_t\}_{0 < t < \infty}$  embedded on this space, taking values in an open subset of  $\mathcal{R}^N$ , say  $D$ , satisfying a stochastic differential equation of the following form:

$$dY_t = \mu(Y_t)dt + \sigma(Y_t)dW_t \quad (1)$$

where  $W_t$  is  $N$ -dimensional Brownian Motion under  $P$ ,  $\mu : D \rightarrow \mathcal{R}^N$  and  $\sigma : D \rightarrow \mathcal{R}^{N \times N}$ . Assumption of absence of arbitrage guarantees the existence of an Equivalent Martingale Measure  $\mathcal{Q}$  under which the prices of bonds discounted by an appropriate deflator are  $\mathcal{Q}$ -martingales.<sup>2</sup> Under the risk neutral measure  $\mathcal{Q}$  the process  $Y$  evolves according to:

$$dY_t = \mu^{\mathcal{Q}}(Y_t)dt + \sigma(Y_t)dW_t^* \quad (2)$$

where  $W_t^*$  is  $N$ -dimensional Brownian Motion under  $\mathcal{Q}$ ,  $\mu^{\mathcal{Q}}(Y_t) = \mu(Y_t) - \sigma(Y_t)\Lambda_t(Y_t)$ , and  $\Lambda_t(\cdot)$  is a  $N$ -dimensional vector representing the market price of risk due to uncertainty in interest rates. Denote by  $\pi_t$  the state price deflator at time  $t$  (See Duffie (2001)). The market price of risk is closely related to the instantaneous volatility of the state price deflator:

$$\frac{d\pi_t}{\pi_t} = -r_t dt - \Lambda_t' dW_t \quad (3)$$

where  $r_t$  represents the short term interest rate. Element  $i$  of vector  $\Lambda_t$  represents the price of risk associated with the  $i$ th coordinate of the Brownian motion  $W_t$ .

According to Duffie and Kan (1996), an affine model of the term structure is basically characterized by two assumptions:

1) The short term interest rate  $r_t$  is an affine function of the state space vector:

$$r_t = \rho_0 + \rho_Y^t Y_t \quad (4)$$

where  $\rho_Y$  is a  $N \times 1$  vector.

2) The drift of the state vector under measure  $\mathcal{Q}$ , and the instantaneous volatility  $\sigma(\cdot)$  should satisfy:

$$\begin{aligned} \mu^{\mathcal{Q}}(Y_t) &= \kappa^{\mathcal{Q}}(\theta^{\mathcal{Q}} - Y_t) \\ \sigma(Y_t) &= \Sigma \sqrt{S_t} \\ S_t^{ij} &= \alpha_i + \beta_i^t Y_t, S_t^{ij} = 0, i \neq j, 1 \leq i, j \leq n \\ \alpha_i &\in \mathcal{R}, \beta_i \in \mathcal{R}^N \end{aligned} \quad (5)$$

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<sup>2</sup>This deflator can be any non-negative process adapted to the Brownian filtration. In particular we use the money market account factor defined by the value of one dollar invested on an account continuously accumulating interest rates on the short term rate.

Condition 2) guarantees that the discounted prices of bonds are  $Q$ -martingales. Duffie and Kan (1996) showed that in this case, the time  $t$  price of a zero coupon bond with time of maturity  $T$ ,  $P(t, T)$  is an exponential affine function of the state space vector at time  $t$ :

$$P(t, T) = e^{A(\tau) + B(\tau)'Y_t}, \quad (6)$$

where  $\tau = T - t$  and  $A(\cdot)$  and  $B(\cdot)$  are solutions of two specific ODEs (for details see Duffie and Kan (1996) or Duffie (2001)). At this point, we are able to see that an affine structure for the diffusion of  $Y$  under  $Q$  guarantees a low computational cost when calculating bond prices. On the other hand, the specification of the drift under measure  $P$  is very important for the calculation of the likelihood function of the model. Note that conditions 1) and 2) say nothing about the drift of the state vector under measure  $P$ . It could be a non-affine function. However, in almost all recent dynamic term structure literature on affine models (see for instance Duffie (2002) and Dai and Singleton (2002)) the econometrician chooses an affine representation for the drift  $\mu(Y_t)$  because it allows for closed-form calculation of various properties of conditional densities of discretely sampled yields (see Singleton (2001))<sup>3</sup>. In particular, in this work we also assume that the drift  $\mu(Y_t)$  is affine. Consequently, a fairly general specification for the market price of risk that is consistent with affine diffusions under both physical and risk neutral measures is<sup>4</sup>:

$$\Lambda_t = \sqrt{S_t} \lambda_0 + \sqrt{S_t^-} \lambda_Y Y_t, \quad (7)$$

where  $\lambda_0$  is a  $N \times 1$  vector and  $\lambda_Y$  is a  $N \times N$  matrix, and  $S_t^-$  is defined by:

$$S_t^{ii-} = \begin{cases} \frac{1}{S_t^{ii}}, & \text{if } \inf(\alpha_i + \beta_i^t Y_t) > 0. \\ 0, & \text{otherwise.} \end{cases} \quad (8)$$

Dai and Singleton (2000) give admissibility conditions on the parameters of the model to guarantee that  $S_t$  stays always positive. According to their notation, the  $A_m(N)$  model is defined as the set of all admissible Affine  $N$ -factor Term Structure Models that have  $m$  factors responsible for the dynamics of the conditional variances of all factors. In their paper, they give a canonical representation for the  $A_m(N)$  model, and define invariant transformations of this canonical model as the ones that preserve admissibility, model identification, and leave the short term rate unchanged. In what follows, we reproduce the parametric form of the  $A_m(N)$  model adopted in this paper, an invariant transformation of the canonical  $A_m(N)$  model proposed in Dai and Singleton (2000)<sup>5</sup>.

**Definition 1.**  $A_m(N)$  Model. Suppose that the short term rate is given by Equation (4), and that the state vector dynamics is represented by the following SDE under  $P$ :

$$dY_t = \kappa(\Theta - Y_t)dt + \Sigma \sqrt{S_t} dW_t \quad (9)$$

where  $W_t$  is an  $N$ -dimensional vector of independent standard Brownian motions under  $P$ , and  $S_t$  is defined in Equation (5). For each  $m$ , partition  $Y_t' = (Y^{B'}, Y^{D'})$ , where  $Y^B$  is  $m \times 1$  and  $Y^D$

<sup>3</sup>Exceptions appear in Duarte (2004) and Duffie and Stanton (2004).

<sup>4</sup>These are basically denominated Essentially Affine models. For a detailed introduction to these models see Duffie (2002). For an even more general treatment of market prices of risk which imply affine dynamics under both measures see Cheredito et al. (2003).

<sup>5</sup>This parameterization, proposed in Dai and Singleton (2002), allows for an easier interpretation of the state variables.

is  $(N - m) \times 1$ . The  $A_m(N)$  canonical representation is the special case of Equations (4) and (9) where:

$$\kappa = \begin{bmatrix} \kappa_{m \times m}^{BB} & \mathbf{0}_{m \times (N-m)} \\ \kappa_{(N-m) \times m}^{DB} & \kappa_{(N-m) \times (N-m)}^{DD} \end{bmatrix} \quad (10)$$

for  $m > 0$ , and  $\kappa$  is upper triangular for  $m = 0$ ,

$$\Theta = \begin{pmatrix} \Theta_{m \times 1}^B \\ \mathbf{0}_{(N-m) \times 1} \end{pmatrix} \quad (11)$$

$$\alpha = \begin{pmatrix} \mathbf{0}_{m \times 1} \\ \mathbf{1}_{(N-m) \times 1} \end{pmatrix} \quad (12)$$

$$\beta = \begin{bmatrix} I_{m \times m} & B_{m \times (N-m)}^{BD} \\ \mathbf{0}_{(N-m) \times m} & \mathbf{0}_{(N-m) \times (N-m)} \end{bmatrix} \quad (13)$$

$\Sigma$  is a diagonal matrix, and  $\rho_y$  is a  $N \times 1$  vector of ones. For models with  $m > 0$  some additional parametric restrictions are described in Dai and Singleton (2000). As we are going to focus on a Gaussian model ( $m = 0$ ) we don't present these restrictions here.

## 2.2 The Market Prices of Risk and Time-Varying Risk Premia

Using the fact that discounted prices of bonds are  $\mathcal{Q}$ -martingales, we are able to derive the dynamics of bond prices under the martingale measure  $\mathcal{Q}$ . Applying Ito's lemma to Equation (6) we can write:

$$\frac{dP(t, T)}{P(t, T)} = r_t dt + B(\tau)' \Sigma \sqrt{S_t} dW_t^* \quad (14)$$

where  $\tau = T - t$ .

