On the Generation of Dual Polar Spaces of Unitary Type over Finite Fields

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It is demonstrated that the generating rank of the dual polar space of type $U_{2n}(q^2)$ is $\binom{2n}{n}$ when $q > 2$. It is also shown that this is equal to the embedding rank of this geometry.

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1. Introduction and Basic Concepts

We assume the reader is familiar with the basic definitions relating to graphs and point–line geometries (as a standard reference see [1]) and in particular, the distance function, a geodesic path, and diameter of a graph; the collinearity graph of a point–line geometry $\Gamma = (P, L)$, a subspace of $\Gamma$, the subspace $(X)_{\Gamma}$ generated by a subset $X$ of $P$, and convex subspace of $\Gamma$. We define the generating rank, $\text{gr}(\Gamma)$, of $\Gamma$ to be $\min\{|X| : X \subseteq P, (X)_{\Gamma} = P\}$, that is, the minimal cardinality of a generating set of $\Gamma$.

We further assume familiarity with the concept of a projective embedding $e : P \to \mathbb{P}G(V)$ of a point–line geometry $\Gamma = (P, L)$ as well as the notion of a relatively universal embedding. We say that $\Gamma$ is embedable if some projective embedding of $\Gamma$ exists. When this is the case we shall define the embedding rank, $\text{er}(\Gamma)$, of $\Gamma$ to be the maximal dimension of a vector space $V$ for which there exists an embedding into $\mathbb{P}G(V)$. An immediate consequence of these definitions is the following:

**Definition.** Let $\Gamma = (P, L)$ be an embedable point–line geometry and let $e : P \to \mathbb{P}G(V)$ be an embedding.

1. $\dim(V) \leq \text{gr}(\Gamma)$. Consequently, $\text{er}(\Gamma) \leq \text{gr}(\Gamma)$.
2. If $\dim(V) = \text{gr}(\Gamma)$ then $e$ is relatively universal.

In general, when we have a subset $X$ of $V$ and some collection of subspaces $\mathcal{A}$ then we will set $\mathcal{A}(X) = \{A \in \mathcal{A} | A \subseteq X\}$.

1.1. Polar Spaces and Dual Polar Spaces of Type $U_{2n}(q^2)$. In this paper we will be interested in two related point–line geometries: the polar and dual polar spaces of type $U_{2n}(q^2)$. The polar space of type $U_{k}(q^2)$ can be described as follows: Let $V$ be a vector space of dimension $k$ over $\mathbb{F}_q^2$ and let $\xi : \mathbb{F}_q^2 \to \mathbb{F}_q^2$ be the automorphism given by $\xi(x) = x^q$. We will usually denote images under this map by the ‘bar’ notation: $\overline{x} = x^q = \xi x$. Now let $f : V \times V \to \mathbb{F}_q$ be a nondegenerate hermitian form, that is, a map which satisfies:

$$f(u + v, w) = f(u, w) + f(v, w); \quad f(av, w) = af(v, w); \quad f(w, v) = \overline{f(v, w)}$$

for $u, v, w \in V, a \in \mathbb{F}_q^2$.

A vector $v$ is isotropic if $f(v, v) = 0$ and a subspace $U$ is isotropic if $f(U, U) = 0$. The maximal dimension of an isotropic subspace is $\lfloor \frac{k}{2} \rfloor$ and all such subspaces are conjugate under the action of

$$G = G(V) = \{T : V \to V \mid f(Tv, Tw) = f(v, w), \forall v, w \in V\}.$$


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We denote the isotropic subspaces of dimension $l$, $1 \leq l \leq \lfloor \frac{k}{2} \rfloor$, by $P_k(l, q^2)$. The points of the unitary polar space are the isotropic one spaces $P_k(1, q^2)$ and the lines are the isotropic two subspaces $P_k(2, q^2)$. We will denote by $\Gamma_k$ this incidence geometry: $(P_k(1, q^2), P_k(2, q^2))$. For convenience throughout this paper we will set $P_k = P_k(1, q^2)$ and $L_k = P_k(2, q^2)$ and when the value of $k$ is understood we will drop this subscript and write simply $P$ and $L$ and do likewise with $\Gamma_k$.

