On rings whose quasi-injective modules are injective or semisimple

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Abstract

Two obvious classes of quasi-injective modules are those of semisimples and injectives. In this paper, we study rings with no quasi-injective modules other than semisimples and injectives. We prove that such rings fall into three classes of rings, namely (i) QI-rings, (ii) rings with no middle class, or (iii) rings that decompose into a direct product of a semisimple Artinian ring and a strongly prime ring. Thus, we restrict our attention to only strongly prime rings and consider hereditary Noetherian prime rings to shed some light on this mysterious case. In particular, we prove that among these rings, QIS-rings which are not of type (i) or (ii) above are precisely those hereditary Noetherian prime rings which are idealizer rings from non-simple QI-overrings.

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1 Introduction

There have been various studies on the classification theory of rings in which the homological property of injectivity plays a crucial role. A number of classes of rings have been investigated by means of injectivity of their certain modules since the latter half of the last century. Among these are semisimple Artinian rings (rings all of whose cyclic modules are injective), right V-rings (rings whose simple right modules are injective), right PCI-rings (rings for which every proper cyclic right module is injective), right SI-rings (rings whose every singular right module is injective), and right QI-rings (rings whose quasi-injective right modules are injective).

In 2010, Alahmadi et al. [1] introduced a new class of rings, in the same direction as above classes of rings arise, for which every (right) module is either injective or the farthest from being

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injective (in a certain sense). Such rings are said to have no (right) middle class and are referred to as (right) *NMC-rings* in this current paper. NMC-rings and some of their variations have been studied in many papers over the last decade where strong connections to aforementioned classes of rings have been discovered (see, for example, [1, 2, 3, 10, 11, 12, 21, 26]). One intriguing property of NMC-rings is that they have only trivial quasi-injective modules. More precisely, if R is a ring with no right middle class, then every quasi-injective right R-module is either injective or semisimple. However, the converse does not hold in general (see Theorem 4.8 of the present paper and [11, Theorem 2 and Proposition 5]). Although this broader class of rings whose quasi-injectives are injective or semisimple appeared in some papers before (see [10, 11, 21]), to the best of our knowledge, it was merely considered in relation to the class of NMC-rings and has never been studied in its own right. In this paper, we examine in detail the structure of rings of the title on their right modules, that we call *right QIS-rings*, and give some characterizations besides several remarkable properties of these rings.

Byrd ([5] and [6]) and Boyle ([4]) studied rings whose quasi-injective right modules are injective. Boyle referred to such rings as right QI-rings and conjectured that they are right hereditary. Observe that right QI-rings are automatically right QIS. We prove that the class of right QIS-rings is much broader than that of right QI-rings (see Theorem 4.8). Note that if R is a hereditary Noetherian ring, then the phrases "R is a right QI-ring", "R is a right V-ring", "R is a right PCI-ring", and "R is a right SI-ring" are all synonymous (see [4, Corollary 10] and [14, Theorem 3.11]).

In Section 2, we classify coatoms of the lattice of left exact preradicals on right modules over a ring R in terms of certain submodules of $E(R_R)$, the injective hull of R_R, and show, in particular, that they are in bijective correspondence with maximal proper fully invariant submodules of $E(R_R)$ (see Proposition 2.2 and Corollary 2.3). We use information obtained on coatoms of the lattice of left exact preradicals for the examination of the structure of right QIS-rings in the next section.

In section 3, we give a classification of right QIS-rings by investigating the internal structure of right QIS-rings which are not right NMC. In particular, we prove the following classification theorem, which leads us to restrict our interest to only strongly prime rings.

Theorem. *Let R be any ring. Then R is a right QIS-ring if and only if one of the following conditions hold:*

(i) R is a right QI-ring.

(ii) R is a right NMC-ring.

(iii) There exists a ring decomposition $R = S \times T$ such that S is either zero or a semisimple Artinian ring and T is a right strongly prime right QIS-ring.

We also obtain that the classes of right QIS-rings and right NMC-rings coincide for right fully bounded rings (in particular, for commutative rings). In this section, we also prove that for any ring R, if $\mathrm{Soc}(R_R) \leq_e R_R$, then R is a right NMC-ring if and only if it is a right QIS-ring, generalizing the fact that right semiartinian right QIS-rings are right NMC, proved in [10, Corollary 5.6] and [21, Proposition 3.1].

In the last section, we turn our attention to right Noetherian rings with restricted right socle condition, namely right Noetherian rings whose singular right modules are semiartinian. Note that rings with restricted right socle condition constitute a broad class including right SI-rings, right NMC rings, and hereditary Noetherian prime rings (HNP rings for short). Thus, it is reasonable to employ this condition while searching for ties of QIS-rings with these classes of rings.

In view of the above theorem, we obtain a complete characterization for right Noetherian rings with restricted right socle condition to be right QIS in the following two propositions.

Proposition. *Let R be any ring which is not semisimple Artinian. Then the following conditions are equivalent:*

(i) R is a right QI-ring with restricted right socle condition.

(ii) R is a right Noetherian right V-ring with restricted right socle condition.

(iii) R is a right SI-ring with $\mathrm{Soc}(R_R) = 0$.

Moreover; if, in addition, R is indecomposable, then (i)–(iii) are equivalent to

(iv) R is a right NMC-ring with $\mathrm{Soc}(R_R) = 0$.

Proposition. *Let R be a prime right Noetherian ring with restricted right socle condition. If R is not a right V-ring, then R is right QIS if and only if the following statements hold:*

(i) There exists a unique isomorphism class of non-injective simple right R-modules.

(ii) If W is a non-injective simple right R-module, then it is the only proper nonzero fully invariant submodule of $E(W)$.

Furthermore, we prove that a basic idealizer ring $\mathbb{I}_S(A)$ from an indecomposable right SI-ring S with zero right socle is, then, a right QIS-ring which is neither right QI nor right NMC (see Proposition 4.7). It follows that there can be found plenty of concrete examples of right QIS-rings which are neither right QI nor right NMC. In particular, if S is chosen to be a non-Artinian HNP ring that is a right V-ring (see, for example, [7], [8], [19], or [24]), then any basic idealizer ring from S will be a right QIS-ring that is neither right QI nor right NMC. In fact, we see that this is the only way for producing such rings among HNP rings (see Theorem 4.8). Moreover, we see that HNP rings R that are right QIS but not right QI are precisely those rings such that there exists a unique (up to isomorphism) non-injective simple right R-module and for any non-injective simple right R-module W, the injective hull of W is a uniserial module of length 2.

Let R be a ring. For a right R-module M, we define the class $\mathfrak{In}^{-1}(M)$, called the injectivity domain of M, consisting of all right R-modules relative to which M is injective. Observe that the injectivity domain of any module contains the class of all semisimple modules (denoted SSMod–R) and that a module M is injective if and only if $\mathfrak{In}^{-1}(M)$ is the entire class of right R-modules, denoted Mod–R. An R-module is said to be poor if its injectivity domain is equal to SSMod–R. Note that poor modules always exist for any ring (see [11, Proposition 1]). R is said to have no right middle class provided that every right R-module is either injective or poor, i.e., there are no injectivity domains other that SSMod–R and Mod–R. We refer to these rings as right NMC-rings in this paper. One can easily see by definition that right NMC-rings are necessarily right QIS. However, Theorem 4.8 below together with [11, Theorem 2 and Proposition 5] shows that the converse is not true, in general. On the other hand, any indecomposable hereditary Noetherian right QI-ring is right NMC by [11, Proposition 5].

Throughout this note, all rings are assumed to be associative rings with identity and all modules are unitary right modules, unless specified otherwise. Given a module M, we will use $E(M)$, $\mathrm{Soc}(M)$, and $Z(M)$ to denote the injective hull, the socle, and the singular submodule of M, respectively. We will also use notations $\leq$ and $\leq_e$ to indicate submodules and essential submodules.

Let M, N be two modules over the ring R. The submodule $\sum\{\mathrm{Im}\, f : f \in \mathrm{Hom}_R(M, N)\}$ of N will be denoted $\mathrm{Tr}_R(M, N)$. Note that $\mathrm{Tr}_R(M, N)$ is a fully invariant submodule of N, i.e., $f(\mathrm{Tr}_R(M, N)) \subseteq \mathrm{Tr}_R(M, N)$ for any R-endomorphism of N. We say that M is N-injective if whenever there is an R-homomorphism $g : N' \to M$, where N' is a submodule of N, then g extends to an R-homomorphism $h : N \to M$. It is not difficult to see that M is N-injective if and only if $\mathrm{Tr}_R(N, E(M)) \subseteq M$.

Let M be a module over the ring R. M is called quasi-injective if it is injective relative to itself. This is equivalent to saying that M is a fully invariant submodule of its injective hull. Moreover, if M is a fully invariant submodule of some injective module, then it is quasi-injective. In particular, if E is an injective R-module, then $\mathrm{Tr}_R(M, E)$ is always a quasi-injective R-module. Also, for a right ideal A (resp., left ideal B) of R and an injective R-module E, the submodule EA (resp., $\mathrm{ann}_E(B)$) of E is fully invariant in E; hence quasi-injective. We refer the reader to [23] for any unexplained terminology and other details on relative injectivity and related concepts.