Naturally, under the physical measure, investors will ask for an instantaneous expected excess return to hold such bond:

$$\frac{dP(t, T)}{P(t, T)} = (r_t + e_{t, \tau}^i) dt + B(\tau)' \Sigma \sqrt{S_t} dW_t \quad (15)$$

A simple application of Girsanov's Theorem to change the measures ( $dW_t^* = dW_t + \Lambda_t dt$ ) reveals that the instantaneous expected excess return  $e_{t, \tau}^i$  is given by:

$$e_{t, \tau}^i = B(\tau)' \Sigma \sqrt{S_t} \Lambda_t = B(\tau)' \Sigma (S_t \lambda_0 + I_t^- \lambda_Y Y_t) \quad (16)$$

where  $I_t^-$  is diagonal defined by:

$$I_t^{ii-} = \begin{cases} 1, & \text{if } \inf(\alpha_i + \beta_i^t Y_t) > 0. \\ 0, & \text{otherwise.} \end{cases} \quad (17)$$

In a gaussian model,  $S_t = I_t$ , and  $I_t^- = I_t$  so that Equations (15) and (16) simplify to:

$$\frac{dP(t, T)}{P(t, T)} = (r_t + e_{t, \tau}^i) dt + B(\tau)' \Sigma dW_t \quad (18)$$

$$e_{t,\tau}^i = B(\tau)' \Sigma (\lambda_0 + \lambda_Y Y_t) \quad (19)$$

Note in Equation (16) or (19) that the term depending on  $\lambda_0$  does not allow change of sign while the sign on the term depending on  $\lambda_Y$  entirely depends on the state vector  $Y_t$  at each instant of time. In addition, we see from Equation (19) that in Affine Gaussian models the only source of time variation of instantaneous expected excess returns is the term depending on matrix  $\lambda_Y$ . As we will see in subsection 3.3, these flexibilities in terms of changes of sign and time dependence will be crucial to match the highly volatile time-varying risk premia in the Brazilian swap market.

### 3 Estimation of an Arbitrage-Free Model for the Term Structure of Brazilian Swaps

#### 3.1 The Brazilian Future on DI and the Swap Market

One of the most important instruments in the Brazilian fixed income market is the future on DI. In order to understand this contract, we first should define what DI yields are. They are yields obtained from interbank borrowing/lending short term market, daily measured by CETIP (Central de Custodia e de Liquidacao Financeira de Titulos). The DI yield at day  $t$  is defined as the average yield obtained in this market in day  $t$ , measured as composed yield with daycount basis  $\frac{\text{workdays}}{252}$ . The future on DI with maturity  $T$  is a derivative that at time  $t$  costs zero by definition, but whose reference price at time  $t$  is obtained by the risk neutral expected value of 100000 Reais discounted by the short term interest rate DI accumulated until maturity  $T$ :  $P_t = 100000 E_t^* \left( e^{-\int_t^T DI_u du} \right)$ , where  $E_t^*()$  represents risk neutral expectation conditional on time  $t$ , and  $DI_u$  denotes the short term interest rate DI at day  $u$ . It is very similar to a zero coupon bond, except that being a future, it presents a continuum of cash flows paid at each day, obtained by the difference between the reference price on the day and the reference price on the day before corrected by the DI short term rate of the day before  $A_t = P_t - P_{t-1} e^{\int_{t-1}^t DI_u du}$ . A swap floater DI-fixed is quoted like a zero coupon bond, in terms of a yield for a fixed maturity, and presents one unique cash flow at its maturity time when the pre-agreed yield (correctly pro-rated) is compared with the short term DI yields cumulated along the life of the swap. At time  $t$ , the yield to maturity  $sw_\tau$  of a swap with maturity at  $T$  is obtained by no arbitrage using  $P_t = e^{-sw_\tau \tau}$ , where  $\tau = T - t$  represents the time to maturity of the swap. Bolsa de Mercadorias e Futuros (BM&F) is the entity that offers the futures on DI and swaps contracts, also acting as intermediate for the over-the-counter swaps contracts. It maintains a historical database with interest rates on swaps with different fixed maturities synthetically constructed from the prices of futures on DI with different maturities. This data is useful to mark to market portfolios on futures on DI and/or on swap floater DI-fixed, as well as useful for risk management of the same instruments. In this work, we assume that the swap rates data represent the term structure of interest rates available.

#### 3.2 Exploring the Data

The data used in this empirical section consists of historical series (with daily frequency) of the yields of the swaps with maturities 30, 60, 90, 120, 180, 270, 360 and 720 days. The series

begin in January 2, 2001 and end in January, 29 2003, with a total of 518 observations for each maturity. Figure 1 presents the historical evolution of this term structure. Figure 2 presents the evolution of the swap yields for three different maturities, 30, 270 and 720 days. Tables 1 and 2 present respectively the correlation structure and the volatilities of the first difference of swaps yields. From Figure 2 and the tables, we note that yields are more volatile for longer maturities, and that volatility changes along time. Figure 2 shows that there have been three periods of high volatility of Brazilian interest rates during the two years sampled, respectively in July 2001, June 2002 and October 2002.

Before proposing a particular model for the evolution of the term structure, we pursue a basic analysis of its correlation structure. We apply principal component analysis to the time series of the first difference of the swaps yields, in order to identify the number of factors necessary to describe the dynamics of interest rates. Table 3 shows the variation of the curve explained by each factor. Note that the first three factors explain 98.7% of the term structure variation along the two years of data. Figure 3 presents the first three principal components loadings. These principal components are very similar to the ones obtained for the US. term structure by Litterman and Scheinkman (1991). The first principal component explains 87.67% of the variations of the curve and can be interpreted as a parallel shift. The second component explains 9.29% of the variation and can be related to changes in the slope. The third principal component explains 1.74% of the variation and is related to changes in the curvature of the term structure.

Guided by the results of the principal component analysis we decided for the estimation of a model with three factors. According to the classification proposed in Section 2, we will choose the model from the  $A_m(3)$  class of affine models. We follow Duffie and Singleton (1997) and Duffie, Pedersen and Singleton (2003) assuming that a subset of 3 maturities among the eight have the swap yields priced without error. Proceeding this way we are able to invert the swap yields of these  $N$  maturities to obtain the state space vector (see Appendix). Based on historical data on the volume of contracts on DI with different maturities,<sup>6</sup> the yields of the swaps with maturities 30, 90 and 360 days are assumed to be measured without error, while the yields of the swaps with maturities 60, 120, 180, 270 and 720 days are measured with iid Gaussian errors along time. The likelihood obtained under these assumptions is presented in the Appendix.

We adopted the  $A_0(3)$  gaussian model because it gives the maximum flexibility to the market prices of risk parametrized in Equation (7).<sup>7</sup> On the other hand, the conditional instantaneous volatilities of the state space variables are simply constants. Table 4 presents values for the parameters of the model and their standard errors. Table 5 shows the mean and standard deviation of the errors, measured as the difference between the model implied yields and the observed yields, for the swaps measured with errors. Note that except for the 2 year swap (longer maturity), the means are all smaller than 10 basis points, and the standard deviations are all smaller than 30 basis points. These results look quite acceptable when compared to results in Duffie et. al (2003) obtained for the US. term structure of swaps, a much less volatile term structure.<sup>8</sup>

Figure 4 shows the state variables extracted during the estimation process, as well as their interpretation relating them to the known variables describing the dynamics of the different swap yields. The interpretation was obtained by the use of multivariate linear regressions ran with state variable  $i$  being the dependent variable, and the swaps yields (sw) being the independent variables:

$$Y_t^i = b_1^i sw_t^{30} + b_2^i sw_t^{60} + b_3^i sw_t^{90} + b_4^i sw_t^{120} + b_5^i sw_t^{180} + b_6^i sw_t^{270} + b_7^i sw_t^{360} + b_8^i sw_t^{720} + b_9^i + \epsilon_t^i \quad (20)$$

<sup>6</sup>This data was collected in the BM&F site and is available on an EXCEL worksheet upon request.

<sup>7</sup>See Dai and Singleton (2002) for a detailed discussion on the different  $A_m(3)$  models,  $m = 0, 1, 2, 3$ .

<sup>8</sup>In their work, they obtained means less than 5 basis points and standard errors less than 10 basis points.

for  $i = 1, 2, 3$ . Table 6 shows the values of the  $b$  coefficients for the three state variables. Due to the simple relation between the swap yields and the state variables in this model, by construction the three state variables are explained by linear combinations of the variables assumed to be measured without error: the 30, 90 and 360 swaps yields. State variable 1 can be approximately written as a translation of 53 basis points added to the weighted sum of the spreads 90-30 days, 360-90 days and the 90 days swap yield, with weights -0.5, 0.12 and -0.1 respectively:

$$Y_t^1 = 0.0053 - 0.1sw_t^{90} - 0.5spr_t^{90-30} + 0.12spr_t^{360-90} \quad (21)$$

where  $spr_t^{90-30} = sw_t^{90} - sw_t^{30}$  and  $spr_t^{360-90} = sw_t^{360} - sw_t^{90}$ . State variable 2 can be approximately written as a negative translation of 53 basis points added to a linear combination of the spread 360-90 days and the 30 days swap yield, with weights 0.24 and 0.1:

$$Y_t^2 = -0.0053 + 0.1sw_t^{30} + 0.24spr_t^{360-90} \quad (22)$$

Finally, state variable 3 can be written as a negative translation of 541 basis points, added to the 30 days swap yield, and to the weighted sum of the spreads 90-30 days and 360-90 days, with weights -0.29, and -0.17 respectively:

$$Y_t^3 = -0.0541 + sw_t^{30} - 0.17spr_t^{90-30} - 0.29spr_t^{360-90} \quad (23)$$