We will denote by $U_k(l, q^2)$ the collection of all nondegenerate subspaces of $V$ which have dimension $l$, $l \leq k$ and identify such a subspace with the isotropic points which it contains, when $l \geq 2$. The hyperbolic lines are the subspaces in $U_k(2, q^2)$ which we will denote by $H_k$ or simply $H$ when the value of $k$ is clear from the context. The geometry $(P_k, H_k)$ will be denoted by $N_k$ or simply $N$.

Now let $k = 2n$ be even. The second geometry which we will be concerned with is the dual polar space of type $U_{2n}(q^2)$. This geometry has as its points the elements of $P_{2n}(n, q^2)$. We will denote this set by $P_n$ or simply $P$ if the $n$ is clear from the context. The lines are in one-to-one correspondence with the elements of $P_{2n}(n - 1, q^2)$ and for such an isotropic subspace, $A$, the line corresponding to it is $l(A) = \{M \in P \mid A \subset M\}$. We will denote the set of lines by $L_n$ or simply $L$. We will use the notation $DU_{2n}(q^2)$ for the isomorphism class of this geometry or simply $D$.

The main theorem of this paper is the following:

**THEOREM A.** The generating rank of $DU_{2n}(q^2)$ is $\binom{2n^2}{n}$.

As we shall show the geometry $DU_{2n}(q^2)$ has an embedding $e$ into a projective space $\mathbb{P}G(M^n)$ where $\dim(M^n) = \binom{2n^2}{n}$. Therefore as a consequence of Theorem A we will have the following, which follows immediately from the definitions:

**THEOREM B.** The embedding into $\mathbb{P}G(M^n)$ is relatively universal. The embedding rank of $DU_{2n}(q^2)$ is $\binom{2n^2}{n}$.

In [3] a basis of an embedable geometry $\Gamma = (P, L)$ is defined to be a subset $X \subset P$ such that $\langle X \rangle_\Gamma = P$ and such that there exists some embedding $e : P \to \mathbb{P}G(V)$ with $e(X)$ an independent set of points. Such sets exist if and only if the embedding rank of $\Gamma$ is equal to the generating rank. Therefore we have the following

**COROLLARY 1.2.** For $q > 2$ bases exist in the unitary dual polar space of type $DU_{2n}(q^2)$.

The layout of this paper is as follows. In Section 2 we consider the geometry $N_k(P_k, H_k)$ and prove that it is possible to find a set of points $p_1, p_2, \ldots, p_k$ in $P_k$ such that for every $s \leq k$ the linear space $V_s$ spanned by $p_1, p_2, \ldots, p_s$ is nondegenerate, and the subspace of $N_k$ generated by $p_1, p_2, \ldots, p_s$ is $P_k(V_s)$. In Section 3 we record some necessary properties of the dual polar space $DU_{2n}(q^2)$. In Section 4 we will define a sequence of numbers by recursion and get a closed expression for these numbers. In Section 5 we prove Theorem A.

2. **Properties of the Unitary Space $U_k(q^2)$**

In this section we consider a nondegenerate unitary space $V$ of dimension $k \geq 2$ over the field $\mathbb{F}_q$. With each isotropic point $x \in P$ let $r_x$ be the group of all transvections with center $x$ and axis $x^\perp$. These are just the elements $\tau$ of $SL(V)$ such that $[\tau, V] \subset x$, $[\tau, x^\perp] = 0$. These are subgroups of $SU(V) = G \cap SL(V)$ and the action of $G$ on $\{r_x \mid x \in P\}$ and $P$ are equivalent. We remark that for $x, y \in P$ either (i) $f(x, y) = 0$ in which case $[r_x, y] = 1$ and $\langle x, y \rangle \cong \mathbb{F}_q \times \mathbb{F}_q$ or (ii) $f(x, y) \neq 0$ and $\langle r_x, r_y \rangle \cong SL(2, q)$. We shall require the following

**LEMMA 2.1.** Let $X \subset P$. Then the group $\langle r_x \mid x \in X \rangle$ leaves $\langle X \rangle_N$ invariant.
Then clearly
\[ x \] is isomorphic to \( y \).

