REALIZABILITY OF \( p \)-POINT GRAPHS WITH PRESCRIBED MINIMUM DEGREE, MAXIMUM DEGREE, AND POINT-CONNECTIVITY

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Received 13 August 1979
Revised 27 May 1980

In a recent paper, we gave a generalization of extremal problems involving certain graph-theoretic invariants. In that work, we defined a \((p, \Delta, \delta, \lambda)\) graph as a graph having \( p \) points, maximum degree \( \Delta \), minimum degree \( \delta \), and line-connectivity \( \lambda \). An arbitrary quadruple of integers \((a, b, c, d)\) was called \((p, \Delta, \delta, \lambda)\) realizable if there is a \((p, \Delta, \delta, \lambda)\) graph with \( p = a \), \( \Delta = b \), \( \delta = c \), and \( \lambda = d \). In this work, we consider the more difficult case of \((p, \Delta, \delta, \kappa)\) realizability, where \( \kappa \) is the point-connectivity. Necessary and sufficient conditions for a quadruple to be \((p, \Delta, \delta, \kappa)\) realizable are derived.

Introduction

In this work, we consider an undirected graph \( G = (V, X) \) with a finite point set \( V \) and a set \( X \) whose elements, called lines, are two-point subsets of \( V \). The number of points \( |V| \) is denoted by \( p \), and the number of lines \( |X| \) is called \( q \). If \( V \) only contains one element, we call the graph trivial. We follow Harary [13] for all notation and terminology; however, we shall reproduce a few basic concepts herein.

The line-connectivity \( \lambda(G) \) or simply \( \lambda \) is the minimum number of lines whose removal results in a disconnected graph. The point-connectivity or more briefly the connectivity \( \kappa(G) \) or \( \kappa \) is the minimum number of points whose removal results in a disconnected or trivial graph. The number of lines connected to a point \( v_i \) of \( G \) is the degree \( d_i(G) \) or \( d_i \) of that point. The minimum degree is denoted by \( \delta(G) \) or \( \delta \), while the maximum degree is denoted by \( \Delta(G) \) or \( \Delta \). A regular graph has \( \delta = \Delta \). The notation \( K_p \) denotes a \( p \) point graph with \( \delta = p - 1 \).

It is well known that certain graph-theoretic extremal questions play a central role in the study of communication network vulnerability[1–12]. A classical result of Harary [14] gives the solution to the problem of finding the maximum value of \( \kappa \) among all graphs with \( p \) points and \( q \) lines. By fixing certain graph invariants and finding a maximum or minimum value of some other invariant, one can generate a variety of interesting extremal questions.

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In a recent paper [11] it was shown that all of the possible extremal problems involving the invariants \( p, \Delta, \delta, \lambda \) could be obtained as a special case of a general realizability question. Namely we defined \( G \) to be a \((p, \Delta, \delta, \lambda)\) graph if \( G \) has \( p \) points, maximum degree \( \Delta \), minimum degree \( \delta \), and line-connectivity \( \lambda \). An arbitrary quadruple of integers \((a, b, c, d)\) is called \((p, \Delta, \delta, \lambda)\) realizable if there exists a \((p, \Delta, \delta, \lambda)\) graph with \((p, \Delta, \delta, \lambda) = (a, b, c, d)\); for simplicity we say the quadruple is realizable. In [11] necessary and sufficient conditions for \((p, \Delta, \delta, \lambda)\) realizability were derived.

In the present paper, we consider the more difficult case of \((p, \Delta, \delta, \kappa)\) realizability.

### Preliminaries

We start by establishing some basic inequalities involving the invariants \( p, \Delta, \delta, \) and \( \kappa \).

**Lemma 1.** If \( \kappa = 0 \) and \( p \geq 2 \), then \( \Delta \leq p - \delta - 2 \) and \( \delta < \lfloor \frac{1}{3}p \rfloor \).