According to the estimated model, the SDEs followed by the state variables are given by:

$$\begin{aligned} dY_t^1 &= (-4.243Y_t^1 - 0.0461Y_t^2)dt + 0.0270dW_t^1 \\ dY_t^2 &= (-0.2854Y_t^2 + 0.1167Y_t^3)dt + 0.0163dW_t^2 \\ dY_t^3 &= -3.761Y_t^3dt + 0.0914dW_t^3 \end{aligned} \quad (24)$$

For interpretation purposes, analyzing the order of magnitude of yields, spreads and the coefficients obtained in the linear approximations described by Equations (21), (22) and (23), we can go further in our approximation to the state variables and consider:

- state variable 1** a linear transformation of the pair **90 days swap yield, spread 90-30 days,**
- state variable 2** a linear transformation of the pair **30 days swap yield, spread 360-90 days,**
- state variable 3** a linear transformation of the **30 days swap yield.**

All the state variables mean revert to the level zero, with variable  $Y^1$  presenting the highest mean reversion speed. Variable  $Y^3$  also presents a relative high mean reversion speed. Moreover its drift is not influenced by the other state variables. On the other hand, variable  $Y^2$  has its drift clearly affected by  $Y^3$ . With our approximation in mind we see that the spread 90-30 days, and the yields of the 30 and 90 days swap present fast mean reversion when compared to the spread 360-90 days. This can be particularly visualized in Figure 4, during the market crisis pointed out before. The spread 360-90 days always takes more time to react to the shock caused by the crisis. From Equation (24) we also note that the instantaneous volatility of the state variable  $Y_3$  (yield level) is much higher than the instantaneous volatilities of the other two state variables (of the order of yield spreads) which are quite comparable (0.0270 and 0.0163).

### 3.3 Excess Returns, Expectation Hypothesis Regressions and the Failure of EHT

An important stylized fact about the term structure of interest rates (see Duffee (2002) for a study on the US. curve) is that the ratio mean expected excess bond returns over standard



deviation of expected excess bond returns is small. Note that this pattern only takes place if expected excess bond returns oscillate considerably and change signs along time. Expected excess bond returns are closely related to risk premia charged by investors due to uncertainty of interest rates along time. As shown by Dai and Singleton (2002), considering models with time-varying risk premia they are able to reproduce the behavior of excess returns and consequently to capture the failure of the Expectation Hypothesis Theory, represented by negative coefficients (increasing with maturity) on regressions of monthly changes in yields on the adjusted slope of the yield curve.<sup>9</sup> In this subsection, we present some empirical evidence in favor of the existence of a time-varying risk premia driving the Brazilian swap market, and show that the Affine model  $A_0(3)$  works in the right direction to capture the failure of the EHT in this market.

In this subsection, we assume for any  $t$ , that the time interval  $\Delta t$  between  $[t, t + 1]$  is of one month. Define  $P_t^n = P(t, t + n)$  as the time  $t$  price of a zero coupon bond with time to maturity of  $n$  months. Let also  $R_t^n = -\frac{\ln(P_t^n)}{n}$  denote its yield to maturity. By expected excess bond returns we mean:

$$e_{t,n} = E_t \left[ \ln \left( \frac{P_{t+1}^{n-1}}{P_t^n} \right) - R_t^1 \right], \quad (25)$$

the expected one month excess return. If we manipulate Equation (25) we obtain:

$$E_t [R_{t+1}^{n-1} - R_t^n] + \frac{e_{t,n}}{n-1} = \frac{(R_t^n - R_t^1)}{n-1} \quad (26)$$

Now, consider the following linear regressions for each maturity  $n > 1$ :

$$R_{t+1}^{n-1} - R_t^n = \alpha_n + \phi_n \frac{(R_t^n - R_t^1)}{n-1} + \epsilon_t^n, t = 1, \dots, T. \quad (27)$$

where  $\epsilon_t^n, t = 1, \dots, T$ s are independent Gaussian error terms. These regressions were suggested to test the EHT (see Fama and Bliss (1987) and Campbell and Shiller (1991)). EHT affirms that the risk premium for bonds with different maturities should be constant along time and just depend on the maturities of these bonds ( $e_{t,n} = e_n$  for all  $t$ ). EHT leads us to the following conclusions: First, Equation (27) turn to be the regression version implied by Equation (26); second, for EHT to hold, the coefficients  $\phi_n$  should be one for all  $n$ . So obtaining coefficients  $\phi_n$  statistically different from 1 in Equation (27) imply the empirical failure of the EHT, or in other words, time-varying risk premia. The statement is also valid in the reverse order: time-varying risk premia imply failure of the EHT.

We measure excess bond returns by assuming that swaps yields are yields of zero coupon bonds, in accordance to the description of the swap market in subsection 3.1. Table 7 shows that excess returns have low mean and high standard deviations, with ratio  $\frac{\text{mean}}{\text{std}}$  decreasing with maturity<sup>10</sup>. Figure 5 presents the evolution along time of excess returns for swaps with different maturities, illustrating that excess returns are volatile and change signs along time. Table 8 presents the slope coefficients  $\phi_n$  of the linear regressions presented in Equation (27). Note that these coefficients are negative and increasing with maturity, presenting a similar pattern to the one obtained in Fama (1984), indicating an empirical rejection of the EHT by the data.

In order to test the capacity of the model  $A_0(3)$  to capture the failure of the EHT in this

<sup>9</sup>The EH implies that these coefficients should be one for all maturities.

<sup>10</sup>This is not formal evidence confirming the stylized fact about expected excess returns described in the first paragraph of this section because mean and standard deviation presented here are for excess returns and not expected excess returns.

market, we follow Dai and Singleton (2002) comparing the model implied  $\phi_n$ 's with their empirical counterparts. The model implied  $\phi_n$ 's are calculated using the parameter vector obtained in the estimation procedure to analitically calculate:

$$\phi_n = \frac{\text{cov} \left( R_{t+1}^{n-1} - R_t^n, \frac{(R_t^n - R_t^1)}{n-1} \right)}{\text{var} \left( \frac{(R_t^n - R_t^1)}{n-1} \right)} \quad (28)$$

In what follows, we present specific steps for the implementation of the calculation of the model implied  $\phi_n$ s.<sup>11</sup>

From the Appendix, we have the relation between  $R_t^n$  and  $Y_t$ :

$$R_t^n = -\frac{A(n, \Gamma)}{n} - \frac{B(n, \Gamma)'}{n} Y_t \quad (29)$$

Using Equation (29) we can write:

$$R_{t+1}^{n-1} - R_t^n = -\frac{A(n-1, \Gamma)}{n-1} + \frac{A(n, \Gamma)}{n} - \frac{B(n-1, \Gamma)'}{n-1} Y_{t+1} + \frac{B(n, \Gamma)'}{n} Y_t \quad (30)$$

$$R_t^n - R_t^1 = -\frac{A(n, \Gamma)}{n} + A(1, \Gamma) - \left( \frac{B(n, \Gamma)'}{n} - B(1, \Gamma)' \right) Y_t \quad (31)$$

Using Equations (30) and (31), we are able to rewrite Equation (28) in terms of the state vector:

$$\phi_n = \frac{(n-1) \left[ -\frac{B(n, \Gamma)'}{n} \text{cov}(Y_t, Y_t) + \frac{B(n-1, \Gamma)'}{n-1} \text{cov}(Y_{t+1}, Y_t) \right] \left( \frac{B(n, \Gamma)}{n} - B(1, \Gamma) \right)}{\left( \frac{B(n, \Gamma)}{n} - B(1, \Gamma) \right)' \text{cov}(Y_t, Y_t) \left( \frac{B(n, \Gamma)}{n} - B(1, \Gamma) \right)} \quad (32)$$

From Equation (51) in the Appendix, we observe that:

$$Y_{t+1} - e^{-\kappa \Delta t} Y_t = e^{-\kappa(t+\Delta t)} \Sigma \int_t^{t+1} e^{\kappa u} dW_u \quad (33)$$

is a gaussian random variable independent of  $Y_t$ .

Now, observe Equation (32) and use Equation (33) to rewrite  $\text{cov}(Y_{t+1}, Y_t)$  just using  $\text{cov}(Y_t, Y_t)$ :

$$\text{cov}(Y_{t+1}, Y_t) = e^{-\kappa \Delta t} \text{cov}(Y_t, Y_t) \quad (34)$$

Note than that in order to obtain the values of the  $\phi_n$ s it is enough to have knowledge of  $\text{cov}(Y_t, Y_t)$ . Using again Equation (51) we obtain:

$$\text{cov}(Y_t, Y_t) = e^{-\kappa t} \Sigma \left[ \int_0^t e^{(\kappa+\kappa')s} ds \right] \Sigma e^{-\kappa' t} \quad (35)$$

Figure 6 presents the empirical coefficients  $\phi_{ns}$  reported in Table 8, the EH theoretical  $\phi_{ns}$  (1 for all n), and the model implied  $\phi_{ns}$  obtained using Equations (32), (34), and (35). Note that the model implied coefficients have the same pattern of the empirical ones: negative and decreasing with maturity. In addition, the model implied coefficients are within the 95% confidence interval of the empirical coefficients. This is a qualitative result, which joint with the results about risk premia described in the next subsection, indicate that the model presents a resonable adjustment both under the historical and the pricing measure.