We now prove that we can choose a certain kind of basis for \( V \):

**Lemma 2.2.** There exists a basis of isotropic vectors \( v_1, v_2, \ldots, v_k \) for \( V \) such that for each \( j < k \), \( V_j = \langle v_1, v_2, \ldots, v_j \rangle \) is nondegenerate and such that for \( j \geq 3 \), \( V_j \cap v_{j+1}^\perp = V_{j-1} \).

**Proof.** If \( V \) has dimension two we can take \( v_1, v_2 \) to be any distinct pair of isotropic vectors. If \( V \) has dimension three we can take any three noncollinear isotropic vectors. Thus we can assume that the dimension of \( V \) is at least four. We can begin by taking \( v_1, v_2, v_3 \) to be any noncollinear triple of isotropic vectors in a non-degenerate three-dimensional subspace of \( V \). Assume now that \( v_1, v_2, \ldots, v_j \) with \( j \geq 3 \) have been chosen so as to satisfy the requirements of the theorem and assume that \( j < n \). It then follows that the dimension of \( V_{j+1} \) is at least two and therefore contains isotropic vectors which do not belong to \( V_j \). Choose \( v_{j+1} \) to be any such vector. To complete the proof it remains only to show that \( V_j + v_{j+1} \) is non-degenerate. Suppose \( x + av_{j+1}, x \in V_j \) is in the radical of \( V_{j+1} \). Then clearly \( x \in V_{j-1} = V_j \cap V_{j+1} \). However, if \( x \neq 0 \) then there is a vector \( y \in V_{j-1} \) with \( (x, y) \neq 0 \) and then \( (x+av_{j+1}, y) = (x, y) \neq 0 \). Therefore \( x = 0 \). However, now we must have \( (v_j, av_{j+1}) = 0 \) which implies \( a = 0 \).

Now let \( v_1, \ldots, v_k \) be a basis for \( V \) as in Lemma 2.2 and set \( p_i = \langle v_i \rangle \) a set of isotropic points and let \( V_j = \langle p_1, p_2, \ldots, p_j \rangle \). Our final result of this section concerns the subspace of the geometry \( N \) generated by \( p_1, p_2, \ldots, p_j \) for \( j \leq k \).

**Lemma 2.3.** Assume \( q > 2 \). Then for \( j \leq k \), \( \langle p_1, p_2, \ldots, p_j \rangle = P(V_j) \).

**Proof.** The case \( j = 1 \) is trivial. So assume that \( j > 1 \). Set \( S = \langle p_i | 1 \leq i \leq j \rangle_N \) and let \( r_i = r_{p_i} \). By [4] \( \langle r_i | 1 \leq i \leq j \rangle = \langle \tau \in SU(V) | \tau(V_j^\perp) = 1 \rangle \cong SU(V_j) \) and is transitive on \( P(V_j) \). \( \langle r_i | 1 \leq i \leq j \rangle \) leaves \( S \) invariant by Lemma 2.1 and therefore \( S = P(V_j) \).

**Remark.** When \( q = 2 \) and \( j \leq 3 \) the result still holds: \( \langle p_1, \ldots, p_j \rangle = P(V_j) \). However, \( \langle p_1, p_2, p_3, p_4 \rangle_N \) is 18 whereas \( P(V_4) = 45 \). It is true, however, if \( j \geq 4 \) then \( \langle P(V_j), p_{j+1} \rangle = P(V_{j+1}) \) since in this case \( r_{p_{j+1}}, r_x | x \in P(V_j) \rangle = N_G(V_{j+1}) \cap C_G(V_{j+1}) \) is isomorphic to \( SU(V_{j+1}) \).

3. Properties of Unitary Dual Polar Spaces

We continue with the notation of the introduction, but now we assume that the dimension \( k \) of our unitary space is even, \( k = 2n \), and record some properties of the geometry \( (P, \mathcal{L}) \) of type \( DU_{2n}(q^2) \) which we require in the sequel. We remark that for points \( x, y \in P \) the distance function defined by the point-collinearity graph of \( DU_{2n}(q^2) \) is given by \( d(x, y) = \dim[x/(x \cap y)] = \dim[y/(x \cap y)] \).