**Proof.** A graph with at least two points has connectivity equal to 0 if it has two or more components. One must contain a point of degree \( \Delta \) so that it must have at least \( \Delta + 1 \) points. Since the minimum degree is \( \delta \) each of the remaining components have at least \( \delta + 1 \) points. Thus \( 2\delta + 2 \leq (\delta + 1) + (\Delta + 1) \leq p \) and the conclusion follows. \( \square \)

**Lemma 2.** For any \((p, \Delta, \delta, \kappa)\) graph with \( p \geq 2 \)

(a) \( \kappa \geq 2\delta - p + 2 \) if the graph is not complete, and

(b) \( \kappa \Delta \geq (p - \kappa)\kappa + (2\delta - p + 2 - \kappa)(p - \delta - 1) \).

**Proof.** Even though the inequality of (a) was established in [3], we shall come upon it here in the process of establishing (b).

If the graph is not complete and has point-connectivity \( \kappa \) then the point set \( V \) may be partitioned into three mutually disjoint sets \( S, T, \) and \( U \) where \( |S| = \kappa, |T| = m, |U| = p - \kappa - m \), and there are no lines joining \( T \) and \( U \). Now if \( t \) is a point of \( T \) with degree \( d_t \), then

\[ \delta \leq d_t \leq m - 1 + \kappa, \]

since \( t \) can only be adjacent to points in \( T \) and \( S \). Thus

\[ 0 < \delta + 1 - \kappa \leq m. \]

Since \( |T| + |U| = p - \kappa \) we have

\[ |U| = p - \kappa - |T| \leq p - \kappa - (\delta + 1 - \kappa) = p - \delta - 1. \]
Therefore,
\[ \delta + 1 - \kappa \leq |T|, \quad |U| \leq p - \delta - 1. \]

Now we can assume that \(|U| > |T| = m\). Thus
\[ (p - \delta - 1) \geq m \geq (\delta + 1 - \kappa). \]

Hence
\[ \kappa \geq 2\delta - p + 2, \quad \text{which is (a)}. \]

Next realize that
\[ d_s \geq d_{ST} + d_{SU} \]
where \(d_{ST}\) denotes the number of lines emanating from \(s\) and terminating in \(T\) and \(d_{SU}\) has a similar meaning. Thus,
\[
\Delta \kappa \geq \left( \max_{s \in S} d_s \right) \kappa \geq \sum_{s \in S} d_s \geq \sum_{s \in S} d_{ST} + \sum_{s \in S} d_{SU} \\
= q_{ST} + q_{SU}
\]
where \(q_{ST}\) denotes the number of lines joining \(S\) and \(T\) and \(q_{SU}\) has the corresponding meaning for \(S\) and \(U\). Now we find a lower bound on \(q_{ST} + q_{SU}\).

We know \(d_s + d_t = |T|^2\) where \(t\) is in \(T\) so that
\[
q_{ST} + \sum_{t \in T} d_{IT} = \sum_{t \in T} d_s + \sum_{t \in T} d_{IT} \geq m\delta.
\]
Similarly
\[
q_{SU} + \sum_{u \in U} d_{uU} \geq (p - \kappa - m)\delta.
\]
Adding, we get
\[
q_{ST} + q_{SU} \geq (p - \kappa)\delta - \left( \sum_{t \in T} d_{IT} + \sum_{u \in U} d_{uU} \right) \\
\geq (p - \kappa)\delta - (m(m - 1) + (p - \kappa - m)(p - \kappa - m - 1)).
\]
Now \(\delta + 1 - \kappa \leq m \leq p - \delta - 1\) and the quadratic in \(m\) has the same value at each of the end points \(\delta + 1 - \kappa\) and \(p - \delta - 1\) so
\[
q_{ST} + q_{SU} \geq (p - \kappa)\delta - ((\delta + 1 - \kappa)(\delta - \kappa) + (p - \delta - 1)(p - \delta - 2)) \\
= (p - \kappa)\kappa + 2\delta - \kappa - p + 2)(p - \delta - 1).
\]
Hence
\[ \Delta \kappa \geq (p - \kappa)\kappa + (2\delta - \kappa - p + 2)(p - \delta - 1). \] \(\square\)
The next result was derived as Lemma 4 of [11]. However, for the sake of completeness we state it here without proof.