<sup>11</sup>These calculations do not appear in Dai and Singleton's (2002) paper.

### 3.4 Interpreting the Model Implied Risk Premia: Time-Varying Market Prices of Risk

The market prices of risk as described in Section 2 have two different components contributing to the description of the instantaneous expected bond excess returns appearing in Equation (16): the vector  $\lambda_0$  contributing with the conditional volatilities terms and the matrix  $\lambda_Y$  contributing with elements in the whole state space vector. In particular in a Gaussian model, vector  $\lambda_0$  only contributes with constant terms. As these instantaneous excess returns represent the risk premia charged by the investors, the only way that risk premia will vary over time in a Gaussian model is to have at least one element in  $\lambda_Y$  different from zero.

Table 4 presents the values of the parameters of the model with their respective standard errors. In particular, note that there are two elements in the matrix  $\lambda_Y$  that are statistically significant:  $\lambda_Y(3, 1)$  (p-value equal to 0.087) and  $\lambda_Y(1, 2)$  (p-value equal to 0.0001). Considering just these significant terms we can use Equation (16) to write the risk premia as:

$$e_{t,\tau} = B(\tau)' \Sigma \begin{pmatrix} \lambda_Y(1, 2) Y_t^2 \\ \lambda_Y(3, 1) Y_t^1 \end{pmatrix} = B^1(\tau) \Sigma_{1,1} \lambda_Y(1, 2) Y_t^2 + B^3(\tau) \Sigma_{3,3} \lambda_Y(3, 1) Y_t^1 \quad (36)$$

Figure 7 presents the functions  $A(\tau)$  and  $B^i(\tau), i = 1, 2, 3$  which appear in Equation (6). They were obtained solving the Ricatti equations which appear from the Affine structure of the state vector (see Duffie (2001)). Basically making the substitution of the parameters values in Equation (36) we obtain:

$$e_{t,\tau} = 3.96 B^1(\tau) Y_t^2 + 45.64 B^3(\tau) Y_t^1 \quad (37)$$

as a function of  $B(\tau)$  which is a known function. Equation (37) shows that **bond risk premia is a function of time**. Moreover, according to Equations (21) and (22), Equation (37) says that risk premia is approximately a linear combination of the spreads 90-30 days and 360-90 days and the yields on the 30 and 90 days swaps. Substituting the approximation described by Equations (21) and (22) in Equation (37) we obtain:

$$e_{t,\tau} = \begin{aligned} & B^1(\tau)(-0.021 + 0.396 sw_t^{30} + 0.95 spr_t^{360-90}) + \\ & + B^3(\tau)(0.2419 - 4.56 sw_t^{90} - 22.82 spr_t^{90-30} + 5.48 spr_t^{360-90}) \end{aligned} \quad (38)$$

Figure 8 presents the model implied bond risk premia for different maturities. Note that the shape of the risk premia term structure is exactly the shape of function  $B^1(\tau)$ . This is easy to explain if we observe that the state variables  $Y_t^1$  and  $Y_t^2$  have opposite signs, functions  $B^1(\tau)$  and  $B^3(\tau)$  have approximately the same shape and opposite signs that offset the effect of the different signs of the state variables. Figure 9 presents the model implied bond risk premia evolving along time. Note that it is very volatile and it is very high during the crisis periods. For instance, consider the one year bond risk premia. According to our approximation it can be written as:

$$e_{t,360} = 2 sw_t^{90} + 5.7 spr_t^{90-30} - 0.11 \quad (39)$$

Figure 10 presents the one year risk premium, the 90 days swap yield and the spread 90-30 days. According to the scales, the main source of contribution to the variation of the one year risk premium is the 90 days swap yield, while the spread 90-30 captures the details in the variation. In terms of term structure dynamic factors, it is the sum of a level factor and a slope factor.

## 4 Application: Risk Management of DI Portfolios

### 4.1 Algorithmic Description

Suppose at time  $t$  we have a portfolio of  $k$  DIs with time to maturities  $\tau_1, \tau_2, \dots, \tau_k$  (measured in years), with current reference prices  $P_t^1, P_t^2, \dots, P_t^k$ , and with number of contracts on each DI equal to  $m_1, m_2, \dots, m_k$ . We adopt as a measure for the portfolio risk the one day 99% Value at Risk (VaR, See Jorion (2000)). In order to be able to calculate the VaR, we have to obtain the conditional distribution of the portfolio Profit and Losses within the next day. We consider as risk factors the yields of the specific swaps which generate by no-arbitrage the reference prices for the different DIs. So the idea is to simulate variations for the reference yields  $y_t^i, i = 1, \dots, k$ , of these swaps. These yields are obtained through the initial set of information:

$$y_t^i = -\frac{\log P_t^i}{\tau_i} \quad (40)$$

In order to come up with a probability distribution for the swap yields, we use a discrete version of the multivariate Equation (24) to simulate the values of the state variables one day in the future, at time  $t + 1$ :

$$\begin{aligned} Y_{t+1}^1 &= Y_t^1 + (-4.243Y_t^1 - 0.0461Y_t^2)\Delta t + 0.0270\Delta W_t^1 \\ Y_{t+1}^2 &= Y_t^2 + (-0.2854Y_t^2 + 0.1167Y_t^3)\Delta t + 0.0163\Delta W_t^2 \\ Y_{t+1}^3 &= Y_t^3 - 3.761Y_t^3\Delta t + 0.0914\Delta W_t^3 \end{aligned} \quad (41)$$

Where  $\Delta W_t^i, i = 1, 2, 3$  are three independent Gaussian random variables with mean zero and variance  $\sqrt{\Delta t}$ , with  $\Delta t = \frac{1}{252}$ . Following that, we fix a number of monte carlo simulations say of  $B$  samples for the state vector at time  $t + 1$ . For each simulated sample  $j$ , for each maturity  $\tau_i$ , we calculate the ratio of the model implied swap yields at times  $t$  and  $t + 1$ :

$$R_t^{j,i} = \frac{R(t+1, \tau_i - \frac{1}{252})}{R(t, \tau_i)} \quad (42)$$

where  $R(t, \tau) = \frac{A(\tau) + B(\tau)'Y_t}{\tau}$  which is obtained from Equation (6).

For each simulated sample  $j$ , the variation in the value of the portfolio will be:

$$\Delta P_t^{port,j} = \sum_{i=1}^k m_i \left[ 100000 e^{-(y_t^i R_t^{j,i})(\tau_i - \frac{1}{252})} - P_t^i (1 + DI_t) \right] \quad (43)$$

where  $DI_t$  is the short term DI yield (defined in Section 3.1) observed at day  $t$ . The reason for using the DI yield is to correct the price of each DI to account for the passage of one day in time. Finally, with the P&L distribution in hands the 99% VaR  $x$  is obtained solving the following equation:

$$Prob \left[ \Delta P_t^{port} < x \right] = 0.01 \quad (44)$$

considering each  $\Delta P_t^{port,j}, j = 1, \dots, B$  with uniform probability  $\frac{1}{B}$ .

## 4.2 Numerical Example

In this subsection, we present an example of estimation of a portfolio risk using the model, and compare it to the same risk estimated by a historical simulation procedure (HS, see Jorion (2000)).

Table 9 presents information regarding the number of open contracts on all different DI futures for the date January 29, 2003. The two most liquid short term DI futures were the ones with maturities in April 1, 2003 and July 1, 2003, with 218903 and 105619 open contracts respectively. On the long term side, the DI with maturity in January 1, 2004 presents a considerable number of open contracts, total of 37833. Suppose we have a portfolio with 5000 contracts of the DI April 03, 1000 contracts of the DI July 03 and 3000 contracts of the DI January 04. In our notation,  $\tau_1 = \frac{43}{252}$  days,  $\tau_2 = \frac{104}{252}$  days and  $\tau_3 = \frac{235}{252}$  days,  $m_1 = 5000$ ,  $m_2 = 2000$  and  $m_3 = 3000$ . The reference prices are respectively  $P_1 = 96089.14$ ,  $P_2 = 90601.65$  and  $P_3 = 79272.00$ , and the reference swap yields are 26.34%, 27.02%, and 28.29%.

Implementing the algorithm introduced in the last subsection, adopting a Monte Carlo sample size  $B = 10000$ , we were able to obtain the individual VaR for each position on DIs, respectively R\$ -367900.00, -119700.00 and -1366800.00 and the overall portfolio VaR which was of R\$ -1780600.00. Now, compare these values with the ones obtained using HS: R\$ -1686100.00, -753610.00, -3725000.00 for the individual positions, and R\$ -6173400.00 for the portfolio VaR. In addition, Figure 11 presents the conditional P&L probability distribution of the portfolio for the next day, using the model and the HS approach. Equations (41), (42) and (43) allow us to identify that the model implied P&L distribution is a mixture of translated log-normal distributions. Note that the HS implied distribution is far more volatile and assymmetric. Observing both the estimated risks and the estimated P&L distributions, we note that the HS approach suggests much higher losses in extreme market conditions than the model approach. This happens because the HS uses a simulation of the size of the historical data sample, that is 518 observations, and as there have been 3 crises in the market during this period, they have a big impact in the percentile calculation of losses. For one hand, this shows that HS is conservative and allows the investor to be aware of risks that might turn into big losses. On the other hand, for instance, from a regulatory capital allocation viewpoint, it asks for an allocation of a big amount of capital to cover the risk of this position, instead of having this money applied in other financial instruments in the market. One possible suggestion would be to maintain both calculations implemented as tools for management decisions.

