Symplectic subspaces of symplectic Grassmannians

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Abstract

Let \( V \) be a non-degenerate symplectic space of dimension \( 2n \) over the field \( F \) and for a natural number \( l < n \) denote by \( C_l(V) \) the incidence geometry whose points are the totally isotropic \( l \)-dimensional subspaces of \( V \). Two points \( U, W \) of \( C_l(V) \) will be collinear when \( W \subset U^\perp \) and \( \dim(U \cap W) = l - 1 \) and then the line on \( U \) and \( W \) will consist of all the \( l \)-dimensional subspaces of \( U + W \) which contain \( U \cap W \).

The isomorphism type of this geometry is denoted by \( C_{n,l}(F) \). When \( \text{char}(F) \neq 2 \) we classify subspaces \( S \) of \( C_l(F) \) where \( S \cong C_{m,k}(F) \).

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1. Introduction and basic concepts

We assume the reader is familiar with the concepts of a partial linear rank two incidence geometry \( \Gamma = (P, L) \) (also called a point-line geometry) and the Lie incidence geometries. For the former we refer to articles in [1] and for the latter see the paper [2].

The collinearity graph of \( \Gamma \) is the graph \( (P, \Delta) \) where \( \Delta \) consists of all pairs of points belonging to a common line. For a point \( x \in P \) we will denote by \( \Delta(x) \) the collection of all points collinear with \( x \). For points \( x, y \in P \) and a positive integer \( t \) a path of length \( t \) from \( x \) to \( y \) is a sequence \( x_0 = x, x_1, \ldots, x_t = y \) such that \( \{x_i, x_{i+1}\} \in \Delta \) for each \( i = 0, 1, \ldots, t - 1 \). The distance from \( x \) to \( y \), denoted by \( d(x, y) \) is defined to be the length of a shortest path from \( x \) to \( y \) if some path exists and otherwise is \( +\infty \).

By a subspace of \( \Gamma \) we mean a subset \( S \) of \( P \) such that if \( l \in L \) and \( l \cap S \) contains at least two points, then \( l \subset S \). \((P, L)\) is said to be a Gamma space if, for every \( x \in P \), \( \{x\} \cup \Delta(x) \) is a subspace. A subspace \( S \) is singular provided each pair of points in \( S \) is collinear, that is, \( S \)

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is a clique in the collinearity graph of $\Gamma$. For a Lie incidence geometry with respect to a “good node” every singular subspace, together with the lines it contains, is isomorphic to a projective space, see [2]. Clearly the intersection of subspaces is a subspace and consequently it is natural to define the subspace generated by a subset $X$ of $\mathcal{P}$, $(X)_\Gamma$, to be the intersection of all subspaces of $\Gamma$ which contain $X$. Note that if $(\mathcal{P}, \mathcal{L})$ is a Gamma space and $X$ is a clique then $(X)_\Gamma$ will be a singular subspace.

A polar space is an incidence geometry $(\mathcal{P}, \mathcal{L})$ which satisfies: (i) For any point $x$ and line $l$ either $x$ is collinear with every point of $l$ or a unique point of $l$; and (ii) For each point $x$ there exists a point $y$ such that $x$ and $y$ are non-collinear. A polar space in which lines are maximal singular subspaces is a generalized quadrangle.

1.1. Ordinary Grassmannians

Let $\mathbb{F}$ be a field and $W$ be a vector space of dimension $m$ over $\mathbb{F}$. For $1 \leq i \leq m - 1$, let $L_i(W)$ be the collection of all $i$-dimensional subspaces of $W$. Now fix $j$, $2 \leq j \leq m - 2$ and set $\mathcal{P} = L_j(W)$.

For pairs $(C, A)$ of incident subspaces of $W$ with $\dim(A) = a$, $\dim(C) = c$ let $S(C, A)$ consist of all the $j$-subspaces $B$ of $W$ such that $A \subset B \subset C$.

