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Kinematic Control of Redundant Robot Manipulators: A Tutorial

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Abstract. In this paper, we present a tentatively comprehensive tutorial report of the most recent literature on kinematic control of redundant robot manipulators. Our goal is to lend some perspective to the most widely adopted on-line instantaneous control solutions, namely those based on the simple manipulator's Jacobian, those based on the local optimization of objective functions in the null space of the Jacobian, those based on the task space augmentation by additional constraint tasks (with task priority), and those based on the construction of inverse kinematic functions.

Key words. Redundant manipulators, robot kinematic control, singularities, pseudoinverses, optimization methods, projection operators.

1. Introduction

The scientific and technological perspectives of robotics can be greatly enhanced by considering redundancy which has been recognized as offering greater flexibility and versatility in today's robot manipulators. It is for this reason that, during the last decade, an ever increasing number of researchers have been directing their efforts towards the adoption of redundancies, not only at the conventional kinematical level but also at the sensing and information handling levels, for the realization of hierarchical intelligent robot control systems. In the present study, we will focus our attention only on kinematic redundancy, although we are aware that the other types of redundancies are equally important to the development of more dexterous robot systems.

Kinematic redundancy occurs when a manipulator possesses more degrees of freedom than the minimum number required to execute a given task. Until recently, it was usually claimed that a six-degree-of-freedom nonredundant robot manipulator is a 'general purpose' device, since it can 'freely' position and orient an object in the Cartesian workspace. Not surprisingly, this claim is still true for the majority of industrial robot manufacturers, and this is one of the reasons, perhaps, why we have registered the realization of kinematically redundant manipulator prototypes mainly in academical research centers. The typical reluctance to produce redundant robot systems observed in industry finds its justifications in that redundancy involves mechanical and control complexity, and then increased costs.

It is not difficult to discover that a six-degree-of-freedom geometry can no longer be considered a general purpose manipulator. This geometry, in fact, has a number of kinematic flaws such as limited joint ranges, workspace obstructions, and kinematic singularities which, in turn, prevent the manipulator from attaining arbitrarily assigned end-effector locations in its workspace. It is then desirable for a 'true' general purpose manipulator to dispose of additional degrees of freedom to overcome the above limitations; the seven-degree-of-freedom human arm constitutes an excellent model of a dexterous redundant structure.

Nonetheless, it should be made clear that manipulator redundancy can be established only with respect to the given task in the sense that a manipulator is termed redundant when the number of active joints exceeds the number of variables which identify the task. For instance, a three-degree-of-freedom planar manipulator becomes redundant if the tip orientation angle is of no concern for a two-dimensional motion task. Analogously, a six-degree-of-freedom manipulator becomes redundant with respect to all those five-dimensional end-effector tasks, such as arc welding, laser cutting, spray painting, which do not require the specification of the sixth roll angle typically encountered in industrial robot geometries.

When a manipulator is redundant, it is anticipated that the inverse kinematic problem admits infinite solutions. This implies that, for a given constant location of the manipulator's end-effector, it is possible to induce a self-motion of the structure, i.e. without changing the location of the end-effector. Thus, the arm can be reconfigured to find better postures for an assigned set of task requirements. More generally, if a motion task trajectory is commanded to the end-effector, it is possible in principle to continuously modify the joint motion in such a manner that not only the end-effector task is correctly executed, but also a suitable constraint task is accomplished at best.

A number of solution techniques for solving the kinematic control problem for redundant manipulators have been suggested by researchers. In this paper, we review some of – what we believe – the most relevant literature that has appeared up until 1988, including our own work, by providing unified frameworks for characterizing the features of each method.