## 5 Conclusion

In this paper we adopt a Gaussian multi-factor term structure model to study the dynamics of the Brazilian swaps market. We estimate the model using Maximum Likelihood, making use of its Affine structure to write closed-form formulas for the pricing of zero coupon bonds and also to write the likelihood function. We propose as an application of the model, the estimation of the risk of a portfolio composed by DIs with different maturities, where the risk is measured by a 99% daily VaR. For a specific portfolio of DIs, we compare the model results to results obtained by a historical simulation (HS), observing that the HS is far more conservative due to the high influence of the market crises contained in the data window.

Regarding the risk premia, we identify that it presents high variation along time, what directly implies a rejection of the Expectation Hypothesis Theory, in perfect accordance with the empirical results obtained for the US. fixed income market. In addition, we find that risk premia is explained by a level factor (a positive combination of the 30 and 90 days swap yield) and a slope factor (the spread 90-30 days). Knowledge of a parametric form for the risk premia consistent with the historical behavior of the term structure of interest rates should be useful for portfolio allocation purposes. An idea for a future work is to simulate the state variables dynamics (once we estimated their SDEs) and obtain the correspondent simulated projected risk premia. According to the projected risk premia, take decisions about allocations along the different maturities of the term structure.

Summarizing, we estimate a parsimonious model, that rules-out arbitrages, fits well the data, presents easy interpretation of the state variables, and can be used as a tool for risk management and portfolio allocation.

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## 7 Appendix - Maximum Likelihood Estimation and Calculation of Conditional Densities Under Affine Models

In this work, we adopt the Maximum Likelihood estimation procedure described in Duffie et al. (2003). We observe the following swap yields along T points in time:  $sw_t^{30}$ ,  $sw_t^{60}$ ,  $sw_t^{90}$ ,  $sw_t^{120}$ ,  $sw_t^{180}$ ,  $sw_t^{270}$ ,  $sw_t^{360}$ , and  $sw_t^{720}$ . Let  $sw_t$  represent the vector containing these swap yields at time  $t$ . Assume that the model parameters are represented by the vector  $\phi$  and that the difference between times  $t-1$  and  $t$  is  $\Delta t$ . From Equation (6), the relation between the yield of a swap with maturity  $\tau$  and the state variables at time  $t$  is:

$$R(t, \tau, \phi) = -\frac{A(\tau, \phi)}{\tau} - \frac{B(\tau, \phi)'}{\tau} Y_t \quad (45)$$

In particular, we assume that the yields of the swaps with maturities 30, 90 and 360 days are observed without error. As the state vector is three dimensional, knowledge of functions  $A(\tau)$  and  $B(\tau)$  allows us to solve a linear system to obtain the values of the state vector at each time  $t$ :

$$\begin{aligned} sw_t^{30} &= -\frac{A(0.0833, \phi)}{0.0833} - \frac{B(0.0833, \phi)'}{0.0833} Y_t \\ sw_t^{90} &= -\frac{A(0.25, \phi)}{0.25} - \frac{B(0.25, \phi)'}{0.25} Y_t \\ sw_t^{360} &= -A(1, \phi) - B(1, \phi)' Y_t \end{aligned} \quad (46)$$

For the swaps with maturities 60, 120, 180, 270 and 720, we assume observation with gaussian errors  $u_t$  uncorrelated along time:

$$\begin{aligned}
sw_t^{60} &= -\frac{A(0.1667, \phi)}{0.1667} - \frac{B(0.1667, \phi)'}{0.1667} Y_t + u_t^{60} \\
sw_t^{120} &= -\frac{A(0.3333, \phi)}{0.3333} - \frac{B(0.3333, \phi)'}{0.3333} Y_t + u_t^{120} \\
sw_t^{180} &= -\frac{A(0.5, \phi)}{0.5} - \frac{B(0.5, \phi)'}{0.5} Y_t + u_t^{180} \\
sw_t^{270} &= -\frac{A(0.75, \phi)}{0.75} - \frac{B(0.75, \phi)'}{0.75} Y_t + u_t^{270} \\
sw_t^{720} &= -\frac{A(2, \phi)}{2} - \frac{B(2, \phi)'}{2} Y_t + u_t^{720}
\end{aligned} \tag{47}$$

After extracting the corresponding state vector at the vector of parameters  $\phi$ , we can write the log-likelihood function of the swap yields as:

$$L(sw, \phi) = \sum_{t=1}^T \log p(Y_t | Y_{t-1}; \phi) - T \log |jac(R(t, \phi))| + \frac{5T}{2} \log 2\pi - \frac{T}{2} \log |\Omega| - \frac{1}{2} \sum_{t=2}^T u_t' \Omega u_t \tag{48}$$

where  $|\cdot|$  denotes determinant, and

$$|jac(R(t, \phi))|^{-1} = \left\| \begin{bmatrix} -\frac{B(0.0833, \phi)'}{0.0833} \\ -\frac{B(0.25, \phi)'}{0.25} \\ -B(1, \phi)' \end{bmatrix} \right\|^{-1} \tag{49}$$

is the jacobian of the function that relates swap yields and the state vector;  $\Omega$  represents the covariance matrix for  $u_t$ , estimated using the sample covariance matrix of the  $u_t$ s implied by the extracted state vector along time.

In order to finish our calculation of the likelihood function we have to identify the conditional probability of the state vector which appears in Equation (48). Under the Gaussian model, the SDE for the state vector is:

$$dY_t = -\kappa Y_t dt + \Sigma dW_t \tag{50}$$

We can integrate this equation in the interval  $[t-1, t]$  to get:

$$Y_t e^{\kappa t} = Y_{t-1} e^{\kappa(t-1)} + \Sigma \int_{t-1}^t e^{\kappa u} dW_u \tag{51}$$

From Equation (51) and the stationarity of the Brownian Motion, we know that conditional on  $Y_{t-1}$  the state vector  $Y_t$  presents a multivariate gaussian distribution with mean  $m_{\Delta t} = e^{-\kappa \Delta t} Y_{t-1}$  and variance  $V_{\Delta t} = \Sigma^2 e^{-\kappa \Delta t} [\int_0^{\Delta t} e^{(\kappa + \kappa') s} ds] e^{-\kappa' \Delta t}$ . Using the formula for a multivariate gaussian distribution with mean  $m$  and variance  $V$  which is given by:

$$\Phi(x, m, V) = ((2\pi)^N |V|)^{-\frac{1}{2}} e^{-\frac{1}{2}(x-m)' V^{-1} (x-m)} \tag{52}$$

we can write:

$$p(Y_t | Y_{t-1}; \phi) = \Phi(Y_t, m_{\Delta t}, V_{\Delta t}) \tag{53}$$

Our final objective is to estimate the vector of parameters  $\phi$  which maximizes function  $L(sw, \phi)$ , what is done by making use of the Nelder-Mead Simplex algorithm for non-linear functions optimization, implemented in the MATLAB *fminsearch* function.



Maturity (days)	30	60	90	120	180	270	360	720
30	1.00	0.94	0.88	0.84	0.78	0.71	0.64	0.57
60	0.94	1.00	0.97	0.94	0.88	0.81	0.74	0.66
90	0.88	0.97	1.00	0.98	0.94	0.87	0.81	0.73
120	0.84	0.94	0.98	1.00	0.98	0.93	0.88	0.80
180	0.78	0.88	0.94	0.98	1.00	0.97	0.93	0.85
270	0.71	0.81	0.87	0.93	0.97	1.00	0.98	0.91
360	0.64	0.74	0.81	0.88	0.93	0.98	1.00	0.95
720	0.57	0.66	0.73	0.80	0.85	0.91	0.95	1.00

Table 1: Correlation Structure of the First Differences of the Swaps Yields.

Maturity (days)	Volatility (bp)
30	43
60	47
90	50
120	52
180	54
270	57
360	60
720	68

Table 2: Volatilities of the First Differences of the Swaps Yields.

Princ. Component	Variance (1.0e-3)	Variation Explained (%)	Accum. Variation Explained (%)
1	0.2072	87.67	87.67
2	0.0220	9.29	96.96
3	0.0041	1.74	98.70
4	0.0018	0.75	99.45
5	0.0206	0.25	99.70
6	0.0003	0.15	99.85
7	0.0002	0.10	99.95
8	0.0001	0.05	100.00

Table 3: Identifying the Main Factors in the Term Structure Dynamics.

Parameter	Value	Standard Error
$\kappa_{11}$	4.243	3.4074
$\kappa_{12}$	0.0461	2.5112
$\kappa_{22}$	0.2854	1.8397
$\kappa_{23}$	-0.1167	0.4050
$\kappa_{33}$	3.7610	1.0331
$\Sigma_{11}$	0.0270	0.0070
$\Sigma_{22}$	0.0163	0.0031
$\Sigma_{33}$	0.0914	0.0020
$\lambda_0(3)$	-0.0627	1.7044
$\lambda_Y(1, 1)$	49.840	126.5493
$\lambda_Y(3, 1)$	499.40	130.6770
$\lambda_Y(1, 2)$	146.90	86.5136
$\lambda_Y(2, 2)$	-25.55	114.3919
$\lambda_Y(3, 2)$	19.58	98.8881
$\lambda_Y(2, 3)$	5.64	24.7774

Table 4: Parameters and Standard Errors for the  $A_0(3)$  Model.