For \( B \in P_{2n}(t, q^2) \), that is a totally isotropic subspace of dimension \( t \), denote by \( U(B) \) those \( p \in P = P_{2n}(n, q^2) \) such that \( B \subset p \). We then have

**Property 3.1.** For \( B \in P_{2n}(t, q^2) \), \( U(B) \) is a convex subspace of \( (P, \mathcal{L}) \). Moreover, \( U(B) \) is the convex closure of any two points in \( U(B) \) whose intersection is \( B \). The diameter of \( U(B) \) is \( n - t \).

Note that for \( B \in P_{2n}(t, q^2) \), \( \tilde{B} = B^\perp/B \) is a non-degenerate unitary space of dimension \( 2(n - t) \). Moreover, the map from \( U(B) \) which takes \( p \) to \( p/B \) is a bijection onto the maximal isotropic subspaces in \( \tilde{B} \). Consequently we have
**Property 3.2.** The geometry of $UDU_{2n}(q^2)$ induced on $U(B)$, $B \in P_{2n}(t, q)$ is $DU_{2n-2}(q^2)$.

Assume now that $p \in P$, $l \in L$. Then $l$ has the form $P(A^\perp)$ for $A \in P_{2n}(n-1, q^2)$, where $P(A^\perp)$ consists of those elements of $P$ which are contained in $A^\perp$. Note that $p \cap A^\perp \neq p \cap A$. This can be seen, by induction on $\dim A \cap p + n$: Suppose first that $A \cap p = 0$. Since $\dim A^\perp = n + 1A^\perp \cap p \neq 0$ and consequently $A^\perp \cap p \neq A \cap p$. Assume then that $A \cap p \neq 0$. Set $B = A \cap p$ and set $\bar{V} = B^\perp/B$ and denote images by using the bar notation. $A, p \subset B^\perp$ so $\bar{A} = A/B, \bar{p} = p/B$. Now $\bar{A} \cap \bar{p} = 0$. By the previous case $\bar{A}^\perp \cap \bar{p} \neq \bar{A} \cap \bar{p}$. Since also $\bar{A}^\perp = A^\perp/B$ the assertion now follows. Since $A^\perp \cap p \neq A \cap p$ there is a unique element on $l$ which contains $A^\perp \cap p$. This implies

**Property 3.3.** For any point–line pair $p,l$ there is a unique point on $l$ nearest $p$.

Therefore $(P, L)$ is a near $2n$-gon in the sense of Shult and Yanushka (see [6]). In a near $2n$-gon we say that **quads exist** if for any two points at distance two the convex closure is a generalized quadrangle (see [5]). By (3.1), (3.2) it follows that quads exist in $(P, L)$. We will refer to the subspace $U(C), C \in P_{2n}(n-2, q^2)$ as quads.

Let $p \in P$ and $Q = U(C)$ be a quad. By arguments similar to those used above it is not difficult to see that the pair $p, Q$ is gated, that is, there is a unique point in $Q$ nearest $p$ and for any point $y \in Q$, $d(p, y) = d(p, x) + d(x, y)$. This implies the well known result that

**Property 3.4.** $(P, L)$ is a classical near $2n$-gon.

**Remark.** Cameron [2] characterized the classical near $2n$-gons and proved that they are precisely the dual polar spaces.

We complete this section with three lemmas.

**Lemma 3.5.** Assume $x$ is an isotropic point of $V$ and $p \in P$ is a maximal isotropic subspace of $V$ and $p \not\subset U(x)$. Then there is a unique point in $U(x)$ at distance one from $p$.

**Proof.** Suppose $q$ is such a point. $p \cap q$ is an $(n - 1)$-dimensional isotropic subspace. Since $x \subset p, q \subset p \cap x^\perp$ and, since $x$ is not a subspace of $p$, $p \cap x^\perp$ is a hyperplane in $p$. Therefore, $q \cap p = p \cap x^\perp$ and $q = (q \cap p, x) = (p \cap x^\perp, x)$. Thus, $q$ is uniquely determined. This also demonstrates that such a point exists.