Most of the proposed approaches are based on the instantaneous or local resolution of redundancy at the velocity level through the use of the manipulator's Jacobian matrix. Global optimization techniques have also been proposed but they involve increased computational complexity which rules them out in practical on-line implementation for which the end-effector trajectory is continuously modified based on sensory feedback information. The local resolution methods can be distinguished in those optimizing a suitable scalar objective function in the null space of the Jacobian matrix, and those adopting a task space augmentation by defining a suitable constraint task in addition to the end-effector task, and eventually resorting to a task priority strategy. Conceptually close to the latter is the method based on the construction of inverse kinematic functions on suitable reduced workspaces. Proper references will be given throughout the paper wherever each method is discussed.

The paper is organized as follows. Section 2 presents the inverse kinematic problem for redundant manipulators. Simple Jacobian-based techniques are discussed in

Section 3. The gradient projection method is illustrated in Section 4 and the augmented task space approach is presented in Section 5. Section 6 concentrates on the inverse kinematic function method. Conclusions are drawn in a final section.

2. The Kinematic Control Problem for Redundant Manipulators

In order to fix the notation throughout the paper, the direct kinematic mappings of interest for robot manipulators, can be written as

$$\mathbf{x} = \mathbf{f}(\mathbf{q}), \tag{1}$$

$$\dot{\mathbf{x}} = \mathbf{J}(\mathbf{q})\dot{\mathbf{q}}, \tag{2}$$

where $\mathbf{q}$ is the $(n \times 1)$ vector of joint variables, $\mathbf{x}$ is the $(m \times 1)$ vector of task variables,* $\mathbf{f}$ is a differentiable nonlinear vector function whose structure and parameters are assumed to be known for any given manipulator, $\mathbf{J}$ is the $(m \times n)$ configuration dependent Jacobian matrix – formally defined as $\partial\mathbf{f}/\partial\mathbf{q}$ – and the upper dot denotes time derivative.

For a given trajectory in the task space $\mathbf{x}(t)$, the *kinematic control* problem can be formulated as to find a joint space trajectory $\mathbf{q}(t)$ such that $\mathbf{f}(\mathbf{q}(t)) = \mathbf{x}(t)$ is satisfied.

It is clear that in the case of a redundant manipulator with respect to a given task $(m < n)$, the inverse kinematic problem admits infinite solutions. This suggests that redundancy can be conveniently exploited to meet additional constraints on the kinematic control problem in order to obtain greater manipulability in terms of manipulator configurations and interaction with the environment.

If the robot is required to move in a cluttered environment, for instance, avoidance of obstacles (Maciejewski and Klein, 1985) and mechanical joint limits (Liégeois, 1977) is usually desired. In other applications, it could be of interest to minimize the joint actuator power consumption (Vukobratović and Kirćanski, 1984).

The other important point in purposely adopting redundancy is the avoidance of kinematic singularities, which occur when the matrix $\mathbf{J}$, at some configuration $\mathbf{q}$, has rank less than m. In this case the manipulator loses its ability to move along or rotate about some direction of the task space, meaning that its manipulability is reduced. The manipulability measure introduced by Yoshikawa (1985b) as $\sqrt{\det(\mathbf{J}\mathbf{J}^T)}$, and the dexterity measures proposed by Klein and Blaho (1987), e.g. the matrix condition number and the minimum singular value of the matrix $\mathbf{J}\mathbf{J}^T$, represent indices of the ability of a manipulator to arbitrarily position and orient its end-effector. Isotropy criteria have been discussed by Angeles (1988). The dynamic manipulability measure (Yoshikawa, 1985a), instead, takes the arm dynamics into account. Related to these measures is also the concept of task compatibility (Chiu, 1988), according to which the matrix $\mathbf{J}\mathbf{J}^T$ is utilized to determine quantitative indices of the ability to perform an exertion/control task along a given direction of the task space.

* Notice that, in the most general case, $\mathbf{x}$ does not necessarily denote the end-effector location but it can be any direct kinematic function of the joint variables expressed in a suitable reference frame.

