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ON PARAMETER FREE INDUCTION SCHEMAS

R. KAYE, J. PARIS AND C. DIMITRACOPOULOS

Abstract. We present a comprehensive study of the axiom schemas \( \Sigma_n^+, B\Sigma_n^+ \) (induction and collection schemas for parameter free \( \Sigma_n \) formulas) and some closely related schemas.

Introduction. This paper is divided into three main sections. In §1 we investigate the relationship between \( \Sigma_n^+, B\Sigma_n^+ \) and their parameter counterparts \( \Sigma_n, B\Sigma_n \). In §2 we prove a series of conservation results which enable us to give axiomatizations of the \( \Sigma_{n+2} \) and \( \Sigma_{n+1} \) consequences of \( \Sigma_n \). Finally, in §3 we investigate the quantifier complexity and finite axiomatizability of these schemas.

§0. Preliminaries and summary of main results. We work in the usual first-order language of arithmetic \( \{0, 1, +, \cdot, <\} \). \( P^- \) denotes a finite set of \( \Pi_1 \) axioms such that if \( M \models P^- \), then \( M \) is the nonnegative part of a commutative discretely ordered ring (see [4] for a precise definition of \( P^- \)). \( \Sigma_n^- \) is \( P^- \) together with the schema
\[
\theta(0) \land \forall x(\theta(x) \rightarrow \theta(x+1)) \rightarrow \forall x\theta(x), \quad \theta \in \Sigma_n.
\]
\( B\Sigma_n^- \) is \( P^- + \Sigma_0 \) together with the schema
\[
\forall x\exists y\theta(x, y) \rightarrow \forall z\exists t\forall x < z \exists y < t\theta(x, y), \quad \theta \in \Sigma_n.
\]
\( L\Sigma_n^- \) is \( P^- \) together with the schema
\[
\exists x\theta(x) \rightarrow \exists x(\theta(x) \land \forall y < x \neg \theta(y)), \quad \theta \in \Sigma_n.
\]
\( I\Pi_n^-, B\Pi_n^-, L\Pi_n^- \) are defined similarly. The corresponding axiom schemas \( \Sigma_n, B\Sigma_n, L\Sigma_n \), etc., where \( \theta \) may contain parameters, were studied by Paris and Kirby. In [4] they proved the following theorem.

Theorem 0.1. For all \( n \),
\[
\begin{align*}
\Sigma_{n+1} \\
\downarrow \\
B\Sigma_{n+1} \iff B\Pi_n \\
\downarrow \\
\Sigma_n \iff I\Pi_n \iff L\Sigma_n \iff L\Pi_n
\end{align*}
\]
Furthermore, the converses to the two vertical arrows are false.

Remark. The collection schema Paris and Kirby considered was different from our \( B\Sigma_n \), but equivalent.
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\[ L\Sigma_{n}^{(k)} \] is \( P^{-} \) together with the schema

\[ \exists x_{1} \cdots \exists x_{k} \theta(x_{1}, \ldots, x_{k}) \rightarrow \exists x_{i} \cdots \exists x_{k} \bigwedge_{i=1}^{k} [\exists y \theta(x_{1}, \ldots, x_{i}, y) \land \forall z < x_{i} \forall y \lnot \theta(x_{1}, \ldots, x_{i-1}, z, y)], \]

with \( \theta \in \Sigma_{n} \) and \( k > 0 \).

\( L\Sigma_{n}^{(-)} \) is just \( P^{-} \) and \( L\Sigma_{n}^{(-)} \) is \( \bigcup_{k \in \mathbb{N}} L\Sigma_{n}^{(k)}. \) \( \Pi_{n}(\mathbb{N}) \) denotes the set of \( \Pi_{n} \) sentences true in the standard model; we define \( \Sigma_{n}(\mathbb{N}), \forall \exists(\mathbb{N}) \) similarly.

The main result of \( \text{§1} \) is

**Theorem 0.2.** For \( n > 0 \),

\[
\begin{align*}
& \vdash B\Sigma_{n+1} & \vdash L\Pi_{n+1} \\
& \vdash I\Sigma_{n} & \vdash \Pi_{n+1} \\
& \vdash B\Sigma_{n} & \vdash L\Pi_{n} \\
& \vdash \Sigma_{n-1} & \vdash \Pi_{n} \\
& \vdash B\Sigma_{n} & \vdash L\Sigma_{n} \\
& \vdash I\Sigma_{n} & \vdash \Pi_{n} \\
& \vdash B\Sigma_{n} & \vdash \Pi_{n} \\
& \vdash I\Sigma_{n} & \vdash \Pi_{n}
\end{align*}
\]

Also \( L\Delta_{0}^{-} \leftrightarrow I\Delta_{0}^{-} \leftrightarrow I\Delta_{0} \leftrightarrow L\Delta_{0} \), and the absence of a directed path from one theory \( T \) to another, \( S \), in the above diagram indicates \( T \nvdash S \).

In \( \text{§2} \) we show the following conservation results:

**Theorem 0.3.** (i) For all \( n \), \( I\Sigma_{n} \) is a \( \Sigma_{n+2} \) conservative extension of \( L\Sigma_{n}^{-} \).

(ii) For \( n > 0 \), \( B\Sigma_{n} \) is a \( \Sigma_{n+2} \) conservative extension of \( B\Sigma_{n}^{-} \).

(iii) For \( n > 0 \), \( I\Sigma_{n} \) is a \( \Sigma_{n+1} \) conservative extension of \( L\Sigma_{n}^{(-)} \).

Furthermore, for \( n > 0 \) all these results are best possible.

The theories \( L\Sigma_{n}^{(-)} \) are of interest in their own right. Using the notion of a set being \( \alpha \)-large for an ordinal \( \alpha \) (see [2]), we prove in \( \text{§2} \)

**Theorem 0.4.** For \( n > 1 \),

\[ I\Delta_{0} + \{ \forall x \exists y([x, y] \text{ is } \omega^{k}_{n-1}\text{-}m\text{-large}) \mid m \in \mathbb{N} \} \]

is an axiomatization of the \( \Pi_{2} \) consequences of \( I\Sigma_{n-1}^{-} + L\Sigma_{n}^{-} \).

**Theorem 0.5.** (i) For \( \theta \in \Sigma_{2} \), if \( I\Sigma_{0} + \exp \vdash \theta \) then \( I\Pi_{1} \vdash \theta \).

(ii) For \( \theta \in \Pi_{2} \), if \( I\Pi_{1} \vdash \theta \) then \( I\Sigma_{0} + \exp \vdash \theta \).

In \( \text{§3} \) we consider axiomatizations of \( I\Sigma_{n}^{-}, B\Sigma_{n}^{-} \) and \( I\Pi_{n}^{-} \), and show

**Theorem 0.6.** Let \( n > 0 \).

(i) \( I\Pi_{n}^{-} \) is \( \Sigma_{n+1} \), but not \( \Pi_{n+1} \)

(ii) \( I\Sigma_{n}^{-} \) is \( \Sigma_{n+1} \lor \Pi_{n+1} \), but not \( \Sigma_{n+1} \lor \Pi_{n+1} \)

(iii) \( B\Sigma_{n}^{-} \) is \( \Sigma_{n+1} \lor \Pi_{n+1} \), but not \( \Sigma_{n+1} \lor \Pi_{n+1} \)

(iv) There is no sentence \( \sigma \) such that \( \Pi_{n+1}(\mathbb{N}) \vdash \sigma \vdash B\Sigma_{n}^{-} \).
Since we know that $\Pi_{n+1}(\mathbb{N}) \models L\Sigma_n^{(\infty)}$, we obtain as a corollary that none of the theories $\Pi_n^0$, $\Sigma_n^0$, $L\Sigma_n^{(\infty)}$, or $B\Sigma_n$ is finitely axiomatizable.

In fact we can improve on Theorem 0.6(i) with the following result:

**Theorem 0.7.** $\Pi_n(\mathbb{N})$ is the only $\Pi_n$ theory (up to deductive equivalence) that implies $\Pi_n^0$.

We conclude this section with some definitions and results which will be important for subsequent work.

The function $\langle x, y \rangle = \frac{1}{2}(x + y) \cdot (x + y + 1) + x$ will serve as a pairing function in $P^- + I\Sigma_0$. Using it we see that, in $P^- + I\Sigma_0$, every $\Sigma_n$ formula is equivalent to a formula of the form $\exists x_1 \forall x_2 \cdots \theta$, with $n$ alternations of quantifiers and $\theta \in \Sigma_0$. $x \in y$ is the formula expressing "$2^x$ appears in the binary expansion of $y$".

**Lemma 0.8 (see [4]).** If $\phi \in \Sigma_n$, then $\forall x < y \phi$ is equivalent in $B\Sigma_n$ to a $\Sigma_n$ formula.

**Definition (see [3]).** Let $M = P^-, n \in \mathbb{N}$, and $a \in M$.

(i) $K^n(M,a)$ is the substructure of $M$ with domain $\{b \in M \mid b$ is $\Sigma_n$ definable in $(M,a)\}$.

(ii) $I^n(M,a)$ is the substructure of $M$ with domain $\{b \in M \mid$ there is $c \in K^n(M,a)$ such that $M \models b < c\}$.

(iii) $K^n(M)$ and $I^n(M)$ denote $K^n(M,0)$ and $I^n(M,0)$ respectively.