Maturity (days)	Mean (bp)	Standard Dev. (bp)
60	-4	9
120	9	17
180	8	29
270	3	24
720	-14	91

Table 5: Statistics for the Errors on the Swaps Fit.

Coefficients	$Y^1$	$Y^2$	$Y^3$
$b_1$	0.5061	0.1103	1.4371
$b_2$	0.0000	0.0000	0.0000
$b_3$	-0.7226	-0.2495	-0.2867
$b_4$	0.0000	0.0000	0.0000
$b_5$	0.0000	0.0000	0.0000
$b_6$	0.0000	0.0000	0.0000
$b_7$	0.1263	0.2397	-0.1711
$b_8$	0.0000	0.0000	0.0000
$b_9$	0.0053	-0.0053	-0.0541

Table 6: Coefficients Relating State Variables and Swap Yields.

Maturity (days)	Variable	Mean (bp)	Standard Dev. (bp)
60	$\hat{e}(t, 2)$	4	14
90	$\hat{e}(t, 3)$	9	31
120	$\hat{e}(t, 4)$	13	49
180	$\hat{e}(t, 6)$	14	86
270	$\hat{e}(t, 9)$	16	145
360	$\hat{e}(t, 12)$	19	224
720	$\hat{e}(t, 24)$	19	654

Table 7: Statistics for the Monthly Excess Returns on Swaps.

Maturity (days)	$\phi_n$	s. e.
60	-1.4250	0.7823
90	-1.8701	0.9434
120	-1.9076	1.1364
180	-2.0416	1.4610
270	-2.2089	1.8973
360	-3.0071	2.4218
720	-7.7812	4.7661

Table 8: Results on the EH Regressions.

DI Maturity	Number of Open Contracts
Febr. 03 2003	94619
March 05 2003	63843
April 01 2003	218903
May 02 2003	24097
July 01 2003	105619
Sept. 01 2003	380
Oct. 01 2003	33129
Nov. 03 2003	5
Dec. 01 2003	5
Jan. 02 2004	37833
April 01 2004	18420
July 01 2004	12445

Table 9: Liquidity in the DI Market in January 29, 2003.

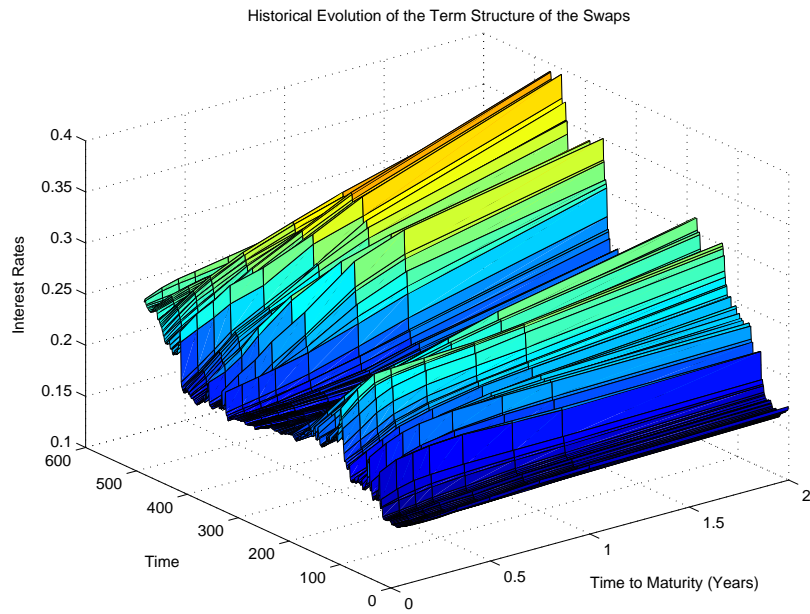


Figure 1: The Term Structure of Interest Rates Swaps.

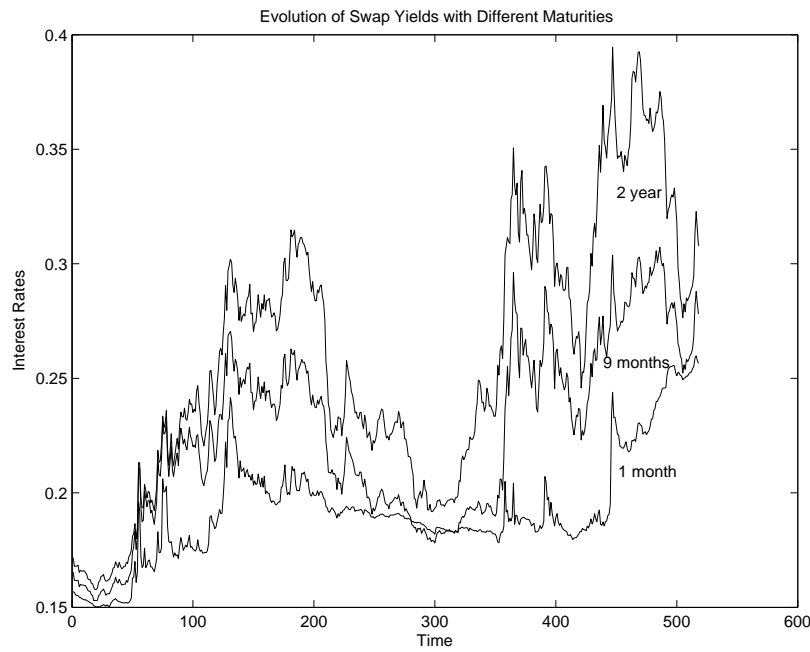


Figure 2: Evolution of the Swaps Yields for Different Maturities.

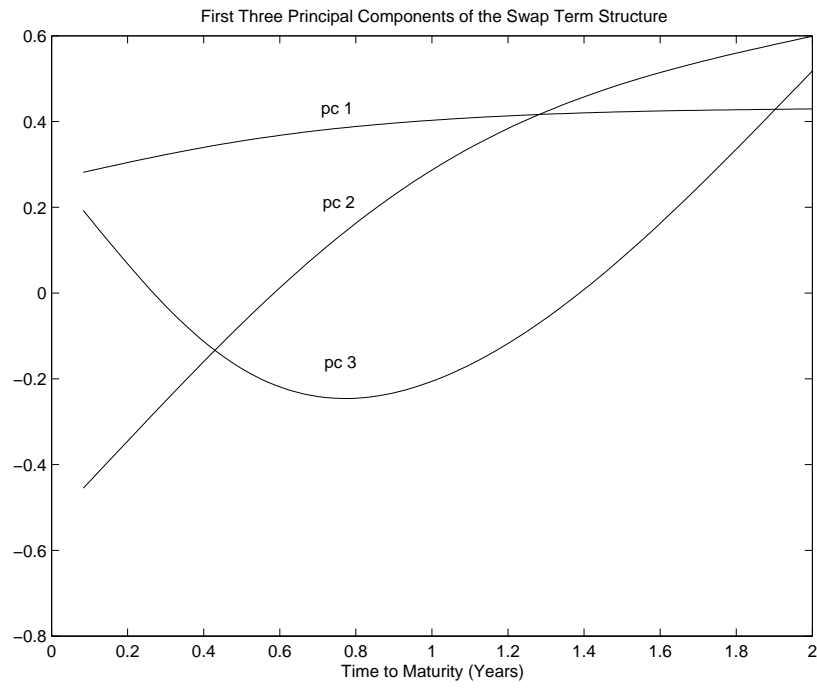


Figure 3: The Main Principal Components Driving the Term Structure Dynamics.