3. Simple Jacobian-Based Techniques

As emphasized in the Introduction, most of the approaches for solving redundancy appeared in the literature are based on the inversion of the mapping (2) in that a solution in terms of the joint velocities is sought as

$$\dot{\mathbf{q}} = \mathbf{K}(\mathbf{q})\dot{\mathbf{x}}, \tag{3}$$

where $\mathbf{K}$ is a suitable $(n \times m)$ control matrix based on the Jacobian matrix. In his pioneering work on resolved-rate control, Whitney (1969) proposed to use the Moore–Penrose *pseudoinverse* of the Jacobian matrix as

$$\dot{\mathbf{q}} = \mathbf{J}^{\dagger}(\mathbf{q})\dot{\mathbf{x}}, \tag{4}$$

where $\mathbf{J}^{\dagger}$ is the matrix defined as $\mathbf{J}^{\dagger} = \mathbf{J}^T(\mathbf{J}\mathbf{J}^T)^{-1}$.

At first glance, this solution is quite attractive since the pseudoinverse has a least squares property that generates the minimum norm joint velocities. However, Baillieul, Hollerback and Brockett (1984) proved that kinematic singularities are not avoided in any practical sense, since joint velocities are minimized only instantaneously and then can become arbitrarily large near singular configurations.

In order to overcome this drawback, Wampler (1986) and Nakamura and Hanafusa (1986) independently proposed the use of a *damped least-square inverse* of the Jacobian matrix in the form of $\mathbf{J}^* = \mathbf{J}^T(\mathbf{J}\mathbf{J}^T + \lambda^2\mathbf{I})^{-1}$, corresponding to a modified Jacobian that is nonsingular in the whole workspace. Under this control, one obtains only an approximate inverse kinematic solution, and the problem is to select suitable values for the damping factor λ which sets the weight of the minimum norm solution, i.e. $\| \dot{\mathbf{q}} \|$, with respect to the minimum task tracking error, i.e. $\| \dot{\mathbf{x}} - \mathbf{J}\dot{\mathbf{q}} \|$. High values of λ give good behaviour but reduced accuracy in the neighbourhood of singular points; it can be recognized that the appropriate choice of λ depends on the minimum singular value of the matrix $\mathbf{J}$ which is a measure of proximity to singularities. Maciejewski and Klein (1988) have recently presented a technique to determine a good estimate of the minimum singular value to set λ; a refinement of the technique is also proposed which performs selective filtering only in the direction of the singular components for a given task tracjectory. Incidentally, we remark that the above method can be used for nonredundant manipulators in the neighbourhood of singular configurations.

Another shortcoming of the solution (4) is that *repeatibility* of joint trajectories for repeated task trajectories is not preserved (Klein and Huang, 1983); this is not desirable in most practical applications, e.g. with industrial robots. Within the framework of generalized inverse methods, Shamir and Yomdin (1988) have established an elegant mathematical condition for the pseudoinverse solution – and more generally for any solution of the type (3) – to be repeatable: For any two columns $\mathbf{k}_i$ and $\mathbf{k}_j$ of $\mathbf{K}$, their Lie bracket $[\mathbf{k}_i, \mathbf{k}_j]$ must be a linear combination of the columns of $\mathbf{K}$.*

* For the reader who is not familiar with differential geometry theory, we recall here that the Lie bracket of two $(n \times 1)$ vectors $\mathbf{u}$ and $\mathbf{v}$, that are both functions of an $(n \times 1)$ vector $\mathbf{w}$, is the vector $[\mathbf{u}, \mathbf{v}] = (\partial\mathbf{v}/\partial\mathbf{w})\mathbf{u} - (\partial\mathbf{u}/\partial\mathbf{w})\mathbf{v}$.

Therefore, this condition can be creatively exploited to find the initial setting $\mathbf{q}(0)$, under the control $\mathbf{K}$, that ensures repeatable task motion in a simply connected region of the manipulator's workspace.* We would like to point out, however, that the above condition, although appealing from a pure mathematical viewpoint, seems less interesting from a real application viewpoint, since it is quite impractical to be exactly satisfied; a slight numerical offset on the initial joint setting satisfying the condition would, in fact, no longer imply repeatability.