**Theorem 0.9 (see [3] and [4]).** Let $M$ be a nonstandard model of $I\Sigma_n$, $n > 0$. Then for all $a \in M$:

(i) $K^n(M,a) \subset_n M$, $I^n(M,a) \subset_{n-1} M$ and $I^n(M,a) \models \Pi_{n+1}(M)$; and

(ii) if $K^n(M,a) \not\subset \mathbb{N}$, then $K^n(M,a) \models I\Sigma_n + \neg B\Sigma_n$ and $I^n(M,a) \models B\Sigma_n + \neg I\Sigma_n$.

**Proposition 0.10.** $I\Sigma_0 \models I\Sigma_0$.

**Proof.** Assume $I\Sigma_0$ and $\theta(b,a)$, $\theta \in \Sigma_0$. By induction on $w$ we can show that

$$\forall w \forall u < w \forall v < w [\langle u, v \rangle \leq w \land \exists x \leq u \theta(x, v)$$

$$\rightarrow \exists x \leq u(\theta(x, v) \land \forall y < x \neg \theta(y, v))]$$

Hence, applying ($+$) for $w = \langle b, a \rangle$,

$$\exists x \leq b(\theta(x, a) \land \forall y < x \neg \theta(y, a))$$

It follows that $L\Sigma_0$, i.e. $I\Sigma_0$, holds.

§1. The relationships between $I\Sigma_n^-$, $III_n^-$, etc. The proof of Theorem 0.2, which sums up the relationships between the various schemas we have introduced, is split into several propositions.

**Proposition 1.1.** For $n > 0$, $I\Sigma_n^- \models I\Sigma_{n-1}^-$.

**Proof.** The proposition is clearly true for $n = 1$. So assume $n > 1$, $I\Sigma_n$ and $\exists \theta(x, a)$ with $\theta \in \Sigma_{n-1}$. We may assume $\theta$ is $\exists z \phi(x, y, z)$, where $\phi \in \Pi_{n-2}$. By induction on $w$ we can show that

$$\forall w \forall t \forall x < t \forall y < t \forall z < t [\langle x, y \rangle \leq w \rightarrow [(\langle x, y, z \rangle \in t \rightarrow \phi(x, y, z))]$$

($+$)

$$\land (\exists z \phi(x, y, z) \rightarrow \exists z < t(\langle x, y, z \rangle \in t))]$$

Now let $b$ be such that $\theta(b, a)$ and let $t_0$ satisfy ($+$) when $w = \langle b, a \rangle$. Then $\{x < t_0 \mid \exists z < t_0(\langle x, a, z \rangle \in t_0)\}$ is a nonempty $\Sigma_0$-definable set and so has a least element $b_0$.  

It is easy to check that
\[ \theta(b_0, a) \land \forall x < b_0 \lnot \theta(x, a). \]

**Proposition 1.2.** For \( n > 0 \), \( B\Sigma_n^- \vdash I\Sigma_{n-1}^- \).

**Proof** (induction on \( n \)). The proposition is clearly true for \( n = 1 \). Now assume \( n > 1 \) and the result is true for \( n - 1 \). Assume \( B\Sigma_n^- \) and \( \exists x \exists y \theta(x, y, a) \) with \( \theta \in \Pi_{n-2}^- \). It suffices to show that there is a least such \( x \). Let \( b \) be such that \( \exists y \theta(b, y, a) \). Clearly,
\[
\forall u \exists y [(\exists t_0 < u \exists t_1 < u (u = \langle t_0, t_1 \rangle \land \theta(t_0, y, t_1)) \land (- \exists w \exists t_0 < u \exists t_1 < u (u = \langle t_0, t_1 \rangle \land \theta(t_0, w, t_1)) \land y = 0)].
\]

Hence, by \( B\Sigma_n^- \), there exists \( c \) such that for \( s = \langle b, a \rangle \)
\[
\forall u < s \exists y < c \varphi(u, y),
\]
where \( \varphi(u, y) \) is the formula in \([\ldots]\) above. Then
\[
\forall x < b [\exists y \theta(x, y, a) \iff \exists y < c \varphi(x, y)],
\]
Since \( \exists y < c \varphi(x, y, a) \) is \( \Pi_{n-2}^- \) (for \( n = 2 \) this is clear and for \( n > 2 \) it follows from the fact that \( I\Sigma_{n-2}^- \vdash B\Sigma_{n-2}^- \)), the required least \( x \) exists by \( I\Sigma_{n-2}^- \).

**Proposition 1.3.** For \( n \geq 0 \), \( L\Pi_n^- \vdash I\Sigma_n^- \) and \( L\Sigma_n^- \vdash L\Pi_n^- \).

**Proof.** Obviously, it suffices to prove \( L\Pi_n^- \vdash I\Sigma_n^- \). Assume \( L\Pi_n^- \) and
\[
\theta(0) \land \forall x (\theta(x) \rightarrow \theta(x + 1)) \land \exists x \lnot \theta(x), \quad \theta \in \Sigma_n.
\]
Then \( \exists x (\lnot \theta(x) \land \forall y < x \theta(y)) \), contradiction. Hence \( I\Sigma_n^- \) holds.

**Proposition 1.4.** For \( n \geq 0 \), \( I\Sigma_n^- \vdash L\Pi_n^- \) and \( L\Pi_n^- \vdash L\Sigma_n^- \).

**Proof.** We use induction on \( n \). Assume \( I\Sigma_n^- \) and
\[
\exists x \theta(x) \land \lnot \exists x \theta(x) \land \forall z < x \lnot \theta(z), \quad \theta \in \Pi_n.
\]
We may assume \( \theta(x) \) is \( \forall y \varphi(x, y) \) with \( \varphi \in \Sigma_{n-1}^- \). By induction on \( w \) we can show that
\[
\forall w \exists t \forall x \leq w \exists y \leq t \lnot \varphi(x, y)
\]
(notice that \( \exists t \forall x \leq w \exists y \leq t \lnot \varphi(x, y) \) is equivalent to a \( \Sigma_n \) formula; if \( n = 1 \), this is immediate, and for \( n > 1 \) it follows by \( B\Sigma_{n-1}^- \)). Hence \( \forall x \exists y \lnot \varphi(x, y) \), i.e. \( \forall x \lnot \theta(x) \), contradiction. The proof of the second part is similar.

**Proposition 1.5.** For \( n > 0 \), if \( M \models \Pi_n^- \) and the \( \Sigma_n \)-definable elements in \( M \) are cofinal in \( M \), then \( M \vDash I\Sigma_n \).

**Proof.** Assume that \( M \models \Pi_n^- \) and the \( \Sigma_n \)-definable elements are cofinal in \( M \). By induction on \( m \leq n \) we show that \( M \models I\Sigma_m \). This is clearly true for \( m = 0 \). Now suppose that \( M \models I\Sigma_m \), \( m < n \), and \( M \models \psi(b, a) \) with \( \psi \in \Pi_{m+1}^- \). We may assume \( \psi \) is \( \forall z \theta(x, y, z) \), where \( \theta \in \Sigma_m \). Let \( c \) be \( \Sigma_n \)-definable, say by the formula \( \varphi(x) \), and greater than \( \langle b, a \rangle \). By considering the least \( r \) such that
\[
M \models \exists x \exists w [\varphi(w) \land x = 2^{w-r}]
\]
we see that \( 2^c \) exists in \( M \). Hence
\[
M \models \exists r \exists s \exists w [\varphi(w) \land \forall x < w \forall y < w (\langle x, y \rangle < w \rightarrow \langle x, y \rangle \in t \lor \exists z < s \lnot \theta(x, y, z))].
\]
By Lemma 0.8 this formula is $\Sigma_n$ (since $M \models B\Sigma_m$ if $m > 0$), and so by $L\Sigma_n^-$ there is a least such $t$, $t_0$ say. Then

$$M \models \forall x < c \forall y < c[\langle x, y \rangle < c \rightarrow (\langle x, y \rangle \in t_0 \leftrightarrow \forall z \theta(x, y, z)]$$

and hence, by $L\Sigma_0$, there is a least $x$ such that $M \models \forall z \theta(x, a, z)$, as required.

**Proposition 1.6.** For $n > 0$, $L\Sigma_n^+ \vdash \Pi_n^+$. 

**Proof.** Assume $L\Sigma_n^-$ and $\theta(0) \land \forall x (\theta(x) \rightarrow \theta(x + 1)) \land \neg \theta(a)$, where $\theta \in \Pi_n$, say $\theta$ is $\forall y \varphi(x, y)$ with $\varphi \in \Sigma_{n-1}$. Then

$$\exists w \exists x < w \exists y < w[\langle x, y \rangle \land \neg \varphi(x, y)]$$

By $L\Pi_n^-$, there is a least such $w$, $w_0$ say. Clearly, $w_0$ is definable by a $\Sigma_n$ formula $\psi(x)$. Using $L\Pi_n^-$, let $t_0$ be the least $t$ such that

$$\forall w[\psi(w) \rightarrow \forall x \leq w(x \in t \lor \forall y \varphi(x, y))]$$

(notice there is some such $t$, as $2^{w_0}$ exists in $L\Sigma_n^-$). Then $\forall x \leq w_0(x \in t_0 \leftrightarrow \neg \theta(x))$ and $L\Sigma_0$ give a contradiction.