**Lemma 3.** Given integers \( p, r, \) and \( e \) with \( 0 \leq e \leq p - 1 \) and \( 0 \leq r \leq p - 2 \), there exists a \( p \) point graph having \( e \) points of degree \( r + 1 \), \( p - e \) points of degree \( r \), and \( \lambda(G) = \kappa(G) = r \) if and only if \( p \cdot r \) and \( e \) have the same parity.

**The \((p, \Delta, \delta, \kappa)\) realizability theorem**

**Theorem.** The quadruple \((p, \Delta, \delta, \kappa)\) of nonnegative integers is realizable if and only if one of the following five mutually exclusive conditions holds:

1. \( \delta < \lfloor \frac{1}{2} p \rfloor \):
   (i) \( \delta < \Delta \leq p - \delta - 2 \) and if \( \delta = \Delta \), then \( p \delta \) is even;
   (ii) \( 1 \leq \kappa \leq \delta < \Delta \leq p - 1 \);
   (iii) \( 1 \leq \kappa \leq \delta = \Delta < \lfloor \frac{1}{2} p \rfloor \), \( p \delta \) is even, and if \( \kappa = 1 \), then \( 2 < \delta < \frac{1}{2} p - 1 \)

2. \( \delta \geq \lfloor \frac{1}{2} p \rfloor \):
   (i) \( 1 \leq 2 \delta - p + 2 \leq \kappa \leq \delta < p - 1 \) and \( \kappa \Delta \geq (p - \kappa) \kappa - (\kappa - [2 \delta - p + 2]) (p - \delta - 1) \);
   (ii) \( \kappa = \Delta = \delta = p - 1 \).

**Proof.** The necessity of (I-i) when \( \kappa = 0 \) follows from some obvious graph-theoretic facts together with Lemma 1. Most of the conditions given in (I-ii) and (I-iii) are obvious and we pause first to note that \( \delta - \Delta - 2 \) must be excluded when \( \kappa = 1 \) and \( 1 < \lfloor \frac{1}{2} p \rfloor \) since the only connected regular graph of degree two is a cycle. To see that when \( \kappa = 1 \) it is impossible for \( \delta = \Delta = \frac{1}{2} p - 1 \) recall from the proof of Lemma 2 that if \( S \) is a disconnecting set of points for a graph leaving sets \( T \) and \( U \) with no lines between them, then

\[
\delta + 1 - \kappa \leq |T|, \quad |U| \leq p - \delta - 1.
\]

Thus if \( \kappa = 1 \) and \( \delta = \Delta \), then the condition \( \delta = \frac{1}{2} p - 1 \) implies

\[
\delta \leq |T|, \quad |U| \leq \delta + 1
\]

so that one of the sets, say \( T \), has \( \delta \) points while the other contains \( \delta + 1 \). In order for the minimum degree to be \( \delta \), \( T \) must be complete and the single point of \( S \) must be adjacent to all points of \( T \). However, if we insist that the graph be regular it follows that the point of \( S \) fails to be adjacent to each point of \( U \). This, of course, implies that the graph has connectivity 0 rather than 1, as was assumed. The necessity of conditions (II-i) and (II-ii) follows from Lemma 2 and the definition of \( \kappa \).

To establish sufficiency we provide constructions for each of the five conditions.