Last but not least, it must be remarked that a solution of the type (3) is inherently *open-loop*: Once the initial joint setting $\mathbf{q}(0)$ is known, Equation (3) is integrated over time to provide the joint path $\mathbf{q}(t)$. This is implemented on the computer, of course, in discrete-time and thus unavoidably causes numerical drifts in the task space. In order to overcome this drawback, a *closed-loop* algorithmic version of the solution (3) can be obtained if the task space vector $\dot{\mathbf{x}}$ is replaced by $\dot{\mathbf{x}} = \dot{\mathbf{x}}_d + \mathbf{\Lambda e}$, where $\mathbf{e} = \mathbf{x}_d - \mathbf{x}$ denotes the error between the desired task trajectory $\mathbf{x}_d$ and the actual task trajectory $\mathbf{x}$ which can be computed from current joint variables via Equation (1), and $\mathbf{\Lambda}$ is a positive definite (diagonal) matrix that suitably shapes the error convergence (Sciavicco and Siciliano, 1987b; Tsai and Orin, 1987). Notice that the current joint variables are not to be confused with the real robot joint sensor measurements used by the dynamic control. This receives the outputs of the kinematic control as reference inputs, indeed.

Furthermore, if a computationally cheaper solution is desired, one may devise a solution based on the *transpose* of the Jacobian matrix, i.e.

$$\dot{\mathbf{q}} = \mathbf{J}^T(\mathbf{q})\mathbf{\Lambda e} \tag{5}$$

which can be shown, via a simple Lyapunov argument, to guarantee limited tracking errors and null steady-state ($\dot{\mathbf{x}}_d = \mathbf{0}$) errors (Sciavicco and Siciliano, 1987a; Slotine and Yoerger, 1987). An intrinsic advantage of the solution (5) is that it may avoid the typical numerical instabilities which occur at kinematic singularities, since no pseudoinverse of the Jacobian matrix is required (Sciavicco and Siciliano, 1988b).

4. Gradient Projection Method

Revisiting the pseudoinverse minimum-norm solution (4), it can be easily shown that a more general solution to Equation (2) is given by

$$\dot{\mathbf{q}} = \mathbf{J}^\dagger(\mathbf{q})\dot{\mathbf{x}} + [\mathbf{I} - \mathbf{J}^\dagger(\mathbf{q})\mathbf{J}(\mathbf{q})]\dot{\mathbf{q}}_0, \tag{6}$$

where $\mathbf{I}$ is the $(n \times n)$ identity matrix and $\dot{\mathbf{q}}_0$ is an $(n \times 1)$ arbitrary joint velocity vector. The solution (4) has been modified by the addition of the homogeneous term created by the projection operator $(\mathbf{I} - \mathbf{J}^\dagger\mathbf{J})$ which selects the components of $\dot{\mathbf{q}}_0$ in the null space of the mapping $\mathbf{J}$; thus, $\dot{\mathbf{q}}_0$ produces only a joint self-motion of the structure

* A simply connected subset of the workspace is that portion of the workspace such that any closed path in the subset can be continuously deformed into smaller and smaller closed paths that eventually shrink to a single point.

but no task space motion. An efficient method to compute a solution of the type (6) while avoiding explicit calculation of $\mathbf{J}^{\dagger}$ can be utilized if a full-rank submatrix of the Jacobian is available (Chevallereau and Khalil, 1988).

One of the most widely adopted approach is to solve redundancy by optimizing a scalar cost function $h(\mathbf{q})$ using the *gradient projection* method, i.e. choosing $\dot{\mathbf{q}}_0 = (\partial h/\partial \mathbf{q})^T$. Notice that any differentiable cost function may be used as long as the function can be reduced to an expression in terms of the joint variables only. Examples of cost functions can be found in Liégeois (1977) for the avoidance of mechanical joint limits, in Yoshikawa (1985a,b) for the maximization of kineto-static and dynamic manipulability measures respectively, and in Dubey, Euler and Babcock (1988) for the maximization of various criteria. Yet another solution based on proper bounds for the rate of change of the Jacobian to be optimized has been proposed by Mayorga and Wong (1988).