**Proposition 1.7.** For $n > 0$, $L\Sigma_n^+ \vdash B\Sigma_n^-$. 

**Proof.** We shall prove an (apparently) finer result, namely that for $\theta(x, y) \in \Pi_{n-1}$

$$L\Sigma_n^- \vdash \forall z (\forall x < z \exists y \theta(x, y) \rightarrow \exists t \forall x < z \exists y < t \theta(x, y))$$

Assume not, say $L\Sigma_n^-$ and $\forall x < a \exists y \theta(x, y)$ hold but $\forall t \exists x < a \forall y < t \neg \theta(x, y)$. Then

$$\exists s \forall t \exists x < s \forall y < t \neg \theta(x, y)$$

and hence, by $L\Pi_n^-$, there is a least such $s$, $s_0$ say. ($\forall t \exists x < s \forall y < t \neg \theta(x, y)$ is equivalent to a $\Pi_n$ formula; for $n = 1$ this is clear and for $n > 1$ it follows from Proposition 1.1, Theorem 0.1 and Lemma 0.8.) But then

$$\exists t \forall x < s_0 - 1 \exists y < t \theta(x, y) \quad \text{and} \quad \exists y \theta(s_0 - 1, y),$$

since clearly $s_0 \leq a$, which gives the contradictory $\exists t \forall x < s_0 \exists y < t \theta(x, y)$.

**Proposition 1.8.** For all $n$, $L\Sigma_n^+ \not\vdash B\Sigma_{n+1}^-$. 

**Proof.** Let $M$ be a nonstandard model of $L\Sigma_{n+1}$ containing nonstandard $\Sigma_{n+1}$-definable elements. By Theorem 0.9(ii), $K^{n+1}(M) \models L\Sigma_n + \neg B\Sigma_{n+1}^-$. We claim that $K^{n+1}(M) \not\models B\Sigma_{n+1}^-$. Indeed, suppose $K^{n+1}(M)$ is a model of $B\Sigma_{n+1}$ and

$$K^{n+1}(M) \models \forall x < a \exists y \varphi(x, y, b),$$

where $\varphi \in \Pi_n$ and $b, a \in K^{n+1}(M)$. Let $\psi_1(x), \psi_2(x) \in \Sigma_{n+1}$ be the formulas defining $a$ and $b$ respectively in $M$ and $K^{n+1}(M)$. Then

$$K^{n+1}(M) \models \forall x \exists y \exists w \exists s[\psi_1(w) \land \psi_2(s) \land (x < w \rightarrow \varphi(x, y, s))]$$

Hence, by $B\Sigma_{n+1}^-$, there exists $c$ such that

$$K^{n+1}(M) \models \forall x < a \exists y < c \exists w \exists s[\psi_1(w) \land \psi_2(s) \land (x < w \rightarrow \varphi(x, y, s))]$$

Therefore, $K^{n+1}(M) \models \forall x < a \exists y < c \varphi(x, y, b)$ and thus $K^{n+1}(M) \models B\Sigma_{n+1}$, a contradiction.

The previous proof shows the following result, which will be needed later.

**Corollary 1.9.** For $n > 0$, if $M \models B\Sigma_n^-$ and all elements of $M$ are $\Sigma_n$-definable, then $M \models B\Sigma_n$. 
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PROPOSITION 1.10. For \( n > 0 \), \( B\Sigma_n \not\subseteq \Pi^n_0 \).

PROOF. Let \( M \) be a nonstandard model of \( \Sigma_n \) containing nonstandard \( \Sigma_n \)-definable elements. Then, by Theorem 0.9(ii) and Proposition 1.5,

\[ I^n(M) \models B\Sigma_n + \neg \Pi^n_0. \]

PROPOSITION 1.11. For all \( n \), \( \Pi^n_{n+1} \not\subseteq B\Sigma_n \).

PROOF. Let \( M \) be a nonstandard model of \( \text{Th}(N) \), \( n > 0 \), and \( a \in M - N \). By Theorem 0.9(i), \( K^n(M, a) \leq_n M \) and hence \( K^n(M, a) \models \Pi^n_{n+1}(M) \). Since \( M \models \text{Th}(N) \), it follows that \( K^n(M, a) \models \Pi^n_{n+1}(N) \) and hence \( K^n(M, a) \models \Sigma^n_{n+2}(N) \). But each instance of \( \Pi^n_{n+1} \) is a \( \Sigma^n_{n+2} \) sentence true in \( N \). Therefore, \( K^n(M, a) \models \Pi^n_{n+1} \). By Theorem 0.9(ii), \( K^n(M, a) \not\models B\Sigma_n \), as required. Since \( B\Sigma_0 \models B\Sigma_1 \), the result holds for \( n = 0 \) as well.

By Propositions 1.1 and 1.11, it follows that \( \Sigma^n_0 \) is strictly stronger than \( \Pi^n_0 \) for \( n > 1 \). This is also true for \( n = 1 \), by the next result and the fact that \( \Sigma^n_1 \models \text{exp} \).

PROPOSITION 1.12. \( \Pi^n_1 \not\models \text{exp} \).

PROOF. Let \( M \) be a nonstandard model of \( \Sigma_1 \) such that \( N < M \). As is well known, there are nonstandard proper initial segments of \( M \) which are not closed under exponentiation. The result follows, since any initial segment of \( M \) is a \( \Delta_0 \)-elementary substructure of \( M \) and hence satisfies \( \Pi_1(M) = \Pi_1(N) \), and so is a model of \( \Pi^n_1 \).

PROPOSITION 1.13. For \( n > 0 \), \( \Sigma^n_{n-1} + \Pi^n_n \not\subseteq B\Sigma^n_0 \).

PROOF. Let \( \{ \exists x \theta_i(x) \mid i \in N \} \) be a recursive axiomatization of \( \Pi^n_0 \) with each \( \theta_i \in \Pi^n_0 \). Let \( f: i \mapsto \theta_i \) have \( \Pi_n \) graph \( f(x) = y \), and let \( M \models \text{PA} \) be countable with a nonstandard element \( \delta \in M \) defined by \( \chi(x) \in \Delta_0 \) and

\[ M \models \forall x \exists! y (f(x) = y \land \forall y \exists x, z (f(y) = z \land \gamma(z, x))) \]

where \( \gamma(\chi, x) \) is the usual \( \Pi_n \) satisfaction relation. Let \( \psi(a) \) be the \( \Pi_n \) formula

\[ \forall w[\chi(w) \rightarrow (\text{len}(a) \geq w \land \forall x \leq \text{len}(a) \forall y (f(x) = y \rightarrow \gamma(y, (a_x))))] \]

where \( (a_x) \) denotes the \( x \)th element of the sequence coded by \( a \), and \( \text{len}(a) \) denotes the length of this sequence. Suppose \( \psi(x) \) is \( \forall y \theta(x, y) \) with \( \theta \in \Sigma^n_{n-1} \), and let \( \phi(a) \) be the formula

\[ \psi(a_1) \land \forall x < a_1 \exists y \leq a_2 \rightarrow \theta(x, y) \land \exists x < a_1 \forall w < a_2 \theta(x, w) \]

where \( a = \langle a_1, a_2 \rangle \) as usual. \( \phi \) is equivalent to a \( \Pi_n \) formula (if \( n = 0 \) this is immediate, and if \( n > 1 \) this only requires \( B\Sigma^n_{n-1} \) and \( M \models \exists! a \phi(a) \)). Let \( a \in M \) be this unique \( a \), and let \( K = K^n(M, a) \). Thus by 0.9 we have

\[ M \models \forall \exists \exists K \models \Pi^n_{n-1} + \neg B\Sigma_n + \theta_i((a_1)_i) \]

for each \( i \in N \); hence \( K \models \Pi^n_n \) also. Furthermore, as \( \phi \) was \( \Pi_n \), \( a \) is the unique element of \( K \) satisfying \( \phi \) in \( K \), so every element of \( K \) is \( \Sigma^n_{n+1} \)-definable in \( K \). We shall use this fact to deduce that \( K \not\models B\Sigma^n_n \), completing the proof.

Suppose, for sake of obtaining a contradiction, that \( K \models B\Sigma^n_n \), \( \xi(x, y, b) \in \Pi^n_{n-1} \) with \( b \in K \), and \( K \models \forall x \exists y \xi(x, y, b) \); since \( b \in K \) it is \( \Sigma^n_{n+1} \)-defined in \( K \) by \( \exists u \forall y \eta(b, u, v) \), say, with \( \eta \in \Sigma^n_{n-1} \). Then

\[ K \models \forall b, u, v \exists y (\xi(x, y, b) \lor \neg \eta(b, u, v)) \]
so by applying $B\Sigma_n^{-}$ in $K$ to the formula $\zeta(x_1, y_1, x_2) \lor \neg \eta(x_2, x_3, y_2)$ we deduce that

$$K \models \forall b, u, t \exists z \forall x \leq t \exists v, y \leq z \zeta(x, y, b) \lor \neg \eta(b, u, v).$$

Hence

$$K \models \forall t \exists z \forall x \leq t \exists y \leq z \zeta(x, y, b),$$

so as $\zeta$ and $b$ were arbitrary we have $K \models B\Sigma_n$, a contradiction.