2 Coatoms in the lattice of left exact preradicals

As we shall see further on, the notion of left exact preradicals of torsion theory plays a key role in our study of rings whose quasi-injective right modules are injective or semisimple (namely, right QIS-rings). Note that left exact preradicals form a coatomic lattice and we shall see, in the next section, that coatoms of this lattice prove crucial for the examination of right QIS-rings. Indeed, we should refer the interested reader to [10] in which several classes of rings including the rings of the title has been characterized by means of left exact preradicals of some specific type. In particular, it has been proved that a ring R is right QIS if and only if every left exact preradical strictly greater than the functor Soc is stable (see [10, Proposition 5.3] in conjunction with [21, Theorem 2.9]). In this short section, we focus on submodules of $E(R_R)$, the injective hull of R_R, for any ring R, determined by coatoms in the lattice of left exact preradicals on Mod–R (denoted lep–R) and see that coatoms of lep–R are in one-to-one correspondence with maximal proper fully invariant submodules of $E(R_R)$.

Let R be a ring. We remind that a subfunctor of the identity functor on Mod–R is called a *preradical* on Mod–R. If a given preradical σ on Mod–R is left exact as a functor, then we say that σ is a *left exact preradical*. Note that given a left exact preradical σ on Mod–R, the class of right R-modules M with $\sigma(M) = M$, denoted $\mathscr{T}_\sigma$, is closed under taking submodules, homomorphic images, and arbitrary direct sums. Any such class of right R-modules is called a *hereditary pretorsion class*. So, every left exact preradical determines a hereditary pretorsion class. Conversely, given a hereditary pretorsion class $\mathscr{T}$, one may define a functor $\sigma^{\mathscr{T}}$ on Mod–R by $\sigma^{\mathscr{T}}(M) = \sum\{N \leq M : N \in \mathscr{T}\}$ for every $M \in$ Mod–R, and see that $\sigma^{\mathscr{T}}$ is a left exact preradical. This yields a bijective correspondence which associates with every left exact preradical on Mod–R a unique hereditary pretorsion class of right R-modules (see [27] for details). If a left exact preradical σ on Mod–R satisfies the property that $\sigma(M/\sigma(M)) = 0$ for any right R-module M, then σ is called a left exact *radical*. Under the aforementioned bijective correspondence, left exact radicals are associated with hereditary *torsion* classes (i.e., hereditary pretorsion classes which are closed under extensions).

It is well known that left exact preradicals on Mod–R also correspond bijectively with certain sets of right ideals of R (see [27]). Thus lep–R corresponds bijectively to a set. Although sets cannot include proper classes as elements, we see no harm to assume that lep–R is a set for our purposes. An important aspect of our interest in left exact preradicals is that lep–R can be given a lattice structure. Given two left exact preradicals σ and τ, we write $\sigma \leq \tau$ provided that $\sigma(M) \subseteq \tau(M)$ for every right R-module M. This defines a partial order on lep–R. On the other hand, one can define arbitrary meets and joins as follows: given a set $\{\sigma_i : i \in I\}$ of left exact preradicals on Mod–R, we have $\left(\bigwedge_{i\in I} \sigma_i\right)(M) = \bigcap_{i\in I} \sigma_i(M)$ for every right R-module M and $\bigvee_{i\in I} \sigma_i$ is the left exact preradical corresponding to the (unique) smallest hereditary pretorsion class in Mod–R containing $\bigcup_{i\in I} \mathscr{T}_{\sigma_i}$. Together with these meet and join operations, the partially ordered set lep–R turns into a lattice. It should be noted that this lattice is a modular coatomic lattice (see [28, Theorem 2]) and that coatoms in lep–R are of importance in the current and the next section.

Given a ring R, there are certain hereditary pretorsion classes of right R-modules which are of particular interest in this work. Among them are the class of semisimple right R-modules (denoted SSMod–R) and the class of singular right R-modules (denoted Sing–R), corresponding to the left exact preradicals, denoted Soc and Z, respectively. We are also interested in a particular type of hereditary pretorsion classes associated to a given right R-module M, namely, the class of all right R-modules relative to which M is injective (denoted $\mathfrak{In}^{-1}(M)$ and called the injectivity domain of M). Clearly, $\mathfrak{In}^{-1}(M)$ contains SSMod–R. It follows that there exists a unique left exact preradical in the sublattice $[\mathrm{Soc}, 1]$ of the lattice lep–R which corresponds to

$\mathfrak{In}^{-1}(M)$ in the lattice of hereditary pretorsion classes of right R-modules. This corresponding left exact preradical is denoted by i_M in [10]. By [21, Theorem 2.9], we see that every left exact preradical in $[\mathrm{Soc}, 1]$ is of the form i_M for a suitable module M. Therefore a ring R is a right NMC-ring if and only if Soc is a coatom in the lattice of left exact preradicals on Mod–R.

Notice that for each right R–module K, $i_M(K)$ is the largest submodule of K relative to which M is injective. This gives that $i_M(K) = \bigcap\{f^{-1}(M) \mid f \in \mathrm{Hom}_R(K, E(M))\}$ (see [10, Lemma 2.3]). Motivated by this identification of i_M, we consider a particular type of preradicals as follows: for a right R-module L and a submodule M of L, the functor $\sigma : \text{Mod–}R \to \text{Mod–}R$ defined by $\sigma(K) = \bigcap\{f^{-1}(M) \mid f \in \mathrm{Hom}_R(K, L)\}$, for every $K \in \text{Mod–}R$, is clearly a preradical. Following [15], we denote this preradical σ by ω^L_M. Notice that $i_M = \omega^{E(M)}_M$ for each $M \in$ Mod–R. We note that preradicals of the form ω^L_M, where M is a fully invariant submodule of L, are treated in [15]. Indeed, one can easily see that M is a fully invariant submodule of L if and only if $\omega^L_M(L) = M$. We also note that the preradical ω^L_M is left exact when L is injective. More generally, a given preradical σ is left exact if and only if $\sigma = \bigwedge\{\omega^E_{\sigma(E)} \mid E \in$ Mod–R and E_R is injective$\}$. Moreover, for any injective module E, the left exact preradical ω^E_0 is also a radical (see [15, Proposition 2.1]).

Lemma 2.1. *Let σ be a coatom in lep–R. Then σ is a radical if and only if $\sigma(E) = 0$ for some nonzero injective module E.*

Proof. If σ is a radical, then $\mathscr{T}_\sigma$ is cogenerated by an injective module, E say. Hence $\sigma(E) = 0$. Conversely, assume that there exists a nonzero injective module E such that $\sigma(E) = 0$. Then, clearly, $\sigma \leq \omega^E_0 \neq \mathrm{id}$, and hence, by maximality of σ, we must have $\sigma = \omega^E_0$, which is a radical. □

The following proposition provides a description of coatoms in lep–R. In particular, we see that some of these coatoms are of the form i_M for some suitable quasi-injective modules M. But before, it is convenient to mention the following two fundamental properties of the preradical ω^L_M for a pair of right R-modules L, M for which M is a fully invariant submodule of L: (i) if σ is a preradical such that $\sigma(L) = M$, then $\sigma \leq \omega^L_M$ in the lattice of preradicals; and (ii) if N is a fully invariant submodule of L containing M, then $\omega^L_M \leq \omega^L_N$ in the lattice of preradicals.

Proposition 2.2. *Let σ be a coatom in the lattice lep–R, $E = E(R_R)$, and $K = \sigma(E)$. Then the following hold:*

(i) $\sigma = \omega^E_K$.

(ii) *K is a maximal proper fully invariant submodule of E.*

(iii) *If K is an essential submodule of E, then it is a maximal proper quasi-injective submodule of E and $\mathscr{T}_\sigma = \mathfrak{In}^{-1}(K)$. In particular, $\sigma = i_K$.*

(iv) *If K is not essential in E, then K is injective and σ is a radical. In this case, $\mathscr{T}_\sigma$ is the torsion class cogenerated by the injective module E/K.*

Proof. (i) Since K is a fully invariant submodule of E, it is quasi-injective. Since $\sigma \neq \mathrm{id}$, $K \neq E$. Also we have $\sigma \leq \omega_K^E < \mathrm{id}$ for $\sigma(E) = K$. Hence, by maximality of σ, we must have $\sigma = \omega_K^E$.

(ii) Let L be a fully invariant proper submodule of E containing K. Then $\sigma = \omega_K^E \leq \omega_L^E < \mathrm{id}$, so that $\omega_K^E = \omega_L^E$, by assumption. This gives that $K = \omega_K^E(E) = \omega_L^E(E) = L$. It follows that K is a maximal fully invariant submodule of E.

(iii) Assume that K is an essential submodule of E. Then any quasi-injective module between K and E must be a fully invariant submodule of E. Hence, by (ii) above, K is a maximal proper quasi-injective submodule of E. Also, since K is essential in E, E is an injective hull of K, and so $\sigma = \omega_K^E = i_K$. This also means that $\mathscr{T}_\sigma = \mathfrak{In}^{-1}(K)$.