It must be remarked that all the above techniques are only local optmization techniques, since they deal with the instantaneous kinematics of motion. Global optimization techniques which minimize some performance index across a whole trajectory have been derived by Nakamura and Hanafusa (1987) based on Pontryagin's maximum principle, by Suh and Hollerbach (1987) using the calculus of variation, and by Kazerounian and Wang (1988) also using the calculus of variation but in a simpler fashion. However, even though global optimization solutions perform better than local optimization solutions, they are impractical for on-line feedback control, due to the heavy computational requirements.

5. Task Space Augmentation

Another method of solving redundancy, conceptually different from the above methods, is that of imposing an additional constraint task to be executed along with the original (end-effector) task. In details, a functional constraint task on the joint variables can be considered in the form

$$\mathbf{y} = \mathbf{f}_y(\mathbf{q}), \tag{7}$$

where $\mathbf{f}_y$ is an $(r \times 1)$ vector with continuous derivatives with respect to $\mathbf{q}$; also it is $r \leqslant (n - m)$ so as to employ at most all redundant degrees of freedom. Consequently, the *augmented task space* can be formally characterized as (Sciavicco and Siciliano, 1987a; Egeland, 1987)

$$\mathbf{x}_A = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{f}(\mathbf{q}) \\ \mathbf{f}_y(\mathbf{q}) \end{bmatrix}. \tag{8}$$

In general, however, it is unlikely that, given any path $\mathbf{x}_A(t)$, the joint paths $\mathbf{q}(t)$ will satisfy both tasks. In other words, it is quite difficult to select an *ad hoc* constraint task to be satisfied together with the original task, unless one has enough insight into the specific problem. An example of such kind can be found in Sciavicco, Siciliano and

Chiacchio (1988), where a snake-like robot arm is commanded to move in a toroidal region.

Another indirect way of choosing the constraint task is that of requiring that the manipulator optimize some cost function of the type discussed above. To the purpose, in the case of $r = (n - m)$, Baillieul (1985) suggested to project the gradient of the cost function onto the null space of the Jacobian and impose that this is zero, i.e.

$$[\mathbf{I} - \mathbf{J}(\mathbf{q})\mathbf{J}^{\dagger}(\mathbf{q})]\left(\frac{\partial h(\mathbf{q})}{\partial \mathbf{q}}\right)^T = \mathbf{0} \tag{9}$$

from which an $(r \times 1)$ constraint task vector can be derived in the form

$$\mathbf{g}(\mathbf{q}) = \mathbf{0} \tag{10}$$

which in turn corresponds to an equation of the type (7). This technique has been later formalized by Chang (1987) who derived the constraint task in the form

$$\mathbf{Z}(\mathbf{q})\left(\frac{\partial h(\mathbf{q})}{\partial \mathbf{q}}\right)^T = \mathbf{0}, \tag{11}$$

where $\mathbf{Z}$ is composed of $(n - m)$ linearly independent row vectors which span the null space of the matrix $\mathbf{J}$. In the scalar case, i.e. $(n - m) = 1$, it is quite straightforward to find a symbolic expression for $\mathbf{Z}$, as already in Baillieul (1985) for a simple three-degree-of-freedom planar arm, but some difficulties may arise in the vector case.