We conclude this section by showing

**Proposition 1.14.** For $n > 0$, $\Pi_n + B\Sigma_n \not\subseteq \Sigma_n$.

**Proof.** Let $M$ and $a$ be as in the proof of Proposition 1.11. By Theorem 0.9(i), $\Pi^n(M, a) \models \Pi_{n+1}(M)$ and hence, as in the aforementioned proof, $\Pi^n(M, a) \models \Sigma_{n+2}(N)$. Since each instance of $\Pi_{n+1}$ is in $\Sigma_{n+2}(N)$, it follows that $\Pi^n(M, a) \models \Pi_{n+1}$. But, by Theorem 0.9(ii), $\Pi^n(M, a) \models B\Sigma_n + \neg \Sigma_n$, and the result follows.

§2. Conservation results. In this section we investigate the proof theoretic strength of $\Sigma_n^-$ and $B\Sigma_n^-$, proving some conservation results and giving nice axiomatizations of the $\Sigma_{n+1}$ and $\Sigma_{n+2}$ consequences of $\Sigma_n$. Theorem 0.3 will emerge from a series of theorems.

**Theorem 2.1.** $\Sigma_n$ is a $\Sigma_{n+2}$ conservative extension of $\Sigma_n^-$. Furthermore, for $n > 0$ this is best possible.

**Proof.** By Proposition 1.11, $\Sigma_n^- \not\subseteq B\Sigma_n$. But, by Theorem 0.1, $\Sigma_n \models B\Sigma_n$ for $n > 0$ and $B\Sigma_n$ is $\Pi_{n+2}$. Hence, for $n > 0$, $\Sigma_n$ is not a $\Pi_{n+2}$ conservative extension of $\Sigma_n^-$. Now assume that $\Sigma_n$ is not a $\Sigma_{n+2}$ conservative extension of $\Sigma_n^-$ ($n > 0$), say $M \models \Sigma_n^- + \forall x \exists y \varphi(x, y)$, where $\varphi \in \Pi_n$ and $\Sigma_n \models \exists x \forall y \neg \varphi(x, y)$. We may assume $M$ is $\omega_0$-saturated. The idea is to define a sequence of $\Pi_n$ formulas $\lambda_i(x_1, \ldots, x_j)$ satisfying the following conditions:

(i) $M \models \exists \vec{x} \lambda_i(\vec{x})$,

(ii) $\vdash \lambda_{i+1}(x_1, \ldots, x_{j_i + 1}) \rightarrow \lambda_i(x_1, \ldots, x_j)$,

(iii) $\forall k \exists \vec{s}$ such that $k, s \leq j$ and $\vdash \lambda_i(\vec{x}) \rightarrow \varphi(x_k, x_s)$.

(iv) If $\theta(y, z, x_1, \ldots, x_m) \in \Pi_{n-1}$, then, for some $i, k, s$ with $k, s, m \leq j$, either

$$\vdash \lambda_i(x_1, \ldots, x_j) \rightarrow \neg \exists y \exists z \theta(y, z, x_1, \ldots, x_m)$$

or

$$\vdash \lambda_i(x_1, \ldots, x_j) \rightarrow \theta(x_k, x_s, x_1, \ldots, x_m) \land \neg \exists y < x_k \exists z \theta(y, z, x_1, \ldots, x_m).$$

The method of constructing such a sequence of formulas is standard, once we have shown how to arrange (iii) and (iv) for $i + 1$, given that (i) holds for $i$. So suppose $M \models \exists \vec{x} \lambda_i(\vec{x})$. Then to arrange (iii) for $k \leq j_i$ simply put

$$\lambda_{i+1}(\vec{x}, x_{j_i+1}) = \lambda_i(\vec{x}) \land \varphi(x_k, x_{j_i+1}).$$

To arrange (iv), if $M \models \neg \exists \vec{x} \exists \exists z \lambda_i(x_1, \ldots, x_{j_i}) \land \theta(y, z, x_1, \ldots, x_m)$, just put

$$\lambda_{i+1}(\vec{x}) = \lambda_i(\vec{x}) \land \neg \exists y \exists z \theta(y, z, x_1, \ldots, x_m).$$
On the other hand, if $M \models \exists \bar{x} \exists y \exists z (\lambda_i(x_1, \ldots, x_j) \land \theta(y, z, x_1, \ldots, x_m)$, then, by $\mathcal{L} \Pi_\omega^n$, in $M$ there is a least $\langle \bar{x}, y, z, u \rangle$ such that

$$\lambda_i(\bar{x}) \land \theta(y, z, \bar{x}) \land \neg \exists t < y \land u \exists s \theta(t, s, \bar{x}).$$

But then clearly for this least $(3 + j_i)$-tuple we must have $u = 0$. Hence we may take

$$\lambda_i(x_1, \ldots, x_j) \land \theta(x_{j_i + 1}, x_{j_i + 2}, x_1, \ldots, x_m) \land \forall t < x_{j_i + 1} \forall s \neg \theta(t, s, x_1, \ldots, x_m)$$

for $\lambda_{i+1}$.

Having constructed this sequence and using the $\omega_0$-saturation of $M$, let $a_1, a_2, \ldots \in M$ be such that

$$M \models \lambda_i(a_1, \ldots, a_j)$$

for $i = 1, 2, \ldots$.

Then, by (iv), the $a_i$'s form the domain of a substructure $K$ of $M$, $K \prec_n M$ and $K \models \Sigma_n$. Finally, (iii) gives $K \models \forall \exists \forall \varphi(x, y)$ and hence the required contradiction.

REMARKS. (i) An alternative proof of this theorem can be given by showing that if $M \models \Sigma_\omega^n$, then $K^{n+1}(M) \prec_n M$ and $K^{n+1}(M) \models \Sigma_\omega^n$.

(ii) Since the axioms of $\Sigma_\omega^n$ are of the form $\Sigma_{n+1} \lor \Pi_{n+1}$, a surprising consequence of this result is that the consequences of $\Sigma_\omega^n$ which are Boolean combinations of $\Sigma_{n+1}$ sentences form an axiomatization of the $\Sigma_{n+2}$ consequences of $\Sigma_\omega^n$. Of course, since $\Sigma_\omega^n$ is itself $\Pi_{n+2}$, any consequence of $\Sigma_\omega^n$ follows from the $\Pi_{n+2}$ consequences of $\Sigma_\omega^n$.

(iii) The first author has generalized Theorem 2.1 by considering the theories $\Sigma_\omega^n$ and $\Pi_\omega^n$ ($E_n$ is the class of bounded $\exists_n$ formulas—see [8]) to obtain the following results:

a) $\Sigma_\omega^n$ is a $\exists_{n+2}$ conservative extension of $\Sigma_\omega^n$.

b) $\Pi_\omega^n$ is a $\forall E_n$ conservative extension of $\Pi_\omega^n$.

Since $\Sigma_\omega^1 \models \Delta_0 + \exp$ (result of the same author) and $\Delta_0 + \exp$ is $\forall \exists$ axiomatizable, it follows that $\Sigma_\omega^1 \models \Delta_0 + \exp$. But $\Delta_0 + \exp$ proves the MRDP theorem (see [1]) and hence that $\Sigma_\omega^1 \models \exists 1$, from which we may conclude that $\Sigma_\omega^1 \models \Pi_\omega^1$.

More on the first author's results can be found in his thesis at Manchester University.

For $\Pi_\omega^n$ we do not have a result as elegant as Theorem 2.1. However, it is possible to obtain a "parameter free" axiomatization of the $\Sigma_{n+1}$ consequences of $\Sigma_\omega^n$, as we now show.

THEOREM 2.2. For $n > 0$, $\Sigma_\omega^n$ is a $\Sigma_{n+1}$ conservative extension of $\Pi_{\omega_0}(\omega)$, and this is best possible.

PROOF. Clearly $\Sigma_\omega^n \models \Pi_{\omega_0}(\omega)$.

First we show that $\Sigma_\omega^n$ is not a $\Pi_{n+1}$ conservative extension of $\Pi_{\omega_0}(\omega)$. For $n > 1$, by repeating the proof of Proposition 1.1.1 we can show that $\Pi_{\omega_0}(\omega)$ $\not\models \Sigma_{n-1}$. The result follows, since $\Pi_{\omega_0}(\omega)$ is $\Pi_{n+1}$ in $\Pi_{\omega_0}^1$, by Propositions 1.3 and 1.4.

For $n = 1$, notice that if $I \prec_\omega M \models P$ and all the $\Sigma_2$-definable elements of $M$ are in $I$, then $I \models \Pi_{\omega_0}(\omega)$. Hence even the $\Pi_2$ consequence $\forall x \exists y (y = 2^x)$ of $\Pi_{\omega_0}(\omega)$ does not follow from $\Pi_{\omega_0}(\omega)$.

Now assume $n \geq 1$ and $\Sigma_\omega^n \models \exists x \forall y \psi(x, y)$ with $\psi \in \Sigma_{n-1}$, whilst $\Sigma_{n-1}(\omega) \not\models \exists x \forall y \psi(x, y)$, say $M \models \Sigma_{n-1}(\omega) + \forall x \exists y \neg \psi(x, y)$, $M$ $\omega_0$-saturated. Let $\theta_i(x_1, \ldots, x_i)$
enumerate all $\Sigma_n$ formulas. We define a sequence of formulas $\eta_i(x_1, \ldots, x_i)$ as follows. $\eta_1$ is $0 = 0$. Suppose we have found $\eta_m(x_1, \ldots, x_m)$ such that

$$M \models \exists x_1 \cdots \exists x_m \bigwedge_{i=1}^m \left[ \eta_i(x_1, \ldots, x_i) \land \forall z < x_i \neg \eta_i(x_1, \ldots, x_{i-1}, z) \right].$$

Set $\eta_{m+1}(x_1, \ldots, x_{m+1})$ to be $\theta_{m+1}(x_1, \ldots, x_{m+1})$, if

$$M \models \exists x_1 \cdots \exists x_{m+1} \left[ \theta_{m+1}(x_1, \ldots, x_{m+1}) \right. \left. \land \bigwedge_{i=1}^m \left( \eta_i(x_1, \ldots, x_i) \land \forall z < x_i \neg \eta_i(x_1, \ldots, x_{i-1}, z) \right) \right],$$

and $x_{m+1} = 0$ otherwise.