(iv) Suppose that K is not essential in E. Let $E_1 = E(K)$. Then there exists a nonzero submodule E_2 of E such that $E = E_1 \oplus E_2$. It follows that $K = \sigma(K) \subseteq \sigma(E_1) \subseteq \sigma(E) = K$, and so $\sigma(E_1) = K$. Since $K = \sigma(E) = \sigma(E_1) \oplus \sigma(E_2) = K \oplus \sigma(E_2)$, we must have $\sigma(E_2) = 0$. Thus σ is a radical by Lemma 2.1. Next, we shall show that K is injective. Since $\mathrm{Tr}_R(E_1, E)$ is a fully invariant submodule of E containing E_1 (and hence K), by (ii) above, we have either $K = \mathrm{Tr}_R(E_1, E)$ or $E = \mathrm{Tr}_R(E_1, E)$. Suppose that $E = \mathrm{Tr}_R(E_1, E)$. Then E is generated by E_1. Thus R can be embedded in a finite direct sum E_1^n of E_1. Then E can be embedded in E_1^n, so there exists a submodule E_2' of E_1^n that is isomorphic to E_2. It follows that $0 = \sigma(E_2') = \sigma(E_1^n) \cap E_2'$. However, since $\sigma(E_1^n) \leq_e E_1^n$, this is impossible. Therefore $K = E_1 = \mathrm{Tr}_R(E_1, E)$; hence K is injective. Now let $M \in \mathscr{T}_\sigma$, i.e., $\sigma(M) = M$. Then, by (i), $\mathrm{Tr}_R(M, E) \leq K$, and so $\mathrm{Hom}_R(M, E_2) = 0$. This gives that $\mathscr{T}_\sigma$ lies in the torsion class cogenerated by $E_2 \cong E/K$; so it is equal to the torsion class cogenerated by E/K, by maximality of σ. This establishes (iv) and the proof is complete. □

Corollary 2.3. *Let R be any ring and $E = E(R_R)$. Then the correspondence defined by $K \longmapsto \omega_K^E$ from the set of maximal proper fully invariant submodules of E to the set of coatoms in the lattice lep–R is a bijection whose inverse is given by $\sigma \longmapsto \sigma(E)$.*

Proof. Let K be a maximal proper fully invariant submodule of E, and let σ be a coatom in lep–R such that $\omega_K^E \leq \sigma$. By Proposition 2.2 (i), there exists a fully invariant proper submodule L of E such that $\sigma = \omega_L^E$. Then $K = \omega_K^E(E) \leq \omega_L^E(E) = L$. By maximality of K, $K = L$, and hence $\omega_K^E = \omega_L^E = \sigma$. It follows that ω_K^E is a coatom in lep–R. This shows that the correspondence defined by $K \longmapsto \omega_K^E$ is a mapping from the set of maximal proper fully invariant submodules of E to the set of coatoms in the lattice lep–R. Denote this mapping by φ. Also, denote by η the correspondence defined by $\sigma \longmapsto \sigma(E)$ from the set of coatoms in lep–R to the set of submodules of E. By Proposition 2.2 (ii), η is a mapping into the set of maximal proper fully invariant submodules of E. Clearly, for each maximal proper fully invariant submodule K of E, $\eta\varphi(K) = K$. On the other hand, by Theorem 2.2 (i), for each coatom σ in lep–R, we have $\varphi\eta(\sigma) = \sigma$. This completes the proof. □

3 Structure of right QIS-rings

We call a ring R a right QIS-ring if every quasi-injective right R–module is either injective or semisimple. As noted in the introductory part, the class of right QIS-rings include right QI-rings and right NMC-rings. In this section we give a classification of right QIS-rings (see Theorem 3.12) by investigating the internal structure of right QIS-rings which are not right NMC. Note that Proposition 3.3 provides a useful argument for distinguishing QIS-rings from NMC-rings with the help of the notion of a semiartinian ring.

Let R be any ring. Every member of the smallest hereditary torsion class ($\mathscr{S}$, say) containing SSMod–R is said to be a semiartinian right R-module. Also, the ring R is called right semiartinian provided $\mathscr{S} = \text{Mod}–R$, i.e, every right R-module is semiartinian. By definition, a nonzero right R-module is semiartinian if and only if every nonzero submodule of every nonzero factor of M contains a simple submodule. This is equivalent to saying that every nonzero factor of M has essential socle. We recall that given a right R-module M, one can form an ascending chain $0 = \mathcal{S}_0 \subseteq \mathcal{S}_1 \subseteq \dots$ of submodules of M, where for every ordinal λ, $\mathcal{S}_{\lambda+1}/\mathcal{S}_\lambda = \text{Soc}(M/\mathcal{S}_\lambda)$ and if λ is a limit ordinal, then $\mathcal{S}_\lambda = \bigcup_{\mu<\lambda} \mathcal{S}_\mu$. Now M is semiartinian if and only if $\mathcal{S}_\nu = M$ for some ordinal ν (see [9, Proposition 1]).

Proposition 3.1. *If R is a right QIS-ring, then one of the following two conditions holds:*

(i) R is a right NMC-ring.

(ii) Every coatom in lep–R is a radical.

Furthermore, if both (i) and (ii) hold, then R is a right QI-ring.

Proof. Note that we may assume that R is not semisimple Artinian. Let $E = E(R_R)$ and let $\sigma \in$ lep–R be a coatom which is not a radical. Then by Proposition 2.2, $\sigma(E)$ is an essential maximal proper fully invariant (hence quasi-injective) submodule of E. Thus, by assumption, $\sigma(E)$ is semisimple, which yields $\text{Soc}(R_R) = \sigma(E) \leq_e E$. We also have $\sigma = \omega^E_{\text{Soc}(R_R)} = i_{\text{Soc}(R_R)}$ by Proposition 2.2. Hence $\text{Soc} \leq \sigma$ in lep–R.

Let D be a nonzero injective module which contains no simple submodules. Suppose that $\sigma(D) = D$. Since $\sigma = i_{\text{Soc}(R_R)}$, $\text{Soc}(R_R)$ is D-injective. This gives that the right R-module $\text{Soc}(R_R) \oplus D$ is quasi-injective which is neither injective nor semisimple, a contradiction. Hence, $\sigma(D) \neq D$. Since $\sigma(D)$ is quasi-injective, $\sigma(D)$ is either injective or semisimple. Suppose that $\sigma(D)$ is an injective module. Since σ is not a radical, $\sigma(D) \neq 0$ by Lemma 2.1. Then $D = \sigma(D) \oplus D_1$ for some nonzero proper submodule D_1 of D. But in this case $\sigma(D_1) = 0$, which contradicts with the fact that σ is not a radical by Lemma 2.1 again. Therefore $\sigma(D)$ cannot be injective; hence it is nonzero semisimple. But this is also a contradiction since D contains no simple submodules. It follows that any nonzero injective right R–module contains a simple submodule. This implies that R is a right semiartinian ring, and so it is right NMC by [21, Proposition 3.1].

Now assume that both (i) and (ii) hold. This implies that the left exact preradical Soc is a radical; hence R is not right semiartinian. Thus, by [11, Theorem 2 and Lemma 5], R is a right QI-ring. $\square$

Lemma 3.2 ([26, Lemma 2.1])**.** *Let R be any ring and let σ be a left exact preradical on Mod–R. Then the class*

$$\mathscr{T}_{\sigma^s} := \{M \in \textit{Mod–}R : M/\sigma(M) \textit{ is semisimple and } \sigma(M/\sigma(M)) = 0\}$$

of right R-modules is a hereditary pretorsion class containing $\mathscr{T}_\sigma \cup$ SSMod–R.

In [21, Proposition 3.1], it is proved that for right semiartinian rings the notions of a right NMC-ring and a right QIS-ring coincide. We generalize this result with the following

Proposition 3.3. *Let R be any ring. If* $\mathrm{Soc}(R_R) \leq_e R_R$*, then R is a right NMC-ring if and only if it is a right QIS-ring.*