**Case (I-i).** Let \( G' \) be a graph, the existence of which is guaranteed by Lemma 3, on \( p - \delta - 1 \) points which is regular of degree \( \delta \) or has exactly one point of
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degree $\delta + 1$ and the rest of degree $\delta$. Choose a point, the one having degree $\delta + 1$ if it exists, and if necessary increase its degree to $\Delta$. Then $G = K_{\delta + 1} \cup G'$ fulfills the requirements. If $\delta = \Delta$ and $p \delta$ is even then $(p - \delta - 1) \delta$ is even so that by Lemma 3 there is a graph $G'$ on $p - \delta - 1$ points which is regular of degree $\delta$. Of course $G = K_{\delta + 1} \cup G'$ is the desired graph.

Case (I-ii). Let $V$ be a point set partitioned into three mutually disjoint subsets $T$, $S$, and $U$ where $|T| = \delta + 1 - \kappa$, $|S| = \kappa$, and $|U| = p - \delta - 1$.

If $\Delta < p - \delta - 1$ construct a graph on $U$ with at least $\kappa$ points of degree $\Delta - 1$, point-connectivity $\Delta - 1$, and all other points of degree $\Delta - 1$ or $\Delta$. The graph exists by Lemma 3. Select $\kappa$ points in $U$, each having degree $\Delta - 1$, pair them with the points of $S$, and join the pairs with lines. Add lines within $T$ to form a complete graph on $T$ and join all points of $T$ with all those of $S$. At this point each point of $T$ has degree $\delta$ while each point of $S$ has degree $\delta + 2 - \kappa$. If $\kappa = 1$ or $2$ then the construction is finished; otherwise, add to the points of $S$ a regular graph of degree $\kappa - 2$ if possible, or a graph with degrees $\kappa - 2$ and $\kappa - 1$. That this is possible follows from Lemma 3. Now the removal of $S$ obviously disconnects the graph and it is easy to see that no smaller disconnecting set exists.

If $\Delta \geq p - \delta - 1$ (which implies $\Delta > \delta$) construct complete graphs on $T$ and $S$ and join each point of $T$ to each point of $S$ so that at present the points of $T$ and $S$ all have degree $\delta$.

First we entertain the possibility of $p$, $\delta$, $\Delta$, and $\kappa$ satisfying the inequality $\Delta - \delta + \kappa - 1 \leq p - \delta - 1$. If this is the case, join a single point of $S$ to $\Delta - \delta$ points of $U$ and the $\kappa - 1$ remaining points of $S$ in a $1 - 1$ fashion to $\kappa - 1$ points of $U$ distinct from the $\Delta - \delta$ points previously used. Next add the complete graph to the points of $U$. Since $\Delta - \delta + \kappa - 1 \geq \kappa$ the removal of fewer than $\kappa$ points from $U$ cannot disconnect the graph. Furthermore, one point of $S$ has degree $\Delta$; the rest have degree $\delta + 1$. Now $\delta \leq p - \delta - 2$ so that the degrees of the points of $U$, being $p - \delta - 2$ or $p - \delta - 1$, lie between $\delta$ and $\Delta$.

On the other hand, it may be that $\Delta - \delta + \kappa - 1 > p - \delta - 1$. In this event two possibilities remain, namely $\Delta \geq p - \delta$ and $\Delta = p - \delta - 1$. If $\Delta \geq p - \delta$ join a single point of $S$ to $\Delta - \delta$ points of $U$ and the $\kappa - 1$ remaining points of $S$ to $\kappa - 1$ points of $U$ in such a way that each point of $U$ receives an additional degree of at most 2. This is possible since

$$\Delta + \kappa \leq (p - 1) + \delta + (p - 1) + (p - \delta - 2) = 2p - \delta - 3$$

so that

$$\Delta - \delta + \kappa - 1 \leq 2\Delta - 2 \delta - 4 = 2(p - \delta - 2).$$

Then add the complete graph to the points of $U$. Thus one point of $S$ has degree $\Delta$ and the remaining $\kappa - 1$ points have degree $\delta + 1$. The points of $U$ have degree $p - \delta - 2$, $p - \delta - 1$, and $p - \delta$, all of which lie between $\delta$ and $\Delta$. As in the previous case removal of fewer than $\kappa$ points cannot disconnect the graph. Finally, if $\Delta = p - \delta - 1$ join each point of $S$ to a point of $U$ in a $1 - 1$ way and add the complete graph to the points of $U$ thereby obtaining the required graph.
Case (I-iii): If $\delta = \Delta = \kappa = 2$, then we may use a cycle on $p$ points. Due to the complexity of this case, we require a division into subcases.