Let $\varphi(x_1, \ldots, x_{m+1})$ be $\bigwedge_{i=1}^{m+1} \eta_i(x_1, \ldots, x_i)$. Then

$$M \models \exists x_1 \cdots \exists x_{m+1} \varphi(x_1, \ldots, x_{m+1}),$$

and so, by $L\Sigma_n^{(\omega)}$,

$$M \models \exists x_1 \cdots \exists x_{m+1} \bigwedge_{i=1}^{m+1} \left[ \exists y \varphi(x_1, \ldots, x_i, y) \land \forall z < x_i \forall y \neg \varphi(x_1, \ldots, x_{i-1}, z, y) \right].$$

By induction on $j \leq m + 1$,

$$M \models \forall x_1 \cdots \forall x_j \left[ \bigwedge_{i=1}^j \left( \exists y \varphi(x_1, \ldots, x_i, y) \land \forall z < x_i \forall y \neg \varphi(x_1, \ldots, x_{i-1}, z, y) \right) \right. \left. \land \bigwedge_{i=1}^j \left( \eta_i(x_1, \ldots, x_i) \land \forall z < x_i \neg \eta_i(x_1, \ldots, x_{i-1}, z) \right) \right],$$

from which it follows that

$$M \models \exists x_1 \cdots \exists x_{m+1} \bigwedge_{i=1}^{m+1} \left[ \eta_i(x_1, \ldots, x_i) \land \forall z < x_i \neg \eta_i(x_1, \ldots, x_{i-1}, z) \right].$$

Since $M$ is $\omega_0$-saturated, there exist $a_1, a_2, \ldots \in M$ such that for $i = 1, 2, \ldots$

$$M \models \eta_i(a_1, \ldots, a_i) \land \forall z < a_i \neg \eta_i(a_1, \ldots, a_{i-1}, z).$$

Let $K$ be the substructure of $M$ with domain $\{a_1, a_2, \ldots\}$. From the construction, $K < \Sigma_n M$ and so $K \models \forall x \exists y \neg \psi(x, y)$. Furthermore, for any nonempty $\Sigma_n$ set in $K$ (and hence in $M$) the construction ensures that the least element of this set in $M$ is also in $K$, which gives $K \models \Sigma_n$ and the required contradiction.

Notice that since $L\Sigma_n^{(\omega)}$ is $\Sigma_n^{+1}$, this result gives us a natural axiomatization of the $\Sigma_n^{+1}$ consequences of $\Sigma_n$ just as $\Sigma_n^{+}$ did for $\Sigma_n^{+2}$.

**COROLLARY 2.3.** Let $n > 0$.

(i) $\Sigma_n^{+}$ is strictly stronger than $L\Sigma_n^{(\omega)}$

(ii) $B\Sigma_n^{+}$ is a $\Delta_n^{+}$ conservative extension of $\Sigma_n^{+}$, and this is best possible

(iii) $B\Sigma_n^{+} + B\Sigma_n^{+}$ is a $\Delta_n^{+}$ conservative extension of $\Sigma_n^{+}$, and this is best possible.

**PROOF.** (i) By Theorems 2.1 and 2.2, noting that $\Sigma_n^{+} \models L\Sigma_n^{(\omega)}$, since $L\Sigma_n^{(\omega)}$ is a $\Sigma_n^{+1}$ consequence of $\Sigma_n$.

(ii) This follows from Theorem 2.1 and the fact that $B\Sigma_n^{+}$ is a $\Pi_n^{+2}$ conservative extension of $\Sigma_n$ (see [5]). Furthermore, both of these results are optimal.
(iii) Immediate by Theorems 2.1 and 0.1. It is best possible, since $\Sigma_{n+1}^n \not\vdash \Sigma_{n+1}^n$ (by Proposition 1.11) and $\Sigma_{n+1}^n$ can be written in the $\Pi_{n+2}$ form.

**Theorem 2.4.** For $n > 0$, $\Sigma_{n+2}^n$ is a $\Sigma_{n+2}^n$ conservative extension of $\Sigma_{n+1}^n$.

**Proof.** The proof is similar to that of Theorem 2.1, but instead of arranging that $K \models \Sigma_n$ and $K \not\models \Pi_{n+1} M$ we arrange that $K \models \Sigma_n$ and $K \not\models \Pi_{n+1} M$. Again $K$ will have domain $\{a_1, a_2, \ldots\}$, where $M \models \lambda_i(a)$ for a suitable sequence of $\Pi_n$ formulas $\lambda_i$. To arrange that $K \not\models \Pi_{n+1} M$, we just ensure that for each $\Pi_n$ formula $\varphi(x, y)$ there is some $i$ such that either

$$M \models \exists \bar{x} (\lambda_i(\bar{x}) \land \exists \bar{y} \varphi(\bar{x}, \bar{y}))$$

or

$$M \models \forall \bar{x} (\lambda_i(\bar{x}) \rightarrow \neg \exists \bar{y} \varphi(\bar{x}, \bar{y})).$$

To arrange that $K \models \Sigma_n$, suppose $\lambda_i(\bar{x})$ is $\Pi_n$, $M \models \exists \bar{x} \lambda_i(\bar{x})$ and we are currently considering the formula $\forall u < x_{j} \exists y \theta(u, y, \bar{x})$ with $\theta \in \Pi_{n-1}$ (this suffices since $B\Pi_{n-1} \leftrightarrow \Sigma_n$). If

$$M \models \exists \bar{x} \forall u (\lambda_i(\bar{x}) \land u < x_{j} \land \forall y \neg \theta(u, y, \bar{x})),$$

let the formula in parentheses be $\lambda_{i+1}(\bar{u}, u)$. Otherwise,

$$M \models \forall \bar{x} (\lambda_i(\bar{x}) \rightarrow \forall u < x_{j} \exists y \theta(u, y, \bar{x}))$$

and hence

$$M \models \forall w \exists y \exists u < w \exists \bar{x} < w[w = \langle u, \bar{x} \rangle \land (u < x_{j} \land \lambda_i(\bar{x}) \rightarrow \theta(u, y, \bar{x}))].$$

By $\Sigma_{n+1}^n$,

$$M \models \forall z \exists t \forall w < z \exists y < t \exists u < w \exists \bar{x} < w[w = \langle u, \bar{x} \rangle \land (u < x_{j} \land \lambda_i(\bar{x}) \rightarrow \theta(u, y, \bar{x}))].$$

Let $M \models \lambda_i(\bar{a})$ and $t_0$ witness the above formula when $z = \langle a_j, \bar{a} \rangle$. Then

$$M \models \lambda_i(\bar{a}) \land \forall u < a_j \exists y < t_0 \theta(u, y, \bar{a}).$$

Now let $\lambda_{i+1}(\bar{x}, s)$ be the $\Pi_n$ formula equivalent to

$$\lambda_i(\bar{x}) \land \forall u < x_{j} \exists y < s \theta(u, y, \bar{x})$$

in $\Sigma_{n+1}^n$. Since $M \models \Sigma_{n+1}^n$ (by Proposition 1.2), $M \models \lambda_{i+1}(\bar{a}, t_0)$. Furthermore, since $K \not\models \Pi_{n+1}$, $M \models \Sigma_{n+1}^n$ and hence $K \models \lambda_{i+1}(\bar{a}, t_0)$, which gives $K \models \Sigma_{n+1}^n$, as required.

Theorems 2.1 and 2.2 give natural axiomatizations of the $\Sigma_{n+2}$ and $\Sigma_{n+1}$ consequences of $\Sigma_n$. These axiomatizations are especially nice in that they themselves have the form of induction axioms. A similar result has long been known for $\Pi_2$, namely

**Theorem 2.5 (see [5]):** For $n > 0$,

$$\Delta_0 + \{\forall x \exists y([x, y] \text{ is } \omega_m^n\text{-large}) \mid m \in \mathbb{N}\}$$

is an axiomatization of the $\Pi_2$ consequences of $\Sigma_n$.

For the definitive source of information on largeness, see [2].

We shall now sketch the proof of
THEOREM 2.6. For \( n > 1 \),
\[
\mathrm{I} \Delta_0 + \{ \forall x \exists y([x, y] \text{ is } \omega_n^{m} - \text{large}) \mid m \in \mathbb{N} \}
\]
is an axiomatization of the \( \Pi_2 \) consequences of \( \mathrm{I} \Sigma_n - 1 + L \Sigma_n^{(k)} \).

PROOF. The result is a consequence of the following two lemmas.