Proof. Since any ring with no right middle class is a right QIS-ring, it suffices to prove the converse. Let $\mathrm{Soc}(R_R) \leq_e R_R$ and assume that R is a right QIS-ring. Note that we may assume that R is not semisimple Artinian. By Lemma 3.1, it is enough to show that there exists a coatom in lep–R which is not a radical. Let $E = E(R_R)$ and let σ be a coatom in lep–R such that $\mathrm{Soc} \leq \sigma$. Since $\mathrm{Soc}(R_R) \leq_e R_R$, we have $\sigma(E) \leq_e E$; so $\sigma(E)$ is a proper quasi-injective submodule of E containing $\mathrm{Soc}(R_R)$. Thus $\sigma(E) = \mathrm{Soc}(R_R)$, by assumption. Therefore $\sigma = \omega^E_{\mathrm{Soc}(R_R)}$ by Proposition 2.2. This shows that σ is the unique coatom in lep–R such that $\mathrm{Soc} \leq \sigma$. Now let $\mathscr{B}$ denote the class of right R–modules M such that $M/Z(M)$ is nonsingular semisimple. By Lemma 3.2, $\mathscr{B}$ is a hereditary pretorsion class containing both SSMod–R and the class of singular right R-modules. Let β be the left exact preradical corresponding to the hereditary pretorsion class $\mathscr{B}$. Then we have $Z \vee \mathrm{Soc} \leq \beta$. Suppose $\beta = \mathrm{id}$. Then $\beta(R) = R$, or equivalently, $R/Z(R_R)$ is a nonsingular semisimple right R-module. It follows that $Z(R_R) \neq 0$. Since any nonsingular semisimple right R-module is projective, $Z(R_R)$ is a direct summand of R_R, which is a contradiction. Therefore $\beta \neq \mathrm{id}$, and so $\beta \leq \sigma$ since σ is the unique coatom in lep–R such that $\mathrm{Soc} \leq \sigma$. Now assume that σ is a radical. Then there exists an injective module E' such that $\sigma(E') = 0$ by Lemma 2.1. Since $Z(E') \leq \beta(E') \leq \sigma(E') = 0$, we obtain that E' is nonsingular. Also since $\sigma \leq \omega_0^{E'} < \mathrm{id}$, the maximality of σ yields $\sigma = \omega_0^{E'}$. Thus $\sigma(E) = \omega_0^{E'}(E) = \bigcap\{\ker(f) \mid f \in \mathrm{Hom}_R(E, E')\}$. It follows that the nonzero singular module $E/\sigma(E)$ embeds in a direct product of copies of E' which is nonsingular, a contradiction. Thus σ is not a radical and the proof is complete. $\square$

Lemma 3.4 ([18, Proposition 3.9.1])**.** *If S is a minimal right ideal of a ring R, then either $S^2 = 0$ or $S = eR$ for some idempotent element $e \in R$.*

In light of Proposition 3.3, we continue our study of right QIS-rings by focusing on rings R whose right socle is not essential in R_R. We first prove in the following lemma that such right

QIS-rings must be semiprime and right nonsingular. But before establishing the lemma, we see it appropriate to refer to a particular type of quasi-injective modules. Let M be a right module over a ring R and let A be a nonempty subset of R. The subset of elements of M annihilated by all elements of A will be denoted by $\mathrm{ann}_M(A)$. Clearly, in case A is a left ideal of R, $\mathrm{ann}_M(A)$ is a fully invariant submodule of M. Thus, if M is an injective right R-module, then for any left ideal A of R, the submodule $\mathrm{ann}_M(A)$ of M is quasi-injective. We also note that if $M = R$ and A is a left ideal of R, then $\mathrm{ann}_R(A)$ is a two-sided ideal of R and, in this case, we write $\mathrm{ann}_l(A)$ instead of $\mathrm{ann}_R(A)$. Analogously, if A is a right ideal of R, then the subset $\{r \in R : Ar = 0\}$ of R, which is obviously a two-sided ideal in R, will be denoted by $\mathrm{ann}_r(A)$.

Lemma 3.5. *Let R be a right QIS-ring such that $\mathrm{Soc}(R_R)$ is not essential in R_R. Then R is semiprime and right nonsingular. In particular $\mathrm{Soc}(R_R) = \mathrm{Soc}({}_RR)$.*

Proof. Let $E = E(R_R)$. Assume that there exists a nonzero two-sided ideal I of R such that $I^2 = 0$. Let J be a right ideal of R such that $I \oplus J \leq_e R_R$. Then $(I \oplus J)I = JI \subseteq I \cap J = 0$; hence $I \oplus J \subseteq \mathrm{ann}_E(I)$. Since $\mathrm{Soc}(R_R)$ is not essential in R_R, $I \oplus J$ cannot be semisimple. Thus $\mathrm{ann}_E(I)$ is an essential quasi-injective submodule of E which is not semisimple. Since R is a right QIS-ring, $\mathrm{ann}_E(I)$ is injective. It follows that $\mathrm{ann}_E(I) = E \supseteq R$, and so $I = 0$, a contradiction. Therefore, R is semiprime.

Since R is semiprime, R contains no singular minimal right ideals by Lemma 3.4. Thus $Z(R_R)$ is not semisimple as a right R-module. It follows that $Z(E_R)$ is a quasi-injective right R-module which is not semisimple, and so, by assumption, $Z(E_R)$ is injective. Then $E = Z(E_R) \oplus W$ for some nonzero submodule W of E. Let $I = W \cap R$. Now I is a nonzero right ideal of R. Since $R/I = R/(W \cap R) \cong (R + W)/W \leq E/W \cong Z(E_R)$, R/I is a singular right R-module. But this is possible only when $Z(R_R) = 0$ since $Z(R_R) \cap I = 0$ and $I \neq 0$. Therefore, R is right nonsingular. The last statement follows from the fact that R is semiprime. □

Lemma 3.6. *Let R be a right QIS-ring such that $\mathrm{Soc}(R_R)$ is not essential in R_R and let I be a right ideal of R. Then $E(R_R)I \neq E(R_R)$ if and only if $\mathrm{ann}_r(I) \neq 0$.*

Proof. Let $E = E(R_R)$. Assume that $\mathrm{ann}_r(I) \neq 0$. If $EI = E$, then $\mathrm{ann}_r(I) \subseteq E.\mathrm{ann}_r(I) = EI\,\mathrm{ann}_r(I) = 0$, a contradiction. Thus $EI \neq E$.

Conversely, suppose that $EI \neq E$. We first assume that EI is semisimple as a right R-module. Then I is semisimple since $I \subseteq EI$. By assumption $\mathrm{Soc}(R_R) \cap A = 0$ for some nonzero right ideal of R. This yields $A.\mathrm{Soc}(R_R) = 0$, and since R is semiprime by Lemma 3.5, we have $\mathrm{Soc}(R_R).A = 0$. Since I_R is semisimple, $I \subseteq \mathrm{Soc}(R_R)$, and so $I.A \neq 0$. It follows that $\mathrm{ann}_r(I) \neq 0$. Now we assume that EI is not semisimple as a right R-module. Observe that EI is a fully invariant submodule of E; hence it is quasi-injective. Since R is a right QIS-ring, EI must be an injective right R-module. In particular, $E = EI \oplus K$ for some nonzero submodule K of E. This gives that $KI \subseteq EI \cap K = 0$, and so $(K \cap R)I = 0$. Since R is semiprime, we have $I(K \cap R) = 0$; hence $0 \neq K \cap R \subseteq \mathrm{ann}_r(I)$, completing the proof. □

Lemma 3.7. *Let R be a right QIS-ring such that* $\mathrm{Soc}(R_R)$ *is not essential in* R_R*. Then for any proper ideal* A *of* R*, either* $\mathrm{ann}_r(A) \neq 0$ *or* R/A *is a semisimple Artinian ring.*

Proof. Let A be a proper ideal of R and let $M = R/A$. Suppose that M is not semisimple as a right R-module. We shall show that $EA \neq E$, where $E = E(R_R)$, and this will complete the proof by Lemma 3.6. Assume the contrary. Since $\mathrm{ann}_{E(M_R)}(A)$ is a quasi-injective submodule of $E(M_R)$ containing M, we must have $E(M_R) = \mathrm{ann}_{E(M_R)}(A)$, by the assumption that R is right QIS. Therefore, $E(M_R)A = 0$. Now, consider the R-homomorphism $f : R \to E(M_R)$ defined by $f(r) = r + A$ for all $r \in R$. f extends to a nonzero R-homomorphism $g : E \to E(M_R)$. But in this case, we have $g(E) = g(EA) = g(E)A \subseteq E(M_R)A = 0$, a contradiction. Therefore, $EA \neq E$. □

Corollary 3.8. *Let R be a right QIS-ring such that* $\mathrm{Soc}(R_R)$ *is not essential in* R_R*. If R is a prime ring, then every (two-sided) ideal of R is idempotent.*

Proof. Let A be a nonzero proper ideal of R. Then $A^2 \neq 0$. By above lemma, the ring R/A^2 is semisimple Artinian. Then, there exists a right ideal B of R such that $A + B = R$ and $A \cap B = A^2$. It follows that $A^2 = A^2 + BA = A$, completing the proof. □

Lemma 3.9. *Let R be a right QIS-ring such that* $\mathrm{Soc}(R_R)$ *is not essential in* R_R *and let I be a right ideal of R. If I_R is not semisimple, then* $E = EI \oplus \mathrm{ann}_E(RI)$*, where* $E = E(R_R)$*.*

Proof. Let $E = E(R_R)$ and let I be a right ideal of R. Assume that I_R is not semisimple. Then, EI is a quasi-injective right R-module which is not semisimple. Hence EI is injective. It follows that there exists a submodule W of E_R such that $E = EI \oplus W$. Since $WI \subseteq EI \cap W = 0$, we have $W \subseteq \mathrm{ann}_E(RI)$. Now let $e \in E$ such that $eRI = 0$. Write $e = x + y$, where $x \in EI$ and $y \in W$. Suppose that $x \neq 0$. Then there exists $r \in R$ such that $0 \neq xr \in R$. Since $er = xr + yr$ and $erRI = 0$, we have $xrRI = 0$. Thus $RIxr = 0$ since R is semiprime by Lemma 3.5. In particular, we have $Ixr = 0$. This gives that $xr \in Exr = Wxr \subseteq W$. But this is a contradiction since xr is a nonzero element of EI and $EI \cap W = 0$. It follows that $x = 0$, and hence $\mathrm{ann}_E(RI) = W$, completing the proof. □

Lemma 3.10. *Let R be a right QIS-ring.*

(i) If A is a proper ideal of R, then R/A is also a right QIS-ring.