**Lemma 3.6.** Assume $B$ is a subspace of $V$. Then $\bigcup_{b \in B} U(b)$ is a subspace of $DU_{2n}(q^2)$.

**Proof.** Set $S = \bigcup_{b \in B} U(b)$. Suppose $p \in U(b), q \in U(c), b, c \in B$ and $p$ and $q$ are collinear. If $b \subset p \cap q$ then $p, q \in U(b)$ then the line $P((p \cap q)^\perp) \subset U(b) \subset S$. We get the same conclusion if $c \subset p \cap q$. We may therefore assume that neither $b$ nor $c$ is contained in $p \cap q$. This implies that $(b, c) \cap (p \cap q) = 0$. Since $b \subset p$ it follows that $b \subset (p \cap q)^\perp$ and similarly $c \cap (p \cap q)^\perp$. It is then the case that $(p \cap q)^\perp = (b, c) \oplus (p \cap q)$. But now, for every $p' \in P((p \cap q)^\perp), p' \cap (b, c) \neq 0$. If $b' = p' \cap (b, c)$ then $p' \in U(b') \subset S$.

**Lemma 3.7.** Suppose $x, y$ are nonorthogonal isotropic points of $V$. Let $h$ be the hyperbolic line spanned by $x, y$. Then

$$(U(x), U(y))_D = \bigcup_{z \in h} U(z).$$

**Proof.** Let $z \in h, z \neq x, y$ and let $p \in U(z)$. Since $z \subset p$ and $(x, z) = (y, z)$ it follows that $p \cap x^\perp = p \cap y^\perp$. Set $B = p \cap x^\perp = p \cap y^\perp$. Also set $B_x = \langle B, x \rangle \in U(x)$ and $B_y = \langle B, y \rangle \in U(y)$. Then $B_x$ and $B_y$ are collinear and the line they determine is $P(B^\perp)$ which contains $p$. Thus $U(z) \subset (U(x), U(y))_D$. However, by Lemma 3.6 $\bigcup_{z \in h} U(z)$ is a subspace and therefore $(U(x), U(y))_D = \bigcup_{z \in h} U(z)$ as asserted.
4. A Recursion Formula

In this section we define for each integer \( n \geq 2 \) a finite sequence of natural numbers \( \{f(n, j)\}_{j=0}^{n} \) by a recursion formula. We then obtain a closed expression for \( f(n, j) \) as well as for \( \sum_{j=0}^{n} f(n, j) \).

For \( n = 2 \) we simply define \( f(2, 0) = f(2, 1) = f(2, 2) = 2 \). Assume for some \( n \geq 2 \) we have defined \( f(n, j), 0 \leq j \leq n \). Set \( \lambda(n) = \sum_{j=0}^{n} f(n, j) \). We now define \( f(n+1, 0) = f(n+1, 1) = f(n+1, 2) = \lambda(n) \). For \( 3 \leq k \leq n+1 \) we define \( f(n+1, k) = \sum_{j=k-1}^{n} f(n, j) \).

The sequences \( f(n, j) \) for \( 2 \leq n \leq 6 \) are as follows.

\[
\begin{array}{cccccc}
2 & 2 & 2 & 6 & 6 & 6 \\
20 & 20 & 20 & 8 & 2 \\
70 & 70 & 70 & 30 & 10 & 2 \\
252 & 252 & 252 & 112 & 42 & 12 & 2 \\
\end{array}
\]

Before proceeding to our main result we first record a necessary lemma.

**Lemma 4.1.** For natural numbers \( m, k \)

\[
\sum_{i=0}^{k} \binom{m+i}{t} = \binom{m+k+1}{k}.
\]

**Proof.** This follows immediately by induction on \( k \) from the identity

\[
\binom{l}{s-1} + \binom{l}{s} = \binom{l+1}{s}.
\]

\( \square \)

**Lemma 4.2.** For \( j = 0, 1, f(n, j) = \binom{2n-2}{n-j} \). For \( 2 \leq j \leq n, f(n, j) = 2^{\binom{2n-1-j}{n-j}} \).

**Proof.** We prove the result by induction on \( n \geq 2 \). By definition, \( f(2, 0) = f(2, 1) = f(2, 2) = 2 \). On the other hand, \( \binom{2}{1} = 2 \binom{1}{0} = 2 \) so that the result holds for \( n = 2 \).