First suppose that $\Delta = \delta \geq 3$, $\delta$ is odd, and $\delta \neq \frac{1}{2}p - 1$. Since $\rho \delta$ is even, $p$ must be even. If $\kappa$ is even, then by Lemma 3 let $\hat{G}$ be a graph on $\frac{1}{2}p$ points with point-connectivity $\delta - 1$, $\kappa$ points of degree $\delta - 1$ and $\frac{1}{2}p - \kappa$ points of degree $\delta$. The desired graph is formed by joining the $\kappa$ points of degree $\delta - 1$ of two copies of $\hat{G}$ by $\kappa$ lines in a one-to-one fashion. Since $\delta - 1 \geq \kappa$ the removal of no fewer than $\kappa$ points disconnects the graph. If $\kappa$ is odd then, since $\delta < \frac{1}{2}p - 1$, we can write $p = p_1 + p_2$ where $p_1$ and $p_2$ are odd and $p_1, p_2 \geq \delta + 2$. By Lemma 3 graphs $\hat{G}_1$ and $\hat{G}_2$ exist each with point-connectivity $\delta - 1$ and $\kappa$ points of degree $\delta - 1$. The remaining points in $\hat{G}_1$ and $\hat{G}_2$ have degree $\delta$. It only remains to join $\hat{G}_1$ and $\hat{G}_2$ with $\kappa$ lines between the points of degree $\delta - 1$ in a one-to-one way.

On the other hand, if $\Delta = \delta > 4$, $\delta$ is even, $\kappa \geq 3$, and $\kappa$ is odd, then let $H_1$ be a graph of the type constructed in the proof of Lemma 3 given in [11] such that $H_1$ has $\delta + 1$ points, point-connectivity $\delta - 1$, one point of degree $\delta$, and $\delta$ points of degree $\delta - 1$. The point $u$ of degree $\delta$ is adjacent to all other points. There are two nonadjacent points $v$ and $w$. Choose any point $t$ different from $u$, $v$, and $w$. Since the degree of $v$ is $\delta - 1$ it follows that $v$ and $t$ are adjacent. Remove the line $xy$ between $v$ and $t$ and insert $xy$ between $v$ and $w$ thereby forming a graph $H_1'$ with two points of degree $\delta$, one point of degree $\delta - 2$, and $\delta - 2$ points of degree $\delta - 1$. Clearly.

$$\delta - 2 \geq \kappa(H'_1) \geq \kappa(H_1 - x) \geq \kappa(H_1) - 1 = \delta - 2;$$

thus $\kappa(H'_1) = \delta - 2$. Let $s = \delta - 2 - (\kappa - 1)$; hence $s$ is even and by examination of $H'_1$ it can be seen that there are $\frac{1}{2}(\delta - 4)$ pairs of nonadjacent points of degree $\delta - 1$ in $H'_1$. We may locate $\frac{1}{2}s$ of these pairs for future reference.