**Lemma 1.** For \( n > 1 \) and \( m \in \mathbb{N} \)
\[
\mathrm{I} \Sigma_{n - 1} + L \Sigma_n^{(k)} \vdash \forall x \exists y([x, y] \text{ is } \omega_n^{m} - \text{large}).
\]

**Proof.** We first introduce some notation. All ordinals referred to are assumed to be less than \( \varepsilon_0 \). If ordinals \( \alpha, \beta \) are in Cantor normal form, say
\[
\alpha = \omega^{\alpha_1} \cdot m_1 + \cdots + \omega^{\alpha_r} \cdot m_r, \quad \beta = \omega^{\beta_1} \cdot n_1 + \cdots + \omega^{\beta_s} \cdot n_s,
\]
with \( \alpha_1 > \cdots > \alpha_r, \beta_1 > \cdots > \beta_s \) and \( \alpha_i \geq \beta_1 > \alpha_{i+1} \), then \( \alpha + \beta \) is taken to be
\[
\omega^{\alpha_1} \cdot m_1 + \cdots + \omega^{\alpha_i} \cdot m_i + \omega^{\beta_1} \cdot n_1 + \cdots + \omega^{\beta_s} \cdot n_s.
\]

Let \( G_1(y) \) be \( \forall x \exists y([x, y] \text{ is } \omega^y - \text{large}) \) and \( G_{m + 1}(y) \) be \( \forall \alpha < \varepsilon_0 (G_m(\alpha) \rightarrow G_m(\alpha + \omega^y)) \). Then \( G_{m}(z) \in \Pi_{m+1} \), and using standard results on largeness (see [2] or [5]) it is straightforward to show that
(i) \( \mathrm{I} \Sigma_{n - 1} \vdash G_n(0) \), and
(ii) \( \mathrm{I} \Sigma_1 \vdash \forall t \ G_m(\delta + \omega^t) \rightarrow G_m(\delta + \omega^{t+1}) \).

We show that
\[
(*) \quad \mathrm{I} \Sigma_{n - 1} + L \Sigma_n^{(k)} \vdash G_{n - 1}(\omega^k \cdot m) \quad \text{for all } m \in \mathbb{N},
\]
which gives the required result, since in \( \mathrm{I} \Sigma_{n - 1} \)
\[
G_{n - 1}(\omega^k \cdot m) \rightarrow G_{n - 2}(\omega^k) \rightarrow \cdots \rightarrow G_1(\omega_{n - 2}^k).
\]

We prove (*) by induction on \( m \), working in \( \mathrm{I} \Sigma_{n - 1} + L \Sigma_n^{(k)} \). The result is clear for \( m = 0 \).

Now assume \( G_{n - 1}(\omega^k \cdot m) \land \neg G_{n - 1}(\omega^k \cdot (m + 1)) \). If \( \forall t \ G_{n - 1}(\omega^k \cdot m + \omega^{k - 1} \cdot t) \), then \( G_{n - 1}(\omega^k \cdot (m + 1)) \) by (ii). Hence \( \exists t \neg G_{n - 1}(\omega^k \cdot m + \omega^{k - 1} \cdot t) \).

Using \( L \Sigma_n^{(k)} \) let \( x_1, \ldots, x_k \) be such that
(iii) \( \bigwedge_{i=1}^{k} [\exists y_{i+1} \cdots \exists y_k \neg G_{n - 1}(\omega^k \cdot m + \omega^{k - 1} \cdot x_1 + \cdots + \omega^{k - i} \cdot x_i + \omega^{k - i - 1} \cdot y_{i+1} + \cdots + \omega^0 \cdot y_k) \land \forall z < x_i \forall y_{i+1} \cdots \forall y_k G_{n - 1}(\omega^k \cdot m + \cdots + \omega^{k - i - 1} \cdot y_{i+1} + \cdots + \omega^0 \cdot y_k)] \).

Then \( \neg G_{n - 1}(\omega^k \cdot m + \omega^{k - 1} \cdot x_1 + \cdots + \omega^0 \cdot x_k) \). If \( x_k > 0 \), then by (iii)
\[
G_{n - 1}(\omega^k \cdot m + \omega^{k - 1} \cdot x_1 + \cdots + \omega^1 \cdot x_{k-1} + \omega^0 \cdot (x_k - 1)),
\]
which contradicts (i). Hence \( x_k = 0 \).

Now if \( x_{k-1} > 0 \), then from \( \neg G_{n - 1}(\omega^k \cdot m + \omega^{k - 1} \cdot x_1 + \cdots + \omega^1 \cdot x_{k-1}) \) we have, by (ii), that \( \neg G_{n - 1}(\omega^k \cdot m + \cdots + \omega^1 \cdot (x_{k-1} - 1) + t) \) for some \( t \), contradicting (iii). Hence \( x_{k-1} = 0 \). Continuing in this way we see that \( x_1 = 0 \), which gives the required contradiction.
Lemma 2. Let $n > 1$, and let $M$ be a recursively saturated countable model of

$$
\mathcal{I}_0 + \{ \forall x \exists y ([x, y] \text{ is } \omega^{m+1}_n \text{-large}) | m \in \mathbb{N} \}.
$$

Then $M$ has arbitrarily large initial segments which are models of $\mathcal{I}_{n-1} + L \Sigma^-(k)$.

Proof. First notice that, since $M \models \forall x \exists y ([x, y] \text{ is } \omega^2 \text{-large})$, $M$ is closed under exponentiation.

Let $a \in M$. Then the recursive saturation of $M$ ensures the existence of $b > a$ and $c > N$ such that $M \models ([a, b] \text{ is } \omega^k r \text{-large})$. Using this we can find by a standard construction (see, for example, the proof of Proposition 13 in [5]) a sequence $a < g_0 < g_1 < \cdots < g_v \leq b$ coded in $M$ such that

(i) $\{g_0, g_1, \ldots, g_v\}$ is $\omega^k \cdot d$-large for some $d > N$, and

(ii) for any formula $\theta(y, x) \in \mathcal{A}_d$, $\bar{a} < g_i$ and $i < j_1 < j_2 < \cdots < j_{n-1} \leq v$, $i < k_1 < k_2 < \cdots < k_{n-1} \leq v$,

$$
M \models \forall x_1 < g_1, \exists x_2 < g_{j_1} \cdot Qx_{j_1-1} < g_{j_1-1} \theta(\bar{a}, \bar{x})
$$

$$
\iff \forall x_1 < g_{k_1}, \exists x_2 < g_{k_2} \cdot Qx_{k_1-1} < g_{k_1-1} \theta(\bar{a}, \bar{x}).
$$

Notice then that if $I \subseteq M$ is a limit of this sequence, then for $\theta(\bar{y}, \bar{x}) \in \mathcal{A}_d$ and $\bar{a} < g_i < g_j < \cdots < g_{j_{n-1}} \in I$

(iii) $I \models \forall x_1 \exists x_2 \cdots Qx_{j_n-1} \theta(\bar{a}, \bar{x}) \iff \forall x_1 < g_{k_1}, \exists x_2 < g_{k_2} \cdots Qx_{k_n-1} < g_{k_n-1} \theta(\bar{a}, \bar{x})$

$$
\iff \forall x_1 < g_{p+1}, \exists x_2 < g_{p+2} \cdots Qx_{j_n-1} < g_{p+n-1} \theta(\bar{a}, \bar{x}),
$$

where $p$ is minimal such that $\bar{a} < g_p$. We denote this last expression by

$$
\forall x_1 \exists x_2 \cdots Qx_{j_n-1} \theta(\bar{a}, \bar{x})*.
$$

Now let $\exists x_0 \chi_j(x_0, \bar{y})$, where $\chi_j(x_0, \bar{y}) \in \Pi_{n-1}$, enumerate all $\Sigma_n$ formulas with free variables $y_1, \ldots, y_k$. Our aim is to find $I$ as in (iii) also satisfying $L \Sigma^-(k)$.

Let $t_j$ be minimal such that

$$
t_j = g_{v-n} \vee \exists x_0 \exists \bar{y} [ t_j = \langle x_0, \bar{y} \rangle \wedge \chi_j(x_0, \bar{y})*].
$$

Then, since there are in $M$ coded sets of arbitrarily small nonstandard size containing $\{t_j | j \in \mathbb{N}\}$, by standard properties of largeness we can find $0 \leq s < r < v-n$ such that $\{g_i | s \leq i \leq r\}$ is $\omega^k \cdot f$-large for some $f > N$ and $[g_s, g_r] \cap \{t_j | j \in \mathbb{N}\} = \emptyset$.

Now for each $y \leq t_j$ let $f_j(y)$ be the least $k$-tuple $\langle x_0, y_2, \ldots, y_k \rangle$ such that

$$
\langle x_0, y_2, \ldots, y_k \rangle = g_{t-n} \vee \chi_j(x_0, y_2, \ldots, y_k)*.
$$

Again there is a coded set $S$ of size at most $g_s \sqrt{f}$ such that $S \equiv \{ f_j(y) \} | t_j < g_s$ and $j \in \mathbb{N}$.

Hence, since $\{g_i | s \leq i \leq r\}$ is certainly $(\omega^{k-1} \cdot g_j \cdot f)$-large, we can find $s \leq s' < r' \leq r$ such that $\{g_i | s' \leq i \leq r'\}$ is $\omega^{k-1} \cdot \sqrt{f}$-large and $S \cap [g_{s'}, g_{r'}] = \emptyset$.