Now assume that we have demonstrated the result for \( n \), that is, for \( j = 0, 1, f(n, j) = \binom{2n-2}{n-j} \) and for \( 2 \leq j \leq n, f(n, j) = 2^{\binom{2n-1-j}{n-j}} \). We now compute

\[
\lambda(n) = \sum_{j=0}^{n} f(n, j)
\]

which, by the inductive hypothesis, is equal to

\[
2^{\binom{2n-2}{n-1}} + \sum_{j=2}^{n} 2^{\binom{2n-1-j}{n-j}} = 2 \sum_{j=0}^{n-1} \binom{n-1+j}{j}.
\]

By Lemma 4.1 this is equal to

\[
2^{\binom{2n-1}{n-1}} = \binom{2n}{n}.
\]

We now prove the result for \( n+1 \). By definition \( f(n+1, 0) = f(n+1, 1) = \binom{2n}{n} \). Also, by definition, \( f(n+1, 2) = \binom{2n}{n} = 2 \binom{2n-1}{n-1} \) and so this case holds as well.

Assume now that \( 3 \leq j \leq n+1 \). Then

\[
f(n+1, j) = \sum_{i=j-1}^{n} f(n, i)
\]
which, by the inductive hypothesis, is
\[ \sum_{i=j-1}^{n} 2 \binom{2n - 1}{n - i} = \sum_{i=0}^{n-j+1} 2 \binom{n - 1 + i}{t}. \]

By Lemma 4.1 this is equal to \( 2 \binom{2(n+1)-1-j}{n+1-j} \) as desired. \( \square \)

5. Proof of the Main Theorem

Let \( \gamma(n, q^2) \) be the generating rank of the geometry \( DU_{2n}(q^2) \). In our first result of this section we prove that \( \gamma(n, q^2) \leq \binom{2n}{n} \). This will be an immediate consequence of

PROPOSITION 5.1. The geometry \( DU_{2n}(q^2) \) has an embedding into a projective space of dimension \( \binom{2n}{n} - 1 \) over \( \mathbb{F}_q \).

PROOF. Let \( x_i, y_j, i = 1, 2, \ldots, n \) be a hyperbolic basis for \( V \), a vector space of dimension \( 2n \) over \( \mathbb{F}_q \), which is equipped with nondegenerate hermitian form \( f \) so that \( f(x_i, y_j) = \delta_{ij} \).

Also set \( \Omega = \{ x_i, y_j | 1 \leq i \leq n \} \). Order \( \Omega \) in the following way: If \( z_i \in \{ x_i, y_i \}, z_j \in \{ x_j, y_j \} \) with \( i < j \) then \( z_i < z_j \). On the other hand we set \( x_i < y_i \). Now let \( M = \wedge^n(V) \). This space has dimension \( \binom{2n}{n} \) over \( \mathbb{F}_q \) and is an irreducible module for \( SU(V, f) \). Let \( \Omega^n \) be the collection of all subsets of \( \Omega \) of cardinality \( n \). Define a map \( \tau : \Omega^n \rightarrow \Omega^n \) by \( \tau(\Phi) = \{ z \in \Omega | z \in (\Phi)^{-1} \} \). This is a bijection and its fixed points are just those \( \Phi \) with \( \Phi \) a totally isotropic subspace of \( V \), alternatively, those \( \Phi \) such that \( \Phi \cap \{ x_i, y_i \} \) has one element for each \( i \).

For such a subset, \( \Phi = \{ z_1 < z_2 < \cdots < z_n \} \in \Omega^n \) set \( \wedge(\Phi) = z_1 \wedge z_2 \wedge \cdots \wedge z_n \). This is a basis for \( M \). Now let \( \sigma \) be the semilinear map from \( M \) to \( M \) given by
\[ \sigma \left( \sum_{\Phi \in \Omega^n} a_\Phi \wedge (\Phi) \right) = \sum_{\Phi \in \Omega^n} a_\Phi \wedge (\tau(\Phi)). \]

The fixed points of \( \sigma \) is a \( \mathbb{F}_q \)-space of dimension \( \binom{2n}{n} \) which we denote by \( M^\sigma \). Now each of the groups \( S_i = \{ g \in SU(V, f) | g((x_i, y_i)) = (x_i, y_i), g((x_i, y_i)^{-1}) = 1 \} \cong SL(2, q) \) fixes \( M^\sigma \).