(ii) If S is a semisimple Artinian ring, then the product $R \times S$ is also a right QIS-ring.

Proof. (i) Let M be a quasi-injective right (R/A)-module which is not semisimple and let $E = E(M_R)$, the injective hull of M as right R-module. Notice that $M \subseteq \mathrm{ann}_E(A) \subseteq E$. In this case, $\mathrm{ann}_E(A)$ is a quasi-injective essential R-submodule of E, which implies by assumption that $\mathrm{ann}_E(A) = E$. Thus E is an injective hull of M as (R/A)-module. It follows that M is quasi-injective as right R-module since it is fully invariant. By assumption, M is injective and the proof is complete.

(ii) Let $T = R \times S$ and let M be a quasi-injective right T-module which is not semisimple. Note that there exists a decomposition $M = M_R \oplus M_S$ into T-submodules where $M_R = \text{ann}_M(S)$ and $M_S = \text{ann}_M(R)$. Let K be an injective hull of M_T. Then we can decompose K in the same way as $K = K_R \oplus K_S$. Here K_R and K_S turn out to be injective hulls of M_R and M_S as R- and S-modules, respectively. Since S is semisimple Artinian, we have $M_S = K_S$. Also, the quasi-injectivity of the right T-module M yields the quasi-injectivity of the right R-module M_R. Since M_T is not semisimple, M_R is not semisimple. Hence, by assumption on R, we must have $M_R = K_R$. Therefore, $M = K$, and so M is an injective right T-module. □

Proposition 3.11. *Let R be a right QIS-ring whose right socle is not essential in R_R. Then there exists a ring decomposition $R = S \times T$ where S is either zero or a semisimple Artinian ring and T is a right QIS-ring with zero right socle.*

Proof. Set $S = \text{Soc}(R_R)$ and $A = \text{ann}_l(S)$. If S is zero, then there is nothing to prove. Thus, suppose that $S \neq 0$. Since R is semiprime by Lemma 3.5, A contains no simple right R-submodules. Hence, by Lemma 3.9, we have $E = EA \oplus \text{ann}_E(A)$. Also, it is not difficult to see that semiprimeness of R yields $S \leq_e \text{ann}_l(A)$ as a right R-submodule. It follows that $S \leq_e \text{ann}_E(A)$; hence $\text{ann}_E(A)$ is an injective hull of S_R. Note that $\text{Hom}_R(EA, \text{ann}_E(A)) = 0$ since EA is a fully invariant submodule of E. This gives that S is EA-injective. Then $EA \oplus S$ is a quasi-injective right R-module. Since A contains no simple submodules, $EA \oplus S$ must be injective. In particular, S is injective; hence $S = \text{ann}_E(A)$. It follows that $E = EA \oplus S$, and so $R = T \oplus S$, where $T = EA \cap R$ is an ideal of R. This gives the desired decomposition of R. □

A ring R is called *right strongly prime* if for any nonzero ideal A of R, there exists a finite subset $S = \{a_1, \ldots, a_n\}$ of A such that $\text{ann}_r(S)$ is nonzero. Domains, prime right Goldie rings and simple rings are natural examples of these rings. Strongly prime rings are studies by Goodearl, Handelman and Lawrence [16, 17], Rubin [25], and Viola-Prioli [29]. Rubin and Viola-Prioli called a ring *absolutely torsion-free* if $\sigma(R) = 0$ for any left exact preradical $\sigma \neq \text{id}$ on Mod–R. Later, in [17], Handelman and Lawrence proved that the class of absolutely torsion-free rings coincides with that of right strongly prime rings. Thus, we see, in particular, that a right strongly prime ring has zero right socle and zero right singular ideal. Moreover, right strongly prime rings are precisely those rings such that Sing–R is the unique coatom in the lattice of hereditary pretorsion classes of right R-modules (see, [17, Proposition V.4]). In the following theorem, we see that the investigation of right QIS-rings which are neither right QI not right NMC reduces to strongly prime rings.

Theorem 3.12. *Let R be any ring. Then R is a right QIS-ring if and only if one of the following conditions hold:*

(i) R is a right QI-ring.

(ii) R is a right NMC-ring.

(iii) There exists a ring decomposition $R = S \times T$ such that S is either zero or a semisimple Artinian ring and T is a right strongly prime right QIS-ring.

Proof. We already know that if (i) or (ii) holds, then R is a right QIS-ring. Also, (iii) implies that R is a right QIS-ring, by Lemma 3.10 (ii). For the converse, we assume that R is a right QIS-ring which is not right NMC. Then by Proposition 3.3, the right socle of R is not essential in R_R. Thus, by Proposition 3.11, there exists a ring decomposition $R = S \times T$ such that S is either zero or a semisimple Artinian ring and T is a right QIS-ring with zero right socle. Note that if T is a right QI-ring, then so is R. It follows that we may assume, without loss of generality, that R is a right QIS ring with zero right socle which is not a right QI-ring. Then we need to prove that R is a strongly prime ring. To this end, we shall first prove that R is a prime ring.

Let A be a nonzero ideal of R. Since R is semiprime by Lemma 3.5, $\mathrm{ann}_r(A) = \mathrm{ann}_l(A)$. We denote this annihilator ideal by $\mathrm{ann}_R(A)$. Assume that $\mathrm{ann}_R(A) \neq 0$. We shall derive a contradiction. Set $B = \mathrm{ann}_R(A)$ and $C = \mathrm{ann}_R(B)$. Then, clearly, $\mathrm{ann}_R(C) = B$ and $B \cap C = 0$. Set $E = E(R_R)$. Then $E = EB \oplus \mathrm{ann}_E(B)$, by Lemma 3.9, and $C = \mathrm{ann}_E(B) \cap R \leq_e \mathrm{ann}_E(B)$ as right R-modules. Since $BC = 0$, we have $\mathrm{ann}_E(C) \supseteq EB \supseteq B$. On the other hand, $B = \mathrm{ann}_R(C) = \mathrm{ann}_E(C) \cap R \leq_e \mathrm{ann}_E(C)$ as right R-modules. Then we have $EB = \mathrm{ann}_E(C)$. Thus we obtain that $E = \mathrm{ann}_E(C) \oplus \mathrm{ann}_E(B)$ and $\mathrm{ann}_E(C)$ is an injective hull of B. Similarly, $\mathrm{ann}_E(B)$ is an injective hull of C. Note that $\mathrm{ann}_E(B)C = \mathrm{ann}_E(B)$. Indeed, $\mathrm{ann}_E(B)C$ is a non-semisimple quasi-injective right R-module, hence an injective right R-module. In this case, $\mathrm{ann}_E(B) = \mathrm{ann}_E(B)C \oplus U$ for some $U \leq E_R$. Then $UC \subseteq \mathrm{ann}_E(B)C \cap U = 0$, and so $U \subseteq \mathrm{ann}_E(C) \cap \mathrm{ann}_E(B) = 0$. Similarly, we have $\mathrm{ann}_E(C)B = \mathrm{ann}_E(C)$.

Let S be a semisimple right (R/B)-module. Suppose that S is not injective as (R/B)-module. Let E' be an injective hull of S_R. Then we have $S \subset \mathrm{ann}_{E'}(B) \subseteq E'$. Since $\mathrm{ann}_{E'}(B)$ is an essential fully invariant submodule E', by assumption that R is right QIS, $E' = \mathrm{ann}_{E'}(B)$, or equivalently, $E'B = 0$. Let $E'' = \mathrm{ann}_E(C)$. Recall that E'' is an injective hull of B and $E''B = E''$. It is now easy to see that $\mathrm{Hom}_R(E'', E') = 0$. Therefore, S_R is E''-injective and hence $S \oplus E''$ is a quasi-injective right R-module which is neither semisimple nor injective, a contradiction. It follows that every semisimple right (R/B)-module is injective. Since R/B is also a right QIS-ring by Lemma 3.10 (i), we obtain that R/B is a right QI-ring. Using symmetry, one can also deduce that R/C is a right QI-ring. In particular, we obtain that R is right Noetherian since it can be embedded in the Noetherian right R-module $(R/B) \oplus (R/C)$. Since R is not prime, there exist finitely many minimal prime ideals $P_1, \ldots, P_n$ of R with $n \geq 2$. Now if we write $B' = P_1$ and $C' = P_2 \cap \ldots \cap P_n$, we have $B' \cap C' = 0$, $\mathrm{ann}_R(B') = C'$ and $\mathrm{ann}_R(C') = B'$. By above arguments, we see that R/B' and R/C' are both right QI-rings. Since R/B' is a prime right QI-ring, it is simple. Thus, $R = B' \oplus C'$. It follows that R is a direct product of right QI-rings, and hence R is right QI. This gives us the desired contradiction. Therefore R is a prime ring.