Now suppose $I$ is a limit point of $\{g_i | s' \leq i \leq r'\}$. If $I \models \exists x_0 \exists \bar{y} \chi_j(x_0, \bar{y})$, then, for some $x_0, \bar{y} \in I$, $M \models \chi_j(x_0, \bar{y})$. Hence $t_j \in I$; so $i < g_s$. Let $a^j_1 < a^j_2 < \cdots < a^j_{r'-1} \leq v$ such that

$$
\exists \langle x_0, y_2, \ldots, y_k \rangle \leq g_r \chi_j(x_0, z, y_2, \ldots, y_k)*.
$$

Since $S \cap [g_{s'}, g_{r'}] = \emptyset$, $a^j_1$ is the least $z$ such that

$$
\exists \langle x_0, y_2, \ldots, y_k \rangle \leq g_s \chi_j(x_0, z, y_2, \ldots, y_k)*.
$$
and hence also the least $z$ such that

$$I \models \exists x_0 \exists y_2 \cdots \exists y_k \chi_j(x_0, z, y_2, \ldots, y_k).$$

Thus we have also $I \models I_{\Sigma_n-1} + L \Sigma_n^{(-1)}$ for any such $I$. If we now repeat the above argument using $\{g_i | s' \leq i \leq r'\}$ in place of $\{g_i | i \leq v\}$ and $\{\chi_j(x_0, a_i, y_2, \ldots, y_k) | j \in \mathbb{N} \}$, we obtain an $\omega^{k-2} \cdot h$-large set $\{g_i | s'' \leq i \leq r''\}$ such that, for any limit $I$ of this set, $I \models I_{\Sigma_n-1} + L \Sigma_n^{(-2)}$.

Clearly this process can be repeated $k$ times to give an $\omega_k \cdot p$-large set $\{g_i | s'' \leq i \leq r''(k)\}$, for some $p > \mathbb{N}$, such that, for any limit point $I$ of this set, $I \models I_{\Sigma_n-1} + L \Sigma_n^{(-k)}$. Since $p > \mathbb{N}$, such a limit point exists, proving the lemma and Theorem 2.6.

**Corollary 2.7.** In $\text{Id}_0 + \exp$, for $n > 1$,

$$Y(a, b) = \max\{c | [a, b] \text{ is } \omega_n^{k+1}\text{-large}\}$$

is an indicator for models of $I_{\Sigma_n-1} + L \Sigma_n^{(-k)}$.

**Corollary 2.8.** For $n > 1$, $I_{\Sigma_n-1} + L \Sigma_n^{(-k+1)}$ is strictly stronger than $I_{\Sigma_n-1} + L \Sigma_n^{(-k)}$.

**Proof.** Let $M$ be a countable nonstandard model of $P$, $N < a \in M$, and let $b > a$ be minimal such that $[a, b]$ is $\omega_n^{k+1}$-large, so by Corollary 2.7 there is $I \subset_e M$, $a \in I < b$, such that $I \models I_{\Sigma_n-1} + L \Sigma_n^{(-k)}$. However, $I \models \neg \exists y ([a, y]$ is $\omega_n^{k+1}$-large) and hence, by Lemma 1, $I \not\models I_{\Sigma_n-1} + L \Sigma_n^{(-k+1)}$.

**Remark.** Theorem 2.6 fails for $n = 1$, since $L \Sigma_1^{(-\omega)} \not\models \exp$ (see the proof of Theorem 2.2) and hence $L \Sigma_1^{(-\omega)} \not\models \forall x \exists y ([x, y]$ is $\omega^2$-large).

We now prove Theorem 0.5.

**Theorem 2.9.** (i) For $\theta \in \Sigma_2$, if $I_{\Sigma_0} + \exp \models \theta$ then $I_{\Pi_1} \models \theta$.

(ii) For $\phi \in \Pi_2$, if $I_{\Pi_1} \models \phi$ then $I_{\Sigma_0} + \exp \models \phi$.

**Proof.** (i) Suppose not, say $I_{\Sigma_0} + \exp \models \exists x \forall y \psi(x, y)$ but $I_{\Pi_1} \not\models \exists x \forall y \psi(x, y)$ has a model, $M$ say, where $\psi \in \Sigma_0$. Then $K^1(M) \subset M$, so $K^1(M) \models I_{\Sigma_0} + \neg \exists x \forall y \psi(x, y)$. To obtain the required contradiction it only remains to show that $K^1(M) \models \exp$. So let $a \in K^1(M)$, say $a$ is defined by the $\Sigma_1$ formula $\eta(x)$ in $M$. Then since $\exists y = 2^x$ is $\Sigma_0$ the formula

$$\exists x \forall y (\eta(x) \wedge y = 2^{x - z})$$

is $\Sigma_1$ and clearly the least element satisfying this formula in $M$ must be 0, i.e. $2^a \in M$. But then this element is in $K^1(M)$ (and still satisfies being $\exists \omega^m$ in $K^1(M)$) as required.

(ii) Suppose not, say $I_{\Pi_1} \models \forall x \exists y \psi(x, y)$ whilst $I_{\Sigma_0} + \exp + \neg \forall x \exists y \psi(x, y)$ has a model, $M$ say, where $\psi \in \Sigma_0$. Let $M \models \forall y \neg \psi(a, y)$. We may assume there is $e \in M$ such that for any $\Sigma_0$ formula $\theta(x)$,

$$M \models \exists x \theta(x) \rightarrow \exists x < e \theta(x).$$

Fix $N < v$ and $d$ much larger than $v$, $e$, $a$. It is well known that in $I_{\Sigma_0} + \exp$ there is $\Gamma(x, y, z, t) \in \Sigma_0$ such that for any $x$, $y < d$ and $\eta(x_1, x_2) \in \Sigma_0$,

$$M \models \Gamma(x, y, \eta(x_1, x_2) \wedge, d^d) \leftrightarrow \eta(x, y).$$
Now for each \( j < \nu \) let \( D_j \) be the set of \( \alpha < d \) such that
\[
\exists x < e \left[ \exists y < \alpha \Gamma(x, y, j, d^{dd}) \land \neg \exists t < x \exists y < \alpha \Gamma(t, y, j, d^{dd}) \land \exists t < x \Gamma(t, x, j, d^{dd}) \right].
\]

Using \( \exp \), \( D_j \) can be coded in \( M \) and \( M \mid |D_j| \leq e \). Hence \( M \models |B| \leq ev \), where \( B = \bigcup_{j < \nu} D_j \). Now if for some \( k \in \mathbb{N} \) it was true that for all \( a < z \in B \cup \{a\} \) either \( z^k > d \) or \( (z, z^k) \cap B = \emptyset \), then \( d < \max(B)^k \leq a^{k\nu+1} \), which contradicts \( d \) being much larger than \( \nu, e \) and \( a \). Hence there must be consecutive elements \( z_1, z_2 \) of \( B \cup \{a\} \) such that \( a < z_1 < z_1^k < z_2 \) for all \( k \in \mathbb{N} \). Hence there is a cut \( I \) lying between \( z_1 \) and \( z_2 \), and it is easy to check that
\[
I \models \Pi_1^1 + \forall y \neg \psi(a, y),
\]
giving the required contradiction.

**Remark.** Theorem 2.9 can be generalized to give

**Theorem 2.10.** Let \( k \geq 1, \theta \in \Sigma_2, \) and \( \varphi \in \Pi_2 \).

(i) If \( (I\Sigma_0 + F_{2+k} \text{ is total}) \vdash \theta, \) then \( L\Sigma_1^{-}(k) \vdash \theta, \)

(ii) If \( L\Sigma_1^{-}(k) \vdash \varphi, \) then \( (I\Sigma_0 + F_{2+k} \text{ is total}) \vdash \varphi. \)

Here the \( F_m(x) \) are defined by \( F_0(x) = x + 1 \) and \( F_{m+1}(x) = F_m^x(x) \).

**Corollary 2.11.** The \( L\Sigma_1^{-}(k) \) hierarchy is proper.

**Proof.** This follows by Theorem 2.10, since for \( k \geq 1 \)
\[
(I\Sigma_0 + F_{2+k} \text{ is total}) \vdash \text{Con}(I\Sigma_0 + F_{2+k} \text{ is total})
\]

but
\[
(I\Sigma_0 + F_{2+k} \text{ is total}) \nvdash \text{Con}(I\Sigma_0 + F_{2+k} \text{ is total})
\]
(see [7]).

**§3. Quantifier complexity of \( I\Sigma^-_n, \Pi^-_n, \) and \( B\Sigma^-_n. \)** In this section we show that \( I\Sigma^-_n, \Pi^-_n, \) and \( B\Sigma^-_n, \) are not finitely axiomatizable for \( n > 0 \) and that, in terms of quantifier complexity, their natural formulations cannot be simplified.

Theorem 0.6 is a consequence of the next four results.

**Proposition 3.1.** For \( n > 0 \), \( \Pi^-_n \) is \( \Sigma_{n+1} \), but not \( \Pi_{n+1} \).

**Proof.** Clearly, \( \Pi^-_n \) is \( \Sigma_{n+1} \). Suppose it were also \( \Pi_{n+1} \). Let \( M \models I\Sigma_n \) with nonstandard \( \Sigma_n \)-definable elements. Then, since \( K^n(M) \prec M \) (Theorem 0.9(i)), \( K^n(M) \models \Pi^-_n \) and so, by Proposition 1.5, \( K^n(M) \models I\Sigma_n \), a contradiction (by Theorem 0.9(ii)).

**Proposition 3.2.** For \( n > 0 \), \( I\Sigma^-_n \) is \( \Sigma_{n+1} \lor \Pi_{n+1} \), but not \( \Pi_{n+1} \lor \Sigma_{n+1} \).