Nonetheless, once the task space augmentation has been set up according to any of the above methods, it is possible to solve an augmented kinematic equation of the type (8) either numerically for each point along the path $\mathbf{x}_A(t)$, as done in Chang (1987), or in the same formal way as in (3). In fact, differentiating Equation (8) with respect to time gives

$$\dot{\mathbf{x}}_A = \mathbf{J}_A(\mathbf{q})\dot{\mathbf{q}}, \tag{12}$$

where

$$\mathbf{J}_A = \begin{bmatrix} \mathbf{J} \\ \mathbf{J}_y \end{bmatrix} \tag{13}$$

is a suitably augmented Jacobian matrix and $\mathbf{J}_y = \partial \mathbf{y}/\partial \mathbf{q}$. One can then solve Equation (12) as

$$\dot{\mathbf{q}} = \mathbf{K}_A(\mathbf{q})\dot{\mathbf{x}}_A \tag{14}$$

which, in the case $r = (n - m)$, reduces to what Baillieul (1985) called the *extended Jacobian* technique; also $\mathbf{K}_A$ becomes $\mathbf{J}_A^{-1}$. Notice that for constraint tasks of the type (11), this method will propagate joint configurations that extremize $h(\mathbf{q})$ provided that the initial joint configuration $\mathbf{q}(0)$ has been chosen to extremize h. Again in this case, to get a closed-loop solution, it is suggested to replace $\dot{\mathbf{x}}_A$ by $\dot{\mathbf{x}}_{Ad} + \mathbf{\Lambda}_A\mathbf{e}_A$ with obvious meaning of the symbols; the pseudoinverse control can be used as in Equation (4) or another option is offered by the use of the Jacobian transpose in the fashion of Equation (5) (Sciavicco and Siciliano, 1987b).

Remarkably, a nice feature of the augmented task space method when the space of redundancy is entirely exploited, i.e. $r = (n - m)$, is that it is repeatable for any initial joint setting, on condition that paths are chosen in a simply connected subset of the workspace (Baker and Wampler, 1988).

On the other hand, a major problem that may be encountered in the application of the solution (14) in a task space augmentation setting is the occurrence of *algorithmic singularities* (Baillieul, 1986) which are the singularities associated with the augmented Jacobian matrix $\mathbf{J}_A$. It can be shown that if the Jacobian matrix $\mathbf{J}$ is full rank m and the constraint Jacobian matrix $\mathbf{J}_y$ is also full rank r, then the augmented Jacobian matrix $\mathbf{J}_A$ is full rank $(r + m)$ if and only if $\mathscr{R}(\mathbf{J}^T) \cap \mathscr{R}(\mathbf{J}_y^T) = 0$, where $\mathscr{R}(\mathbf{A})$ denotes the range space of matrix $\mathbf{A}$ (Sciavicco and Siciliano, 1988a). A useful analytical tool to aid in the analysis of algorithmic singularities has been established by Baillieul (1987) via the adoption of suitable constrained (coordinate-free) manipulability indices.

An effective solution to handle multiple tasks is constituted by the *task priority* strategy formalized by Nakamura, Hanafusa and Yoshikawa (1987), but implicitly adopted by Maciejewski and Klein (1985) to perform obstacle avoidance. The original (end-effector) task and the constraint task are assigned different priorities in that the task of lower priority is satisfied only if it does not conflict with the task of higher priority. With reference to the above augmented task space formulation and assuming that the constant task $\mathbf{y}$ is given lower priority than the original task $\mathbf{x}$, the joint velocity solution (14) can be modified into – referring to the pseudoinverse solution and suppressing the $\mathbf{q}$-dependence for notation compactness –

$$\dot{\mathbf{q}} = \mathbf{J}^\dagger\dot{\mathbf{x}} + (\mathbf{I} - \mathbf{J}^\dagger\mathbf{J})\tilde{\mathbf{J}}_y^\dagger(\dot{\mathbf{y}} - \mathbf{J}_y\mathbf{J}^\dagger\dot{\mathbf{x}}) + (\mathbf{I} - \mathbf{J}^\dagger\mathbf{J})(\mathbf{I} - \tilde{\mathbf{J}}_y^\dagger\tilde{\mathbf{J}}_y)\mathbf{z} \tag{15}$$