Likewise, the monomial group of \( SU(V, f) \) with respect to the basis \( x_i, y_i, 1 \leq i \leq n \) leaves \( M^\sigma \) invariant. Since these subgroups generate \( SU(V, f) \) this group leaves \( M^\sigma \) invariant as well. \( SU(V, f) \) is also irreducible on \( M^\sigma \). (If we identify \( SU(V, f) \) with the twisted group \( \mathbb{A}_{2n-1}(q^2) \) then \( M^\sigma \) is the module with highest weight \( \lambda_n \) which is fixed under the graph automorphism and has field of definition \( \mathbb{F}_q \). For details see [7].) We now show that \( \mathbb{P}G(M^\sigma) \) affords an embedding of \( DU_{2n}(q^2) \).

First note that the \( \mathbb{F}_q \)-span of \( x = x_1 \wedge x_2 \wedge \cdots \wedge x_n \) meets \( M^\sigma \) in a one-dimensional \( \mathbb{F}_q \) subspace, namely the \( \mathbb{F}_q \)-span of \( x \). Since \( SU(V, f) \) is transitive on all maximally isotropic subspaces of \( V \) it follows that for any such subspace, i.e. point \( p \in \mathcal{P} \) of the geometry \( DU_{2n}(q^2) \) the one-dimensional \( \mathbb{F}_q \)-space \( \wedge^n(p) \) meets \( M^\sigma \) in a one-dimensional \( \mathbb{F}_q \) subspace. Now consider the line \( l = U(\{ x_1, x_2, \ldots, x_{n-1}, x_n \}) \). Then \( l = \{ (x_1, x_2, \ldots, x_{n-1}, x_n), (x_1, x_2, m, \ldots, x_{n-1}, ax_n + y_n) | a \in \mathbb{F}_q \} \). Set \( y = x_1 \wedge x_2 \wedge \cdots \wedge x_n \wedge y_n \). Then \( M^\sigma \) contains the \( \mathbb{F}_q \)-span of \( x \) and \( y \). However, this span meets each of the one-dimensional \( \mathbb{F}_q \)-spaces spanned by \( x_1 \wedge x_2 \wedge \cdots \wedge x_{n-1} \wedge ax_n + y_n, a \in \mathbb{F}_q \) in one space. Since \( SU(V, f) \) is transitive on the lines of the geometry \( DU_{2n}(q^2) \) this holds for all lines. It therefore follows that \( \mathbb{P}G(M^\sigma) \) affords an embedding for \( DU_{2n}(q^2) \) as asserted. \( \square \)

COROLLARY 5.2. The generating rank of \( DU_{2n}(q^2) \) is at least \( \binom{2n}{n} \).

PROOF. This follows from the definition of an embedding and the generating rank of a geometry. \( \square \)
We can now prove our main result:

THEOREM 5.3. Assume $q > 2$. Then the generating rank, $\gamma(n, q^2)$, of the unitary dual polar space, $DU_{2n}(q^2)$, is $\left(\frac{2n}{n}\right)$.

PROOF. We make use of the notation previously introduced. As in Section 2 let $p_1, p_2, \ldots, p_{2n}$ be isotropic points such that for each $j \geq 2$, $\langle p_1, p_2, \ldots, p_j \rangle = V_j$ is nondegenerate and for $j \geq 4$, $p_j^\perp \cap V_{j-1} = V_{j-2}$. Now set $C = V_{n+1}$. By Lemma 2.3 the subspace $\langle p_1, p_2, \ldots, p_{n+1} \rangle_N = P(C)$. From this and Lemma 3.7 it follows that

$$\langle U(p_1), U(p_2), \ldots, U(p_{n+1}) \rangle_D \supset \bigcup_{z \in P(C)} U(z).$$

By Lemma 3.6 the latter is a subspace of $DU_{2n}(q^2)$ and therefore

$$\langle U(p_1), U(p_2), \ldots, U(p_{n+1}) \rangle_D = \bigcup_{z \in P(C)} U(z).$$

On the other hand, suppose $p \in P$ so that $p$ is an $n$-dimensional isotropic subspace of $V$. Since $\dim(V) = 2n$ and $\dim(C) = n + 1$ it follows that $p \cap C \neq 0$. Therefore there is an $x \in P(C)$, $x \subset p$ and then $p \in U(x) \subset \langle U(p_1), U(p_2), \ldots, U(p_{n+1}) \rangle_D$. Consequently, the set of subspaces $U(p_i)$, $1 \leq i \leq n + 1$ generate $P$. We will make use of the recursion of Section 4 to demonstrate that these subspaces can be generated by $\binom{2n}{n}$ points. Together with Lemma 5.2 this will imply that the generating rank of $DU_{2n}(q^2)$ is exactly $\binom{2n}{n}$.