We complete the proof by showing that R is strongly prime. Let σ be a left exact preradical on Mod–R such that $\sigma \neq \mathrm{id}$. We shall show that $\sigma(R) = 0$, which completes the proof. Assume contrarily that $\sigma(R) \neq 0$. Then $\sigma(E) \neq 0$. Since E contains no simple right R-submodules, $\sigma(E)$ is a fully invariant submodule of E which is not semisimple. Thus, $\sigma(E)$ is an injective right R-module by assumption on R. It follows that $E = \sigma(E) \oplus Y$ for some nonzero right R-submodule Y of E. Since $\sigma(E)$ is a fully invariant submodule of E, $\mathrm{Hom}_R(\sigma(E), Y) = 0$. Hence we have $\mathrm{Hom}_R(J, I) = 0$, where $I = Y \cap R$ and $J = \sigma(E) \cap R$. It follows that $IJ = 0$, which is a contradiction since R is prime and both I, J are nonzero right ideals of R. Therefore, $\sigma(R) = 0$. □

Remark 3.13. We note that right strongly prime rings have already some limitations in possessing quasi-injective modules. Let R be a right strongly prime ring. Then Sing–R is the unique coatom in the lattice of hereditary pretorsion classes by Proposition V.4 in [17]. Let H be a quasi-injective right R-module which is not injective. Then $\mathfrak{In}^{-1}(H) \neq$ Mod–R; hence $H \in \mathfrak{In}^{-1}(H) \subseteq$ Sing–R. It follows that any quasi-injective right R-module that is not injective must be singular. In particular, nonsingular indecomposable injective right R-modules do not contain proper nonzero quasi-injective submodules.

Proposition 3.14. *Let R be a right fully bounded ring. Then R is a right QIS-ring if and only if R is a right NMC-ring.*

Proof. Let R be a right QIS-ring which is not a right NMC-ring. Then we may assume, by Theorem 3.12 and [3, Lemma 2.4], that R is either a right QI-ring or a prime right QIS-ring with $\mathrm{Soc}(R_R) = 0$. In the former case, R must be a semisimple Artinian ring since it is a right fully bounded ring which is a finite direct product of simple rings (see [13, Theorem 2]). But this yields a contradiction since semisimple Artinian rings are right NMC. Thus, we may suppose that R is a right fully bounded prime right QIS-ring with $\mathrm{Soc}(R_R) = 0$. Since R modulo any nonzero proper ideal is semisimple by Lemma 3.7, every singular right R-module is semisimple, i.e., R is a right SI-ring. Hence, Sing–R = SSMod–R. It follows that SSMod–R is the unique coatom in the lattice of hereditary pretorsion classes of right R-modules. However, by [21, Theorem 2.9], this just means that R is a right NMC-ring, again a contradiction. Therefore R must be a right NMC-ring. This completes the proof since the converse is straightforward. □

Corollary 3.15. *Let R be a commutative ring which is not semisimple Artinian. Then R is a right QIS-ring if and only if there exists a ring decomposition $R = S \times T$ such that S is either zero or a semisimple Artinian ring and T is a ring with exactly one nonzero proper ideal.*

Proof. Follows from Proposition 3.14 and [3, Theorem 4.3]. □

4 Noetherian rings with restricted socle condition

This section is devoted to the investigation of right Noetherian right QIS-rings with restricted right socle condition. Following [13], we say that a ring R satisfies the *restricted right socle condition* if, whenever I is a proper essential right ideal of R, then R/I has at least one simple submodule. Notice that this is equivalent to the condition that every singular right R-module is semiartinian. Right SI-rings and hereditary prime Noetherian rings (HNP for short) are natural examples of rings with restricted right socle condition. Also, it is shown in [26, Propositon 2.2] that any right NMC-ring satisfies this condition. On the other hand, Faith [13] proved that any right QI-ring with restricted right socle condition must be right hereditary, a partial verification of Boyle's conjecture.

In light of Theorem 3.12, we divide our investigation of right Noetherian right QIS-rings with restricted right socle condition into two steps. In the first step, we consider QI-rings and NMC-rings and give the following

Proposition 4.1. *Let R be any ring which is not semisimple Artinian. Then the following conditions are equivalent:*

(i) R is a right QI-ring with restricted right socle condition.

(ii) R is a right Noetherian right V-ring with restricted right socle condition.

(iii) R is a right SI-ring with $\mathrm{Soc}(R_R) = 0$.

Moreover; if, in addition, R is indecomposable, then (i)–(iii) are equivalent to

(iv) R is a right NMC-ring with $\mathrm{Soc}(R_R) = 0$.

Proof. Note that right QI-rings (resp., right Noetherian right V-rings) are precisely those rings which are finite products of simple right QI-rings (resp., of simple right Noetherian right V-rings), by [13, Theorem 2 and Corollary 3]. Also, any ring with zero right socle is right SI if and only if it is a finite product of rings Morita equivalent to a right SI-domain, by [14, Theorem 3.11]. On the other hand, it is not difficult to see that a product $R_1 \times \cdots \times R_n$ of rings $R_1, \ldots, R_n$ satisfies the restricted right socle condition if and only if so is each R_i. Thus, we may assume that R is an indecomposable ring.

(i) $\Rightarrow$ (ii): Straightforward.

(ii) $\Rightarrow$ (iii): Since R is indecomposable, it is simple; hence $\mathrm{Soc}(R_R) = 0$. Let M be a singular right R-module. Since R satisfies the restricted right socle condition, $\mathrm{Soc}(M) \leq_e M$, and so $\mathrm{Soc}(M) = M$ by assumption. Thus R is right SI.

(iii) $\Rightarrow$ (iv): By [14, Theorem 3.11], R is Morita equivalent to a right SI-domain. So, R is a right NMC-ring by [26, Theorem 4.2].

(iv) $\Rightarrow$ (i): Note that a right SI-domain is a (right Noetherian) right PCI-domain; hence a right QI-domain by [4, Theorem 7]. Also, any right NMC-ring satisfies the restricted right socle condition. Then by [26, Theorem 4.2], the proof is complete. □

Lemma 4.2. *Let R be a right strongly prime ring. Then every nonzero injective right R-module is faithful.*

Proof. Let E be a nonzero injective right R-module. Since R is right strongly prime, we have $\omega_0^E \leq Z$, where $Z(M)$ denotes the singular submodule of any right R-module M. It follows that $\text{ann}_R(E) = \omega_0^E(R) \subseteq Z(R_R) = 0$. □

In the second step, we focus on right strongly prime (or, equivalently, prime) right Noetherian right QIS-rings which are neither right QI nor right NMC. The following proposition provides a useful characterization for a class of these rings.

Proposition 4.3. *Let R be a prime right Noetherian ring with restricted right socle condition. If R is not a right V-ring, then R is right QIS if and only if the following statements hold:*

(i) There exists a unique isomorphism class of non-injective simple right R-modules.

(ii) If W is a non-injective simple right R-module, then it is the only proper nonzero fully invariant submodule of $E(W)$.

In this case, R contains at most one nonzero proper (two-sided) ideal.

Proof. Let R be a right QIS-ring. Since R is not a right V-ring, there exists a non-injective simple right R-module, W say. By assumption, every nonzero singular right R-module contains a simple submodule, which yields, in particular, that $E(W)/W$ has a simple submodule. It follows that there exists a uniserial submodule U of $E(W)$ of length 2. Note that $\text{Tr}_R(U, E(W))$ is a non-semisimple quasi-injective submodule of $E(W)$. Since R is right QIS, we have $\text{Tr}_R(U, E(W)) = E(W)$. Since any nonzero homomorphic image of U in $E(W)$ contains W, $E(W) = \sum_{\lambda\in\Lambda} U_\lambda$ for some index set Λ and submodules U_λ isomorphic to U for all $\lambda \in \Lambda$. In this case, $E(W)/W = \sum_{\lambda\in\Lambda}(U_\lambda/W)$ is a homogeneous semisimple right R-module. Let W' be a non-injective simple right R-module which is not isomorphic to W. If W is $E(W')$-injective, then $W \oplus E(W')$ is quasi-injective right R-module which is neither semisimple nor injective. Thus W is not $E(W')$-injective, and hence $\text{Tr}_R(E(W'), E(W)) \nsubseteq W$. Let $f : E(W') \to E(W)$ be a nonzero R-homomorphism such that $W \subset \text{Im}(f)$. Since $W' \ncong W$, $W' \subseteq \ker(f)$. Thus, by above arguments, $E(W')/\ker(f)$ is nonzero semisimple. But $E(W')/\ker(f)$ is embedded in $E(W)$, and so $\text{Im}(f) \subseteq W$, a contradiction. Therefore W is the unique non-injective simple right R-module, establishing (i). Finally, (ii) follows immediately from the assumption that R is a right QIS-ring.