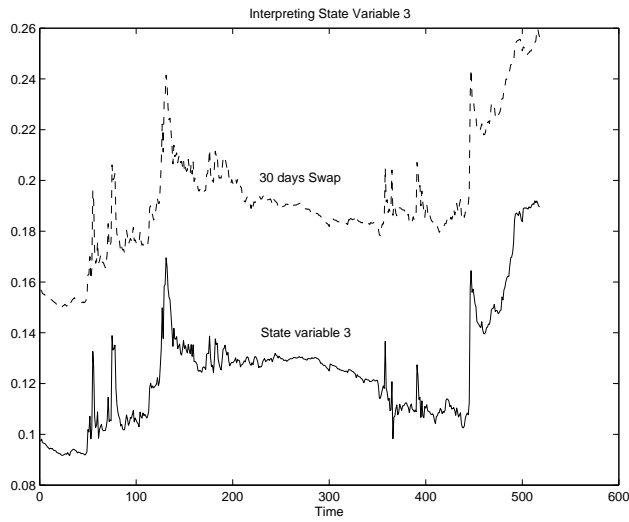
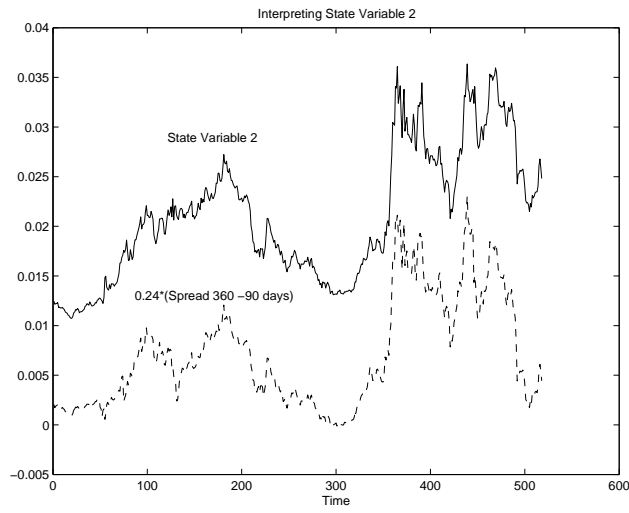
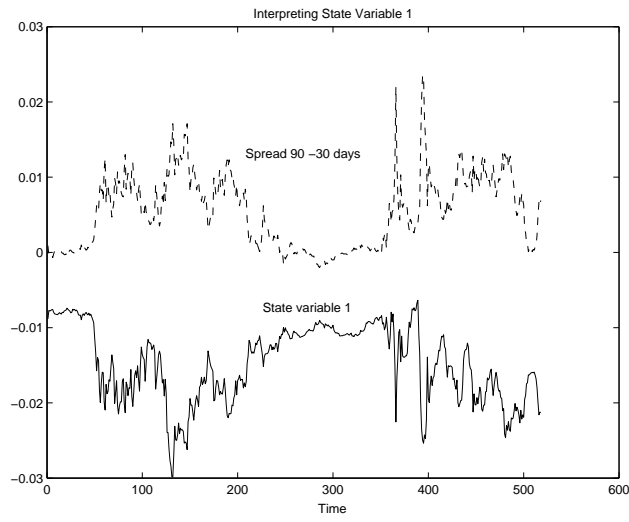


Figure 4: Interpreting the State Variables.

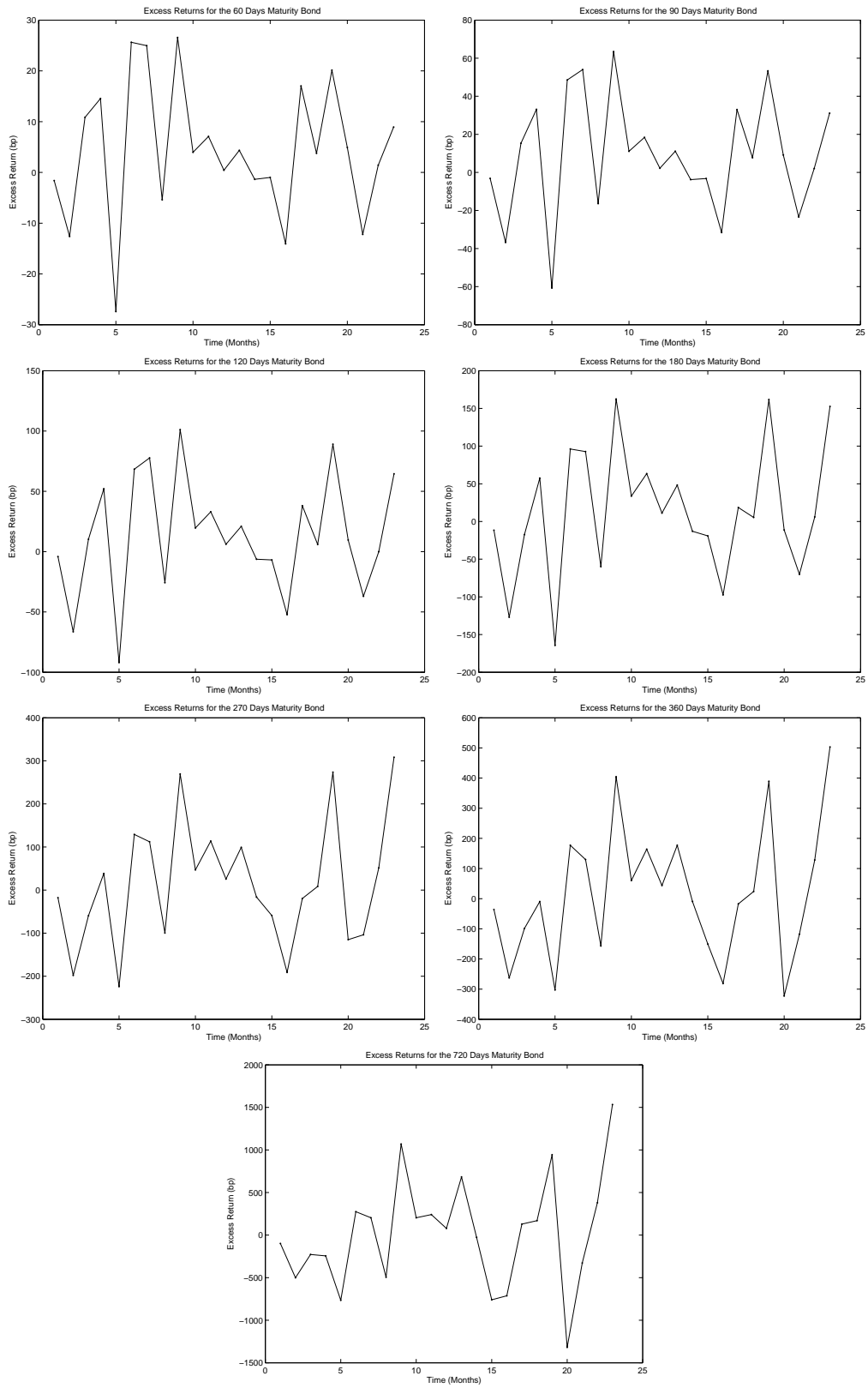


Figure 5: Excess Returns Historical Behavior for the Swap Market.

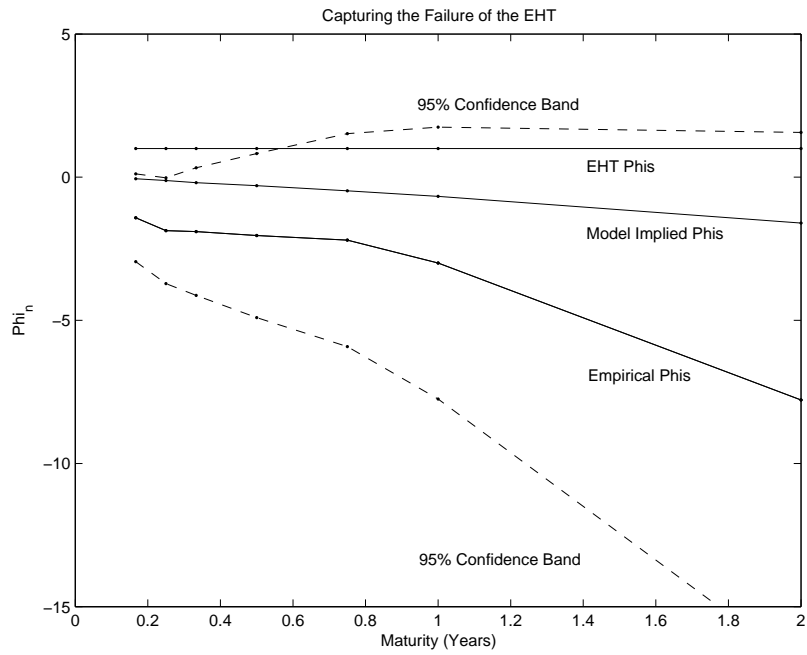


Figure 6: Model Implied  $\phi_n$ s coefficients for EHT Regressions.