Suppose $n = 2$. Then each of $U(p_i)$, $i = 1, 2, 3$ is a $DU_2(q^2)$ geometry, that is, a hyperbolic line which requires two points to generate. As a result, by the above argument, we can generate $DU_3(q^2)$ with $2 + 2 + 2 + 6 = 6 \binom{2}{2}$ points.

Suppose now that $n > 2$. For $j = 1$ set $G(n, j) = \{A \subset U(p_1) \mid \langle A \rangle_D = U(p_1)\}$. For $1 < j \leq n + 1$. Let $B_{j-1} = \langle U(p_i) \mid i(j) \rangle_D$ and set $G(n, j) = \{A \subset U(p_j) \mid \langle A, B_{j-1} \cap U(p_j) \rangle_D = U(p_i)\}$.

Finally, set $g(n, j) = \min(|A| \mid A \in G(n, j))$. We claim that $g(n, j) = f(n, j - 1)$ for every applicable pair $n, j$ where $f(n, j)$ are the numbers defined recursively in Section 4. We will prove this inductively on $n$ and $j$.

Suppose for some $n \geq 2$ that we have demonstrated that $g(n, j) = f(n, j - 1)$ for $j = 1, 2, \ldots, n + 1$. Then by the definition of the numbers $f(n, j)$ it follows that

$$\gamma(n) \leq \sum_{j=0}^{n+1} g(n, j) = \sum_{j=0}^{n} f(n, j) = f(n + 1, 0) = \binom{2n}{n}$$

from Lemma 4.2. However, from Corollary 5.2 $\gamma(n, q^2) \geq \binom{2n}{n}$ and therefore we get equality. Note that the theorem will now be a consequence of the equality of $g(n, j)$ and $f(n, j)$ for all $n, j$. Notice also that as a result of this if $A_j \in G(n, j)$, $j = 1, 2, \ldots, n + 1$ then $A = \bigcup_{j=1}^{n+1} A_j$ is a generating set for $DU_{2n}(q^2)$. This implies the following:

Assume $j < n + 1$. Set $C(n, j) = \{C \subset P \mid \langle B_{j-1}, C \rangle_D = P\}$ and $c(n, j) = \min(|C| \mid C \in C(n, j))$. Then

$$c(n, j) = \sum_{i=j}^{n+1} g(n, i).$$

We now prove that $g(n + 1, j) = f(n + 1, j - 1)$ for $j = 1, 2, \ldots, n + 2$. Since $U(p_1)$ is isomorphic to $DU_{2n}(q^2)$, $m(n + 1, 1) = \gamma(n) = \binom{2n}{n}$. Also, for $j = 2$ or $3$, $m(n + 1, j) = \binom{2n}{n}$ since $B_{j-1} \cap U(p_j) = \emptyset$ as $p_j^\perp \cap V_{j-1}$ is either 0 or a single nonisotropic point in the
respective cases. Assume then that $j \geq 4$. Now $B_{j-1} \cap U(p_j) = B_{j-2}$ and, of course, $U(p_j)$ is a subspace of $DU_{2n+2}(q^2)$ isomorphic to $DU_{2n}(q^2)$. From Lemma 5.3 we have that

$$g(n + 1, j) = c(n, j - 1) = \sum_{i=j-1}^{n+1} g(n, i).$$

By the induction hypothesis

$$\sum_{i=j-1}^{n+1} g(n, i) = \sum_{i=j-2}^{n} f(n, i)$$

which, by Lemma 4.2 is equal to $f(n + 1, j - 1)$ so that $g(n + 1, j) = f(n + 1, j - 1)$ as required.

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