Conversely assume that the conditions (i) and (ii) are satisfied. Let H be a quasi-injective right R-module which is not semisimple and let $G = E(H)$, an injective hull of H. Then $G = \bigoplus_{\lambda\in\Lambda} G_\lambda$ for some index set Λ and indecomposable injective right R-modules G_λ for all $\lambda \in \Lambda$. Since H is quasi-injective, we have $H = \bigoplus_{\lambda\in\Lambda} H_\lambda$, where $H_\lambda = H \cap G_\lambda$. Note that for each $\lambda \in \Lambda$, H_λ is a nonzero quasi-injective submodule of G_λ and G_λ is an injective hull of H_λ. Given any $\mu \in \Lambda$, there are two cases for G_μ; namely, G_μ is nonsingular or G_μ has nonzero singular submodule. In the former case we have $H_\mu = G_\mu$ since R is strongly prime which implies that G_μ contains no nonzero proper quasi-injective submodules (see Remark 3.13). Now assume that G_μ has nonzero singular submodule. Then G_μ contains a simple submodule. Since G_μ is indecomposable, it is an injective hull of a simple right R-module. By assumptions, either G_μ is simple or it is isomorphic to $E(W)$, where W is the unique (up to isomorphism) non-injective simple right R-module. In the latter case, either $H_\mu = G_\mu$ or H_μ is the unique simple submodule of G_μ by (ii). Let $\Lambda_1 = \{\lambda \in \Lambda : H_\lambda = G_\lambda\}$. We shall show that $\Lambda = \Lambda_1$. We assume the contrary and look for a contradiction. Let $\nu \in \Lambda \setminus \Lambda_1$. By above arguments, we have $G_\nu \cong E(W)$ and H_ν is the unique simple submodule of G_ν. Let $\mu \in \Lambda_1$. Then $H_\mu = G_\mu$. By [23, Proposition 1.18], H_ν is G_μ-injective. Thus G_μ cannot be nonsingular since R is right strongly prime. It follows from above arguments that G_μ is either simple or isomorphic to $E(W)$. The latter case is impossible since $H_\nu \cong W$ is not injective relative to $E(W)$. Therefore H_μ is simple for each $\mu \in \Lambda_1$. On the other hand, H_ν is simple (necessarily isomorphic to W) for all $\nu \in \Lambda \setminus \Lambda_1$. It turns out that $H = \bigoplus_{\lambda \in \Lambda} H_\lambda$ is semisimple, a contradiction. Thus $\Lambda = \Lambda_1$, and so $H = G$. Therefore R is a right QIS-ring.

For the last statement, let R be a prime right Noetherian right QIS-ring with restricted right socle condition. If R is simple, then we are done. Thus assume that R is not simple and let A be a nonzero proper ideal of R. Since R is right strongly prime, $\mathrm{Soc}(R_R) = 0$. By Lemma 3.7, the ring R/A is semisimple Artinian. By (ii) and Lemma 4.2, R/A is a simple Artinian ring; hence A is maximal. Therefore, every nonzero proper ideal of R is maximal. Since R is prime, this gives that R has a unique nonzero proper ideal. □

By Proposition 4.3 we see how a right QIS-ring can be close to a right V-ring. Now, in the following lemma, we see that a right QIS-ring is also close to being a right NMC-ring.

Lemma 4.4. *Let R be a right strongly prime ring with restricted right socle condition. If R is a right QIS-ring, then there are no hereditary pretorsion classes properly between SSMod–R and Sing–R, i.e., $[SSMod\text{–}R, Mod\text{–}R] = \{SSMod\text{–}R, Sing\text{–}R, Mod\text{–}R\}$ in the lattice of hereditary pretorsion classes of right R-modules.*

Proof. We let $\sigma \in$ lep–R such that $\mathrm{Soc} < \sigma < Z$ and look for a contradiction, by which the result follows by correspondence. Now there exists right R-modules M, N such that $\mathrm{Soc}(M) \subsetneq \sigma(M)$ and $\sigma(N) \subsetneq Z(N)$. If $L = M \oplus N$, then $\mathrm{Soc}(L) \subsetneq \sigma(L) \subsetneq Z(L)$. Set $E = E(L)$. Then we also have $\mathrm{Soc}(E) \subsetneq \sigma(E) \subsetneq Z(E)$. Since $\sigma(E)$ is quasi-injective that is not semisimple, it must be injective; hence $\sigma(E)$ is a direct summand of $Z(E)$, a contradiction since $Z(E)$ has essential socle by assumption. □

Recall that a preradical σ on Mod–R is said to be *stable* if $\sigma(E)$ is injective for every injective right R-module E.

Lemma 4.5. *Let R be any ring. Then R is a right QIS-ring if and only if every left exact preradical σ with* $\mathrm{Soc} < \sigma$ *is stable.*

Proof. By [21, Theorem 2.9], for any left exact preradical σ such that $\mathrm{Soc} \leq \sigma$, there exists a right R-module M such that $\sigma = i_M$. Thus the lemma follows from [10, Proposition 5.3]. □

Proposition 4.6. *Let R be a prime right Noetherian ring with restricted right socle condition. Then the following conditions are equivalent:*

(i) R is a right QIS-ring.

(ii) There are no hereditary pretorsion classes properly between SSMod–R and Sing–R.

(iii) Every singular right R-module is either injective or poor.

Proof. (i) $\Rightarrow$ (ii): By Lemma 4.4.

(ii) $\Rightarrow$ (i): If SSMod–R = Sing–R, then R is a right NMC-rings (since R is right strongly prime), and so it is right QI, by Proposition 4.1. Thus assume that SSMod–$R \neq$ Sing–R (or equivalently, Soc $< Z$). Now Z and id are the only left exact preradicals strictly greater that Soc and, since R is right nonsingular, Z is stable; hence R is a right QIS-ring by Lemma 4.5.

(i) $\Rightarrow$ (iii): If R is a right V-ring, then it is right SI by Proposition 4.1. Thus assume that R is not a right V-ring. Again by Proposition 4.1, this gives that R is not right NMC; hence SSMod–$R \neq$ Sing–R. Note that there exists a unique (up to isomorphism) non-injective simple right R-module, W say, by Proposition 4.3. We first show that W is poor. Assume the contrary. Since W is not injective, $\mathfrak{In}^{-1}(W)$ lies between SSMod–R and Sing–R. It follows that $\mathfrak{In}^{-1}(W)$ = Sing–R by Lemma 4.4. But, in this case, W must be $E(W)$-injective, a contradiction. Thus W is poor. Now let M be any singular module which is not injective. Then $\mathfrak{In}^{-1}(M)$ lies between SSMod–R and Sing–R. We shall show that $\mathfrak{In}^{-1}(M)$ = SSMod–R. Assume the contrary. Then by Lemma 4.4, $\mathfrak{In}^{-1}(M)$ = Sing–R, and hence M is quasi-injective. Since R is right QIS and M is not injective, M must be semisimple. Note that the only non-injective semisimple right R-modules are those containing a copy of W since W is the only non-injective simple right R-module and R is right Noetherian. Therefore M has a copy of W as a direct summand, and so it is poor. But this yields a contradiction since, in this case, we have SSMod–R = $\mathfrak{In}^{-1}(M)$ = Sing–R, which implies that R is right NMC.

(iii) $\Rightarrow$ (i): First we shall show that there exists a unique isomorphism class of non-injective simple right R-modules. Let W and W' be non-injective simple right R-modules. By the restricted socle condition, there exists a uniserial module U' of length 2 which contains W'. Note that every simple right R-module is singular since $\mathrm{Soc}(R_R) = 0$. Hence W is poor by our assumption. Since U' is not semisimple, W cannot be U'-injective; so $\mathrm{Tr}_R(U', E(W)) \nsubseteq W$. Thus there exists an R-homomorphism $f : U' \to E(W)$ such that $\mathrm{Im}(f) \supset W$. Since U' has length 2, f must be an isomorphism. Indeed, if f is not an isomorphism, then we must have $\ker(f) = W'$, which gives that $\mathrm{Im}(f) \cong U'/W'$ is simple, a contradiction. Therefore $W' \cong W$.

Next we show that if W is a non-injective simple right R-module, then it is the only nonzero proper fully invariant submodule of $E(W)$. To this end, let M be a proper nonzero fully invariant submodule of $E(W)$. Then, by assumption, M is poor. Since $M \in \mathfrak{In}^{-1}(M)$ = SSMod–R, we must have $M = W$, as desired, completing the proof by Proposition 4.3. □

In Theorem 3.12, we see that indecomposable right QIS-rings fall into three classes of rings, namely, right QI-rings, right NMC-rings, and strongly prime rings. The following proposition provides us with a way for producing concrete examples of right QIS-rings which are neither right QI nor right NMC using basic idealizer rings. Recall that given a ring S and a right ideal A in S, one can form the largest subring of S, denoted $\mathbb{I}_S(A)$, containing A as a two-sided ideal. If A is such that $SA = S$ and $(S/A)_S$ is isotypic (or, in other words, homogeneous) semisimple, then we call the ring $R = \mathbb{I}_S(A)$ a *basic idealizer*. Let $R = \mathbb{I}_S(A)$ be a basic idealizer. Then R/A is a simple Artinian ring, A is an idempotent (maximal) ideal, and if U is a simple right S-module such that $(S/A)_S \cong U^{(n)}$ for some $n \in \mathbb{N}$, then U_R is a uniserial module of length 2. If $W = \mathrm{Soc}(U_R)$ and $V = U/W$, then $V \ncong W$ and $\mathrm{ann}_R(W) = A$. In this case, we say that $R = \mathbb{I}_S(A)$ is a basic idealizer of type $U = [VW]$. Note that any simple right S-module not isomorphic to U is also a simple right R-module. In fact, a simple right R-module is of the form V, W, or X, where X is a simple right S-module not isomorphic to U. We refer the interested reader to the book of Levy and Robson [20] in which a very detailed account of idealizer rings is given.