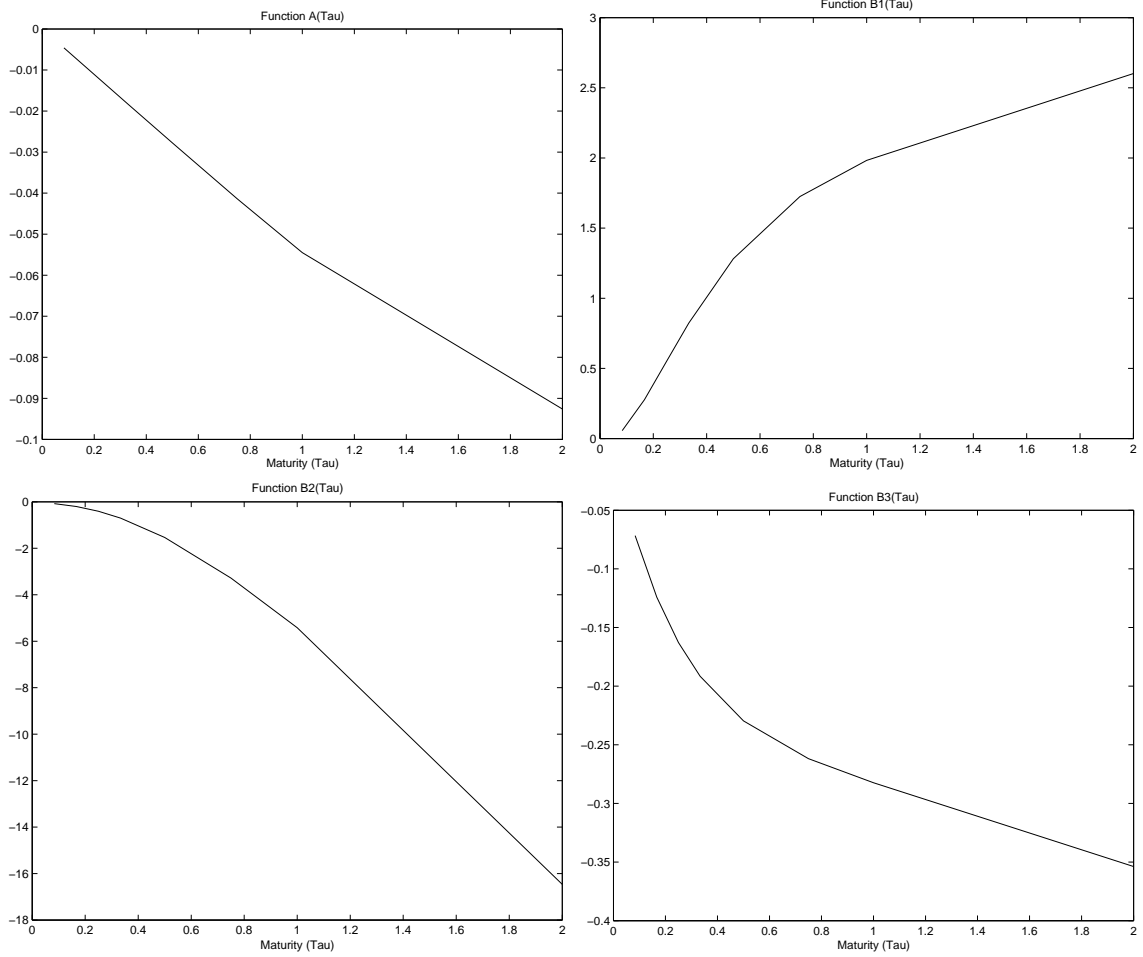


Figure 7: Functions  $A(\tau)$  and  $B(\tau)$  which Determine Model Implied Swaps Yields.

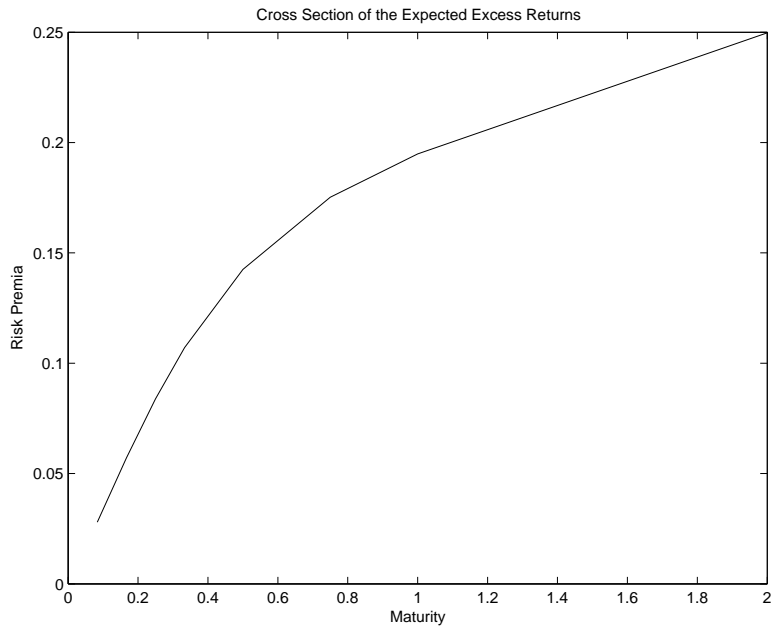


Figure 8: Cross Section of the Model Implied Instantaneous Excess Returns on a Specific Day.

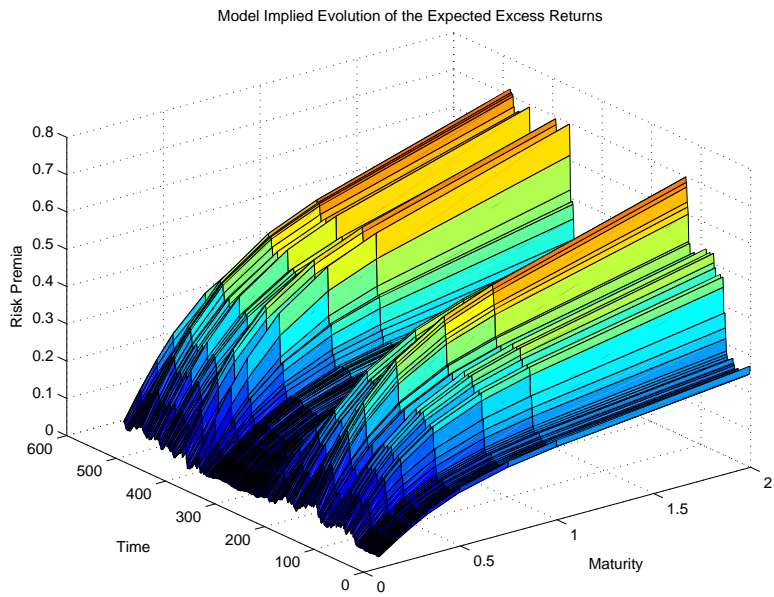


Figure 9: Dynamic Evolution of the Model Implied Instantaneous Excess Returns.

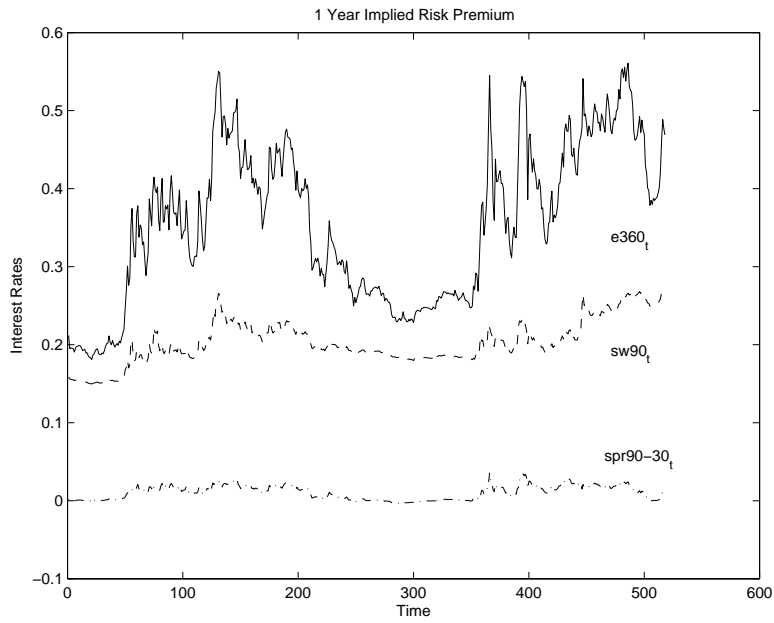


Figure 10: Dynamic Evolution of the One Year Model Implied Instantaneous Excess Returns.

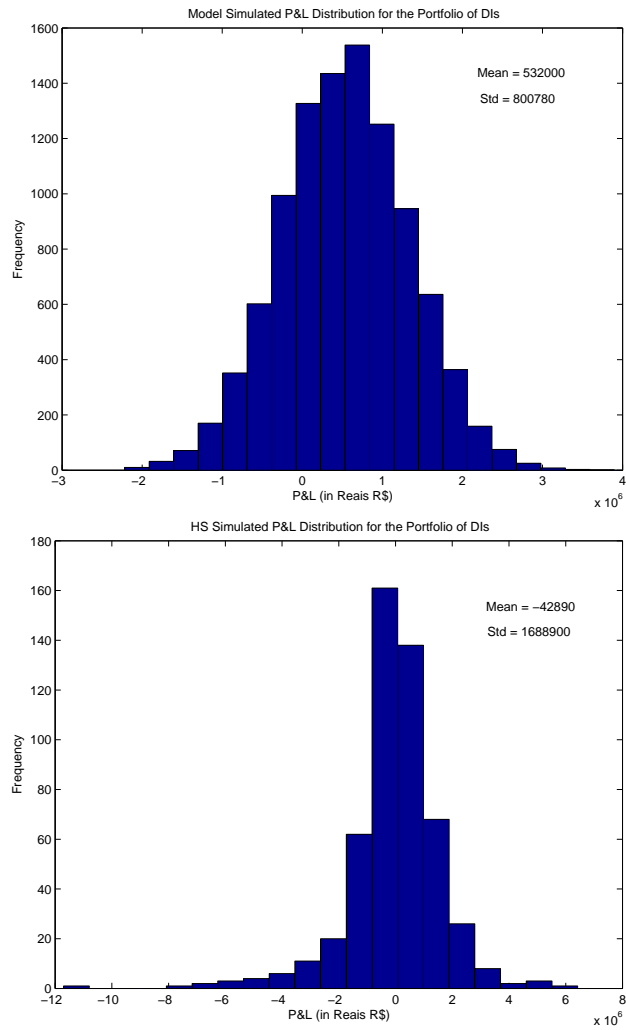


Figure 11: One Day Conditional P&L Distributions Implied by the Model and HS.