Now as $p - \delta - 1 \geq \delta + 1$, we can define $H'_2$ as a graph of the type constructed in the proof of Lemma 3 where $H'_2$ has $p - \delta - 1$ points, point-connectivity $\delta - 1$, $\delta$ points of degree $\delta - 1$, and the rest of degree $\delta$. Again by direct examination of $H'_2$ it can be seen that the $\delta$ points of degree $\delta - 1$ may be arranged in pairs where the members of each pair are nonadjacent. Locate $\frac{1}{2}s$ of these pairs for future reference. Now form the graph $G'$ by first joining in a one-to-one way $\delta - 2$ points in $H'_1$ of degree $\delta - 1$ to $\delta - 2$ points in $H'_2$ of degree $\delta - 1$ such that the collection of $s$ points in $H'_1$ which were organized in pairs are joined to the corresponding $s$ points in $H'_2$. Then join the point in $H'_1$ of degree $\delta - 2$ to the two remaining points of $H'_2$ of degree $\delta - 1$. Of course removal from $G'$ of the collection of the $\delta - 1$ points in $H'_1$, joined to $H'_2$ disconnects $G'$. Suppose that the removal of a set $D$ of fewer than $\delta - 1$ points disconnects $G'$. It is easy to see from the fact that $\kappa(H'_1) = \delta - 2$ and $\kappa(H'_2) = \delta - 1$ that the points of $D$ must all belong to $H'_1$ and must disconnect $H'_1$. Furthermore, the subgraph $\hat{H}'_1$ of $H'_1$ obtained upon the removal of $D$ from $H'_1$ must consist of two connected components since $\kappa(H'_1) = \delta - 1$ and therefore the addition of the line joining $v$ and $t$ to $H'_1$ yields a connected graph. Let these two components be denoted by $J$ and $K$. Then it
follows that \( J \) must contain one of the points \( v \) or \( t \) and \( K \) the other, else the removal of \( D \) from \( H_1 \) yields a disconnected graph. But the degree of \( t \) is \( \delta - 2 \) and the degree of \( v \) is \( \delta - 1 \) so that both points are joined to \( H_2 \). Thus the components \( I \) and \( K \) are each connected to \( H_2 \) so that \( G' - D \) is connected. This contradiction proves that \( k(G') = \delta - 1 \). It only remains to remove the \( s \) lines joining the \( s \) points of \( H'_i \) arranged in pairs of nonadjacent points from the corresponding \( s \) points in \( H'_2 \) and join the pairs within \( H'_1 \) and \( H'_2 \) to restore the degrees to \( \delta \). Since the removal of a line from a graph can reduce the point-connectivity by at most 1 the resulting graph \( G \) has point-connectivity \( k \).

The third alternative is \( \Delta = \delta \geq 4 \), \( \delta \) even, and \( \kappa \) even. By Lemma 3 there are graphs \( G_1 \) and \( G_2 \) on \( p_1 \) and \( p_2 \) points, respectively, with \( p = p_1 + p_2 \), \( p_1, p_2 \geq \delta + 1 \), \( \kappa(G_1) = \kappa(G_2) = \delta - 1 \), and each having \( \kappa \) degree \( \delta - 1 \) points and the rest of degree \( \delta \). Then join the \( \kappa \) points of \( G_1 \) having degree \( \delta - 1 \) with those of \( G_2 \) by adding \( \kappa \) lines.

Now suppose that \( \delta = \frac{1}{2}p - 1 \), \( \delta \) is odd, and \( 2 \leq \kappa \leq \delta \). Then consider a point set \( V \) partitioned into three mutually disjoint subsets \( T, S, \) and \( U \) where \( |T| = \delta + 2 - \kappa \), \( |S| = \kappa \), and \( |U| = \delta \). First add complete graphs on the sets \( T, S, \) and \( U \) consecutively (\( t_1, t_2, \) etc.), join \( t_1 \) to \( s_1, \ldots, s_{\kappa - 2}; f_2 \) to \( s_\kappa, s_1, \ldots, s_2 \); and so on in a cyclic fashion so that \( (\delta + 2 - \kappa) \times (\kappa - 1) \) lines have been added between \( T \) and \( S \). Next remove an internal path of length \( \kappa - 1 \) from \( S \) which begins at \( s_{\kappa + 1} \) and ends at \( s_1 \). It is clear that the point-connectivity of the resulting graph is \( \kappa \).