with $\tilde{\mathbf{J}}_y = \mathbf{J}_y(\mathbf{I} - \mathbf{J}^\dagger\mathbf{J})$. Notice that the second term can be simplified to $\tilde{\mathbf{J}}_y^\dagger(\dot{\mathbf{y}} - \mathbf{J}_y\mathbf{J}^\dagger\dot{\mathbf{x}})$ since the projection operator $(\mathbf{I} - \mathbf{J}^\dagger\mathbf{J})$ is both Hermitian and idempotent. As in the case of solution (6), this second term is in the null space of the primary task Jacobian so as to satisfy the secondary task while producing no motion for the primary task. A third term is also present which allows for the inclusion of yet another task with lower priority. A result similar to (15) has also been obtained by Walker and Marcus (1988). Again, as emphasized above, the vectors $\dot{\mathbf{x}}$ and $\dot{\mathbf{y}}$ can be feedback-corrected into $\dot{\mathbf{x}}_d + \mathbf{\Lambda e}$ and $\dot{\mathbf{y}}_d + \mathbf{\Lambda}_y\mathbf{e}_y$ so as to gain numerical robustness at low computation expenses (Sciavicco and Siciliano, 1988b).

With reference to the above-debated problem of algorithmic singularities of the solution (14), we can recognize that the solution (15) performs better in that an algorithmic singularity now no longer interfers with the whole augmented task. As long as the primary task Jacobian is guaranteed to be singularity-free, in fact, an algorithmic singularity – which will occur when $\mathbf{J}_y$ is full rank r but $\mathscr{R}(\mathbf{J}^T) \cap \mathscr{R}(\mathbf{J}_y^T) \neq 0$ – will not affect the primary task, but, eventually, only the secondary task (Sciavicco and Siciliano, 1988a).

6. Inverse Kinematic Functions

Another method of solving redundancy is to specify an *inverse kinematic function* to (1) as a function $\mathbf{g}$ defined on a suitable portion of the workspace, namely an *invertible workspace* (Wampler 1988a), such that

$$\mathbf{f}(\mathbf{g}(\mathbf{x})) = \mathbf{x} \tag{16}$$

for all $\mathbf{x}$ contained in the invertible workspace. A useful tool derived from topology, namely the *winding number*, can be used to discover restrictions on invertible workspaces (Wampler, 1988b). In the case of simple nonredundant geometries, indeed, it is well-known that inverse kinematic functions in closed-form exists. Thus, the problem is to find closed-form expressions in the redundant case since a continuum of possible inverse functions $\mathbf{g}$ satisfying (16) exist. The most direct way of accomplishing this purpose is undoubtedly to dispose of an inverse function for $(n - m)$ of the joint coordinates, say

$$\mathbf{q}_1 = \mathbf{g}_1(\mathbf{x}) \tag{17}$$

where $\mathbf{g}_1$ is defined on a *feasible workspace*, and then solve for the remaining joint coordinates $\mathbf{q}_2$ from

$$\mathbf{f}(\mathbf{q}_1, \mathbf{q}_2) = \mathbf{x}. \tag{18}$$

According to the definition of Wampler (1987), a feasible workspace is that subset of the reachable workspace, for a differentiable inverse kinematic function $\mathbf{g}$, such that a finite bound on the ratio of joint speed to speed in the workspace exists.

Like any of the above kinematic control methods, the inverse kinematic function method allows for on-line implementation, i.e. given a task path $\mathbf{x}(t)$, the joint path is derived from $\mathbf{q}(t) = \mathbf{g}(\mathbf{x}(t))$. The other nice property of the method is repeatability, or cyclic behaviour according to the coinage of Wampler (1987). Also, an important result has been obtained by Baker and Wampler (1988): Any kinematic control method that allows on-line path corrections and enjoys the cyclic property is equivalent to an inverse kinematic function, that is there exists an inverse function that produces the same joint paths. Thus, for instance, the extended Jacobian method can be regarded as a viable computational way of implementing an inverse kinematic function, whenever that is difficult to obtain in closed-form.