Proposition 4.7. *Let S be an indecomposable right SI-ring with* $\mathrm{Soc}(R_R) = 0$*. If $R = \mathbb{I}_S(A)$ is a basic idealizer for a right ideal A of S, then R is a right QIS-ring which is neither right QI nor right NMC.*

Proof. By Proposition 4.1, S is a simple right Noetherian right V-ring that is not semisimple Artinian. Assume that the basic idealizer ring $R = \mathbb{I}_S(A)$ is of type $U = [VW]$. First, we show that R is a prime right Noetherian ring. By [20, Theorem 4.19], we immediately see that R is right Noetherian. Since S is simple, it does not contain a simple right ideal. It follows that A is essential in S_S, and so there exists a regular element $c \in S$ contained in A. Then $cS \subseteq R$, and hence R is prime and the right quotient rings of R and S coincide (see [22, 2.3.6, 3.1.4, and 3.1.6]).

Now we show that every singular right R-module is semiartinian. Note that it suffices to consider only cyclic singular modules. Let B be a proper essential right ideal of R. We want to show that $(R/B)_R$ is Artinian. First, we assume that B is contained in A. If $B = A$, then there is nothing to prove. Thus we assume $B \neq A$. Note that we have $B \subseteq BS \subseteq A$. Suppose that $B = BS$. In this case, B is a right ideal of S. Since R is prime right Noetherian, B contains a regular element d of R. Note that d is also a regular element of S since the right quotient rings of R and S coincide, and so S/B is a cyclic singular right S-module. Therefore, $(S/B)_S$ is Noetherian and semiartinian by assumption. Thus $(S/B)_S$ has finite length. It follows from [20, Corollary 4.18 (i)] that $(S/B)_R$ has finite length. In particular, $\mathrm{Soc}_R(A/B) \neq 0$. Now suppose that $B \neq BS$. Then BS/B is a nonzero semisimple right R-module by [20, Lemma 4.10]. Therefore, in any case, we have $\mathrm{Soc}_R(A/B) \neq 0$. Let $\mathrm{Soc}_R(A/B) = B_1/B$. Then B_1 is an essential right ideal of R such that $B \subset B_1 \subseteq A$. Thus replacing B with B_1, above arguments show that $B_2/B_1 := \mathrm{Soc}_R(A/B_1) \neq 0$. Continuing in this fashion, we obtain a strictly ascending chain $B_0 = B \subset B_1 \subset B_2 \subset \ldots$ of essential right ideals of R contained in A where B_{i+1}/B_i is of finite length for each $i \geq 0$. Since R is right Noetherian, this process must terminate (necessarily at A) after a finite number of steps, which yields that $(A/B)_R$ has finite length. Since $(R/A)_R$ is finitely generated semisimple, it is of finite length; hence $(R/B)_R$ has finite length. Now let B be an arbitrary proper essential right ideal of R. Then $A \cap B$ is an essential right ideal of R contained in A. By above, $R/(A \cap B)$ is an Artinian right R-module, and so is R/B, as desired.

Next we prove that W is the unique (up to isomorphism) non-injective simple right R-module and it is the only nonzero proper fully invariant submodule of $E(W)$. Since U_S is injective, U_R is injective by [20, Proposition 4.13 (ii)]. This gives that $U_R \cong E(W)$. Hence W_R is a non-injective simple right R-module and it is the only nonzero proper fully invariant submodule of $E(W)$. Let X be a simple right R-module which is not isomorphic to V or W. Then by [20, Corollary 4.8 (iii)], X is a simple right S-module not isomorphic to U_S. Since S is a right V-ring, X_S is injective. Hence X_R is also injective again by [20, Proposition 4.13 (ii)]. Thus it remains to prove that V_R is injective. Since every singular right R-module contains a simple submodule, in order to show that V_R is injective, it is enough to prove that $\operatorname{Ext}^1_R(Y,V) = 0$ for every simple right R-module Y. Suppose on the contrary that there exists a simple right R-module Y such that $\operatorname{Ext}^1_R(Y,V) \neq 0$. Since $\operatorname{Ext}^1_R(W,V) \cong \operatorname{Ext}^1_S(U,U) = 0$ by [20, Theorem 5.7 (iii)] and the fact that U_S is injective, we have $Y \ncong W$. Also by [20, Theorem 5.7 (i)], $\operatorname{Ext}^1_R(V,V) = 0$; hence $Y \ncong V$. It follows from [20, Corollary 4.8 (iii)] that Y is a simple right S-module. Therefore, $\operatorname{Ext}^1_R(Y,V) \cong \operatorname{Ext}^1_S(Y,U) = 0$, by [20, Theorem 5.3 (iii)], a contradiction. It follows that V_R is injective, and so W_R is the only non-injective simple right R-module. Therefore, R satisfies the conditions (i) and (ii) of Proposition 4.3, proving that R is a right QIS-ring. Note that the ring R is neither right QI nor right NMC, by Proposition 4.1, since R is an indecomposable ring with zero right socle which is not a right V-ring. □

Let S be a non-Artinian HNP ring which is also a right V-ring (see, for example, [7], [8], [19], or [24]) and take any right ideal A of S such that $(S/A)_S$ is isotypic semisimple. Then $R = \mathbb{I}_S(A)$ is a basic idealizer and, by above proposition, R is a right QIS-ring which is neither right QI nor right NMC. The following theorem shows that this is the only way in which we can produce examples of such rings R in the class of HNP rings.

Given an HNP ring R, we mean by an overring of R any intermediate ring between R and the ring of quotients of R. Note that every non-Artinian overring of an HNP ring is also HNP (see [20, Theorem 13.5]).

Theorem 4.8. *Let R be an HNP ring which is not simple. Then the following statements are equivalent:*

(i) R is a right QIS-ring which is not right QI.

(ii) There is exactly one isomorphism class of non-injective simple right R-modules and if W is a non-injective simple module, then $E(W)$ is a uniserial module of length 2.

(iii) R is a basic idealizer ring from a right QI overring that is not simple Artinian.

Proof. (i)⇒(ii): Let R be a right QIS ring which is not a right QI ring. By [4, Theorem 5], R is not a right V-ring. Then by Proposition 4.3 (ii), there is exactly one isomorphism class of non-injective simple right R-modules. Since R is not simple, there exists a nonzero maximal ideal A of R. Let W be a simple right R-module annihilated by A. Then by [20, Theorem 15.2] W is not injective and there exists a unique (up to isomorphism) simple right R-module V such that $\mathrm{Ext}^1_R(V,W) \neq 0$. This means that there exists a uniserial right R-module U such that $W \subset U$ and $U/W \cong V$. In particular, we have $W \subset U \subseteq E(W)$. Since R contains no invertible ideals by Lemma 3.8, it follows from [20, Proposition 15.8] that $V \ncong W$, and hence V is injective. This yields that V is faithful by [20, Theorem 15.2] (or by Lemma 4.2 of the present paper). It follows from [20, Corollary 24.11] that U is the unique submodule of $E(W)$ of length 2. Observe that for any nonzero homomorphism $f : U \to E(W)$ is an isomorphism since $V \ncong W$. This shows that U is a fully invariant submodule of $E(W)$. Hence $U = E(W)$, by Proposition 4.3 (ii). This establishes (ii).

(ii)⇒(i): Immediate by Proposition 4.3 (ii).

(ii)⇒(iii): Since R is not simple, there exists a nonzero maximal ideal A of R. Let W be a simple right R-module annihilated by A. By (ii) together with [20, Theorem 15.2], W is the unique (up to isomorphism) non-injective simple right R-module and there exists a unique simple module V such that $\mathrm{Ext}^1_R(V,W) \neq 0$. Then there exists a uniserial right R-module U of length 2 such that $W \subset U$ and $U/W \cong V$. By (ii), $U \cong E(W)$. Then V is injective, and so $V \ncong W$. This gives that A is not invertible by [20, Proposition 15.8 (iii)]; hence it is idempotent. Thus, by [20, Theorem 14.9], $R = \mathbb{I}_S(A)$ is a basic idealizer of type $U = [VW]$, where S is the subring of the quotient ring of R consisting of elements x such that $Ax \subseteq A$. Combining Theorem 4.4 and Proposition 4.13 (ii) in [20] we see that S is a right V-ring. Therefore S is a right QI ring.

(iii)⇒(i): Immediate by Proposition 4.7. □

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