Finally, let \( \kappa = \delta - 3 \leq \Delta \leq \frac{1}{2}p - 1 \). If \( \delta \) is odd, then the construction given at the beginning of the proof of Case (I-iii) suffices here. If \( \delta \) is even we consider 2 cases: \( p = 2\delta + 3 \) and \( p \geq 2\delta + 4 \). In the event that \( p = 2\delta + 3 \) let \( \delta = \delta_1 + \delta_2 \) where \( \delta_1 \) and \( \delta_2 \) are even and \( G_1 \) and \( G_2 \) be graphs each on \( \delta + 1 \) points with \( \delta_1 \) and \( \delta_2 \) points of degree \( \delta - 1 \) respectively and all others of degree \( \delta \). Finally, join a new point to the \( \delta_1 \) and \( \delta_2 \) points of degree \( \delta - 1 \) in \( G_1 \) and \( G_2 \). If \( p \geq 2\delta + 4 \) first let \( H_1 \) be a graph on \( \delta + 1 \) points with \( \delta - 2 \) points of degree \( \delta - 1 \) and 3 points of degree \( \delta \). Then form \( G_1 \) by adjoining a single point \( s \) to the \( \delta - 2 \) points of degree \( \delta - 1 \). Next let \( G_2 \) be a graph on \( p_2 = \delta + 2 \) points with 2 points of degree \( \delta - 1 \) and the rest of degree \( \delta \). Finally, join \( s \) of \( G_1 \) to the 2 points of \( G_2 \) having degree \( \delta - 1 \).

**Case (II-i).** We begin with a point set \( V \) partitioned into three mutually
disjoint subsets $S$, $T$, and $U$ with $|S| = \kappa$, $|T| = \delta + 1 - \kappa$, and $|U| = p - \delta - 1$. Add complete graphs to the points of $T$ and the points of $U$ and join each point of $S$ to each point of $U$ and each point of $T$. Next, with the points of $S$ and $U$ labeled $s_1, \ldots, s_\kappa$ and $u_1, \ldots, u_{p - \delta - 1}$, if $k = \kappa - (2\delta - p + 2) > 0$ remove the lines between $s_1$ and $u_1$, $s_2$ and $u_1$, $s_1$ and $u_2$, $s_2$ and $u_2$, $s_3$ and $u_3$, and so on until we have removed $k$ lines from each point of $U$. In so doing a total of $k(p - \delta - 1)$ lines are removed leaving the first $e$ points of $S$ with degree $p - \kappa - [(p - \delta - 1)k/K] - 1$ and the rest with degree $p - \kappa - [(p - \delta - 1)k/K]$. Each point of $U$ has degree $K$ following this cyclic procedure. Since the inequality $\Delta \geq (p - \kappa) - (p - \delta - 1)k$ is satisfied here it follows that $\Delta \geq p - \kappa - [(p - \delta - 1)k/K]$. Now if $p - \kappa - [(p - \delta - 1)k/K] > 0$ we must provide an increase of at least $r = \delta + \kappa - p + [(p - \delta - 1)k/K]$ on $\kappa - e$ points of $S$ and $r + 1$ on the remaining $e$ points of $S$ in order for the points of $S$ to have degree at least $\delta$. In Case (II-ii) we must provide exactly these increases in degree. Note that

$$r = \delta + \kappa - p + \left\lfloor \frac{(p - \delta - 1)k}{\kappa} \right\rfloor$$

$$= \kappa - \left( p - \delta - \left\lfloor \frac{(p - \delta - 1)k}{\kappa} \right\rfloor \right) < \kappa - 2.$$