**Proof.** Clearly, \( I\Sigma^-_n \) is \( \Sigma_{n+1} \lor \Pi_{n+1} \). As in the proof of Proposition 3.1, it follows that \( I\Sigma^-_n \) is not \( \Pi_{n+1} \). To show it is not \( \Sigma_{n+1} \), first suppose \( n > 1 \) and that \( I\Sigma^-_n \) is equivalent to \( \{ \exists x \theta_i(x) \mid i \in \mathbb{N} \} \), where \( \theta_i(x) \in \Pi_n \).

Let \( M \models I\Sigma_n, \) \( M \) \( \omega_0 \)-saturated. Then there exists \( a \in M \) such that \( a = \langle a_0, \langle a_1, \langle \cdots \rangle \rangle \rangle \) and \( M \models \theta_i(a_i) \) for all \( i \in \mathbb{N} \). Since \( M \models I\Sigma_{n-1}, \) \( K^{n-1}(M, a) \prec M \) by Theorem 0.9, and hence \( K^{n-1}(M, a) \models \theta_i(a_i) \) for all \( i \in \mathbb{N}, \) i.e. \( K^{n-1}(M, a) \models I\Sigma_{n-1}, \) contradiction. Finally, if \( n = 1 \) and \( I\Sigma^-_n \) were \( \Sigma_2, \) then as in the proof of 1.12, \( I\Sigma^-_1 \) would hold in any initial segment \( I \subset M, \) where \( \mathbb{N} < M. \) But such \( I \) need not satisfy \( \exp, \) whereas \( I\Sigma^-_1 \vdash \exp, \) contradiction.
PROPOSITION 3.3. For \( n > 0 \), \( B^\Sigma_n \) is \( \Sigma_{n+1} \lor \Pi_{n+1} \), but not \( \Sigma_{n+1} \lor \Pi_{n+1} \).

PROOF. To see that \( B^\Sigma_n \) is \( \Sigma_{n+1} \lor \Pi_{n+1} \), for \( n > 1 \), recall that \( B^\Sigma_n \vdash B\Sigma_{n-1} \), so that in the natural formulation of \( B\Sigma_n \) the bounded quantifiers can be pushed through; and that this \( \Sigma_{n+1} \lor \Pi_{n+1} \) schema, together with the \( \Pi_{n+1} \) schema \( B\Sigma_{n-1} \), implies \( B\Sigma_n \).

To show that \( B\Sigma_n \) is not \( \Pi_{n+1} \) we mimic the proof of Proposition 3.1 together with Corollary 1.9. If \( B\Sigma_n \) was \( \Pi_{n+1} \), then it would be axiomatized by \( T = \{ 3x \theta(x) \mid i \in N \} \), where each \( \theta_i \) is \( \Pi_n \) and \( T \) is recursive. But then by the method of proof of Proposition 1.13 we can find a model \( K \models T + \neg B\Sigma_n \), contradiction.

REMARKS. Notice that \( I\Sigma^{-}_{0} \) and \( \Pi\Pi^{-}_{0} \) are \( \Pi_1 \). Clearly Proposition 3.1 holds for \( L\Sigma^{-}_{n} \), \( n > 0 \), also. By constructions similar to that in the proof of 1.13 Kaye has shown that for \( n > 0 \), neither \( I\Sigma^{-}_{0} \) nor \( B\Sigma_n \) is \( \Sigma_{n+1} \lor \Pi_{n+1} \). See his thesis at Manchester University for more details.

PROPOSITION 3.4. For \( n > 0 \), there is no sentence \( \sigma \) such that \( \Pi_{n+1}(N) \vdash \sigma \vdash B\Sigma_n \).

PROOF. Suppose not, say \( \tau \in \Pi_{n+1}(N) \) and \( \tau \vdash \sigma \vdash B\Sigma_n \). Let \( M \models I\Sigma_n + \tau \) with \( M \) containing nonstandard \( \Sigma_n \)-definable elements. By Theorem 0.9, \( K''(M) \not\models B\Sigma_n \). Hence, by Corollary 1.9, \( K''(M) \not\models B\Sigma_n \), contradiction, since \( M \models \tau \) so that \( K''(M) \models \tau \) and \( \tau \vdash B\Sigma_n \).

COROLLARY 3.5. For \( n, k > 0 \), none of the following is finitely axiomatizable: \( L\Sigma_{n+1}^{-}(\omega), L\Sigma_{n+1}^{-}(k), \Pi\Pi_{n+1}^{-}, I\Sigma_n, B\Sigma_n \).

PROOF. \( L\Sigma_{n+1}^{-}(\omega) \) is \( \Sigma_{n+2} \) and true in \( N \), so

\[ \Pi_{n+1}(N) \vdash L\Sigma_{n+1}^{-}(\omega) \vdash L\Sigma_{n+1}^{-}(k) \vdash \Pi\Pi_{n+1}^{-} \vdash I\Sigma_n \vdash B\Sigma_n. \]

The proof Proposition 3.4 shows that no recursive set \( \Gamma \subseteq \Pi_{n+1}(N) \) can prove \( B\Sigma_n \). Our final theorem says that, for \( n > 0 \), \( \Pi_n(N) \) is the only \( \Pi_n \) theory (up to deductive equivalence) that proves \( \Pi\Pi_n \).

THEOREM 3.6. For \( n > 0 \), if \( T \) is a consistent \( \Pi_n \) theory such that \( T \vdash \Pi\Pi_n \), then \( T \models \Pi_n(N) \).

PROOF. \((\Rightarrow)\) Assume \( T \) is consistent, \( \Pi_n \) and that \( T \vdash \Pi\Pi_n \), but \( T \not\models \sigma \), i.e. \( T + \neg \sigma \) is consistent, for some \( \sigma \in \Pi_n(N) \). Then \( T + \neg \sigma + B\Sigma_n \) is consistent. Indeed, if, for some \( \tau \in T \), \( B\Sigma_n \vdash \tau \land \neg \sigma \rightarrow 0 = 1 \), then, by Corollary 2.3. \( I\Sigma_{n-1} \vdash \tau \land \neg \sigma \rightarrow 0 = 1 \), i.e. \( I\Sigma_{n-1} + T + \neg \sigma \) is inconsistent, a contradiction, since \( T \vdash \Pi\Pi_n \vdash I\Sigma_{n-1} \). Now let \( M \models T + \neg \sigma + B\Sigma_n \). Since \( M \models I\Sigma_{n-1}, \neg \sigma \in \Sigma_n \) and \( N \not\models \neg \sigma \), \( M \) contains nonstandard \( \Sigma_n \)-definable elements. We may assume that the set of \( \Sigma_n \) sentences true in \( M \) is coded in \( M \). By an easy extension of the proof of Theorem 0.9, we have \( I''(M) \prec_{n-1} M \) and \( I''(M) \models B\Sigma_n + \neg I\Sigma_n \). Hence \( I''(M) \models I\Pi_n \) since \( T \models \Pi_n \), \( T \models \Pi\Pi_n \) and \( M \models T \).

Furthermore the \( \Sigma_n \)-definable elements of \( I''(M) \) are cofinal in \( I''(M) \). For suppose \( a \in I''(M) \), so \( a \preceq b \) for some \( \Sigma_n \)-definable element \( b \) of \( M \), say \( M \models \exists y \chi(y, b) \land \exists x \exists y \chi(y, x) \) with \( \chi \in \Pi_{n-1} \). Then, using \( B\Sigma_n \), the formula

\[ \exists z_0, z_1[z = \langle z_0, z_1 \rangle \land \chi(z_0, z_1) \land \forall y < z_0 \neg \chi(y, z_1)] \]

is \( \Sigma_n \) in both \( M \) and \( I''(M) \) and defines, in both, some \( c \geq b \geq a \) as required.

The required contradiction now follows by Proposition 1.5.
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(=) If $T$ satisfies the hypotheses, then, by the first part of the proof, $T \vdash \Pi _n (N)$. Now let $\sigma \in T$ and $M \models T$. Since $M \models \Pi _n (N)$, $N < _n M$ and hence $N \models \sigma$, i.e. $\sigma \in \Pi _n (N)$. Therefore, $\Pi _n (N) \vdash T$.

COROLLARY 3.7. For $k > 0$, none of $L \Sigma _1 ^{(\omega )}, L \Sigma _1 ^{(k)}$, or $L \Pi _1$ is finitely axiomatizable.

PROOF. Assume not. Then there exists $\sigma \in \Pi _1 (N)$ such that $\sigma \not\vdash \Pi _1$. But then, by Theorem 3.6, $\sigma \not\vdash \Pi _1 (N)$, which is impossible.

REMARK. By Remark (ii) after Theorem 2.1, the previous theorem and the observation that $\forall \exists (N)$ proves the MRDP theorem, we deduce that, for all $n \geq 1$, $\forall _n (N)$ is the only $\forall _n$ theory (up to deductive equivalence) that implies $\forall _n ^{-}$. This answers a question in [6] for $n > 1$. (A. Wilkie proved it for $n = 1$.)

We conclude with two open problems:
1. Is the collection schema in the proof of Proposition 1.7 equivalent to $B \Sigma _n ^{-}$?
2. In Theorem 2.6 can we replace $\Sigma _n ^{-} + L \Sigma _n ^{(k)}$ by just $L \Sigma _n ^{(k)}$?

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