7. Summary and Discussion

In the above sections we have attempted to bring out unified frameworks for the analysis of the most widely adopted methods for kinematic control of redundant robot manipulators. We have restricted our survey to the instantaneous solution techniques which allow on-line path modification.

As it has emerged above, the first three classes of methods, namely simple Jacobian-based techniques, gradient projection method, and task space augmentation, are based on the joint velocity solution of the mapping (2) (the joint displacements can then be found via numerical integration). The inverse kinematic function method aims, instead, at finding the joint displacements directly.

We have tried to point out the advantages and disadvantages of each method. Singularity avoidance, joint path repeatability and satisfaction of general constraint tasks are the main issues that should guide the choice of the most effective solution technique. Simple Jacobian-based methods, though computationally simpler, hardly ensure repeatability. The use of the damped least-squares pseudoinverse, or even of the transpose, may provide remedy for singularity occurrence. Local optimization techniques do not supply practical guarantees if a local minimum is not achieved. It is our confidence that the task space augmentation method with task priority offers the most benefits over the other methods, since it allows the user to set a number of independent constraints, e.g. singularity avoidance, while systematically ensuring that the primary task is not affected by the constraint task. Also, repeatibility in the joint space is often obtained. The inverse kinematic function method, perhaps, is the most logical one to apply – and conceptually corresponds to an augmented task space method – except for the inherent difficulties in finding closed-form expressions for such inverse functions.

In the present study, we have not addressed the *dynamic control* problem for redundant manipulators; at least, a few words are in order. The usual approach to robot control is as follows: Solve the inverse kinematics for a finite number of task configurations, by using any of the above methods. From the joint variable solutions, construct continuous joint paths, e.g. via splining functions, which constitute the reference inputs to some dynamic joint space controller.

Another approach is to design a dynamic control directly in the task space which allows on-line path modification based on sensory feedback data. Unless the goal is to design a simple positional controller, in which case simple PD independent joint controllers (with gravity compensation) are seen to lead to stable behaviour (Asada and Slotine, 1986), a tracking task space controller requires that reference joint accelerations are available. This implies that the second-order kinematic mapping must be considered, which can be derived by further differentiating Equation (2), i.e.

$$\ddot{\mathbf{x}} = \mathbf{J}(\mathbf{q})\ddot{\mathbf{q}} + \dot{\mathbf{J}}(\mathbf{q})\dot{\mathbf{q}}. \tag{19}$$

The most general solution to Eq. (19) can be written as

$$\ddot{\mathbf{q}} = \mathbf{J}^{\dagger}(\mathbf{q})[\ddot{\mathbf{x}} - \dot{\mathbf{J}}(\mathbf{q})\dot{\mathbf{q}}] + [\mathbf{I} - \mathbf{J}^{\dagger}(\mathbf{q})\mathbf{J}(\mathbf{q})]\ddot{\mathbf{q}}_0, \tag{20}$$

where $\ddot{\mathbf{q}}_0$ is an arbitrary joint acceleration vector which is projected onto the null space of $\mathbf{J}$. The choice of $\ddot{\mathbf{q}}_0$ is not immediate. Hsu, Hauser and Sastry (1988) stressed that $\ddot{\mathbf{q}}_0$ must be chosen to control the components of the joint velocities – unobservable at the output of the system – which may lead to internal unstable behaviour.

We remark here that another possibility would be to derive the joint accelerations by symbolic differentiation of the joint velocity solutions obtained with any of the above-illustrated methods. This issue along with other refinements of the kinematic control methods – or yet innovative approaches – open a research area mature for further investigation.

We would like to conclude this tutorial with the following consideration: If the robotics research community believe that the addition of extra degrees of freedom can improve on conventional nonredundant manipulator's performance, it is probably about time to provide the robotics industry community with convincing motivations in favour of a greater use of redundancy!

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