Let's consider $\delta = \Delta \neq p - 1$ first. If $\kappa$ is even then $p$ and $k$ have the same parity. If $p$ is odd so that $\delta$ is even and $p - \delta - 1$ is even then $(p - \delta - 1)k$ is even. Since $(p - \delta - 1)k = [(p - \delta - 1)k/\kappa] + e$ and $\kappa$ is even so is $e$. Thus $kr$ and $e$ have the same parity. If $p$ is even then $k$ is even and so $(p - \delta - 1)k$ is even. As in the previous case $e$ is even so that $kr$ and $e$ have the same parity. If $\kappa$ is odd then $p$ and $k$ have opposite parity. If $p$ is odd then $k$ and $\delta$ are even and it follows as in the preceding cases that $kr$ and $e$ have the same parity. If $p$ is even then consider $p - \delta - 1$. If $p - \delta - 1$ is even then $[(p - \delta - 1)k/\kappa]$ and $e$ have the same parity. But $r$ and $[(p - \delta - 1)k/\kappa]$ have the same parity so that $kr$ and $e$ have the same parity. If $p - \delta - 1$ is odd then $p$ and $\delta$ have the same parity. Since $p\delta$ is even, $p$ and $\delta$ must both be even. Now $\kappa$ is odd so $k$ and $p$ have opposite parity and $k$ is odd. But then $(p - \delta - 1)k$ is odd and $[(p - \delta - 1)k/\kappa]$ and $ke$ have opposite parity. Also because $\kappa$ is odd, $[(p - \delta - 1)k/\kappa]$ and $e$ have opposite parity. Finally $r$ and $[(p - \delta - 1)k/\kappa]$ have opposite parity so that $kr$ and $e$ have the same parity. Thus, by Lemma 3, we may add a graph on the points of $S$ such that the degrees of the points of $S$ all become $\delta$.

When $\delta < \Delta \leq p - 1$, if $kr$ and $e$ have opposite parity then add a graph with $e + 1$ points of degree $r + 1$ and the rest (if any) of degree $r$ so that all but one of the points has degree $\delta$, the remaining point having degree $\delta + 1$. Of course if $kr$ and $e$ are even we may, as in the previous case, add a graph to $S$ which forces each point of $S$ to have degree $\delta$. Finally choose a point of $S$ and, if necessary, increase its degree to $\Delta$ by adding lines internal to $S$ and/or lines from $S$ to $T \cup U$. 
It is clear from the fact that $T \cup U$ are complete that the point-connectivity of any of the graphs constructed in Case (II-i) must be $\kappa$.

**Case (II-ii):** Use the graph $K_p$. □

**Conclusion**

This theorem contains the solutions to a variety of extremal problems. Namely if any three of the parameters $p$, $\Delta$, $\delta$, and $\kappa$ are given, then one can calculate the complete range of possible values for the unspecified parameter. The end points of such a range are the solutions to the maximum and minimum extremal problems.

Unfortunately the extrema cannot be given by simple, explicit formulas for all the parameters considered here. However, the most interesting extremal values are probably those concerning $\kappa$. An example of the complexity of the extremal solution is given in the following corollary, the proof of which follows by algebraic manipulation of the inequalities of the theorem.

**Corollary.** Let $\min \kappa(p, \Delta, \delta)$ denote the minimum value of $\kappa$ among all $(p, \Delta, \delta)$ graphs.

1. For $\delta < \frac{1}{2}p$ and $\Delta \leq p - \delta - 2$, 
   $$\min \kappa(p, \Delta, \delta) = 0.$$

2. For $\delta < \frac{1}{2}p$ and $\Delta > p - \delta - 2$, 
   $$\min \kappa(p, \Delta, \delta) = 1.$$

3. For $p - 1 \neq \delta < \frac{1}{2}p$, 
   $$\min \kappa(p, \Delta, \delta) = \min \left\{ \left[ \frac{1}{2}(-\Delta - \delta - 1) + \sqrt{(\Delta - \delta - 1)^2 + 4(p - \delta - 1)(2\delta - p + 2)} \right], \right. \left. (2\delta - p + 2) \right\}.$$

4. For $\delta = p - 1$, 
   $$\min \kappa(p, \Delta, \delta) = p - 1.$$

**References**