Finally, let $\mathcal{L}$ consist of all the sets $S(C, A)$ where $\dim A = j - 1$, $\dim C = j + 1$ and $A \subset C$. The rank two incidence geometry $(\mathcal{P}, \mathcal{L})$ is the incidence geometry of $j$-Grassmannian of $W$, denoted by $\mathcal{G}_j(W)$. We also use the notation $\mathcal{G}_{m,j}(\mathbb{F})$ for the isomorphism type of this geometry and sometimes $A_{m-1,j}(\mathbb{F})$.

We note that the incidence geometry $\mathcal{G}_{4,2}(\mathbb{F})$ is a polar space which is isomorphic to the incidence geometry of singular one spaces and totally singular two spaces on a hyperbolic orthogonal space in a vector space of dimension six, $D_{3,1}(\mathbb{F}) \cong Q^+(6, \mathbb{F})$.

1.2. The symplectic Grassmannians

Let $V$ be a space of dimension $2n$ over the field $\mathbb{F}$, $f$ a non-degenerate alternating form on $V$ so that $(V, f)$ is a non-degenerate symplectic space. For $X \subset V$ let $X^\perp = \{ v \in V : f(x, v) = 0, \forall x \in X \}$. Recall that a subspace $U$ of $V$ is totally isotropic if $U \subset U^\perp$.

For $1 \leq k \leq n$, let $\mathcal{I}_k$ consist of all totally isotropic $k$-dimensional subspaces of $W$. We will set $P = \mathcal{I}_1$, the collection of all one-dimensional subspaces of $V$ and $L = \mathcal{I}_2$, the collection of totally isotropic two spaces (projective lines). The incidence geometry $(\mathcal{P}, \mathcal{L})$ is the symplectic polar space of rank $n$ over the field $\mathbb{F}$, which we will denote by $C_{n,1}(\mathbb{F})$.

Now fix $l$ with $2 \leq l \leq n - 1$ and set $\mathcal{P} = \mathcal{I}_l$. For a pair of subspaces $C \subset D \subset C^\perp$ (so $C$ is totally isotropic) where $\dim C = c < l < d < \dim D$ let $T_l(D, C)$ consist of all the $l$-dimensional totally isotropic subspaces $U$ such that $C \subset U \subset D$. When $c = l - 1$, $d = l + 1$ we set $\lambda_l(D, C) = T_l(D, C)$ and $\mathcal{L} = \{ \lambda_l(D, C) : C \subset D \subset C^\perp, \dim C = l - 1, \dim D = l + 1 \}$. In this way we obtain a rank 2 incidence geometry $\Gamma = (\mathcal{P}, \mathcal{L})$ which we refer to as the symplectic $l$-Grassmannian of $V$. We denote the isomorphism type of this geometry by $C_{n,l}(\mathbb{F})$. Note that two totally isotropic $l$-subspaces, when viewed as points of $\Gamma$, are on a line if and only if they span a totally isotropic $l + 1$ dimensional subspace.

When the subspace $C$ has dimension $l - k$ and $D$ is totally isotropic and has dimension $l + m - k$ then $T_l(D, C)$ is an ordinary Grassmannian isomorphic to $\mathcal{G}_{m,k}(\mathbb{F})$. Subspaces arising this way are said to be parabolic since their stabilizers in $\text{Aut}(\Gamma) \cong PSp_{2n}(\mathbb{F})$ are parabolic subgroups. In [3] we classified subspaces of $C_{n,l}(\mathbb{F})$ which are isomorphic to $\mathcal{G}_{m,k}(\mathbb{F})$:.
Theorem. Let $S$ be a subspace of $C_{n,l}(F)$, $S \cong G_{m,k}(F)$ with $l < n$ and $k \leq \left\lfloor \frac{n}{2} \right\rfloor$.

Then either $S$ is parabolic or char$(F) = 2$, $(m, k) = (4, 2)$ and $S$ is a subspace of $T(U \perp, U)$ for some totally isotropic subspace $U$, dim $U = l - 1$. Moreover, if $Y$ is the subspace of $U \perp$ spanned by all the elements of $S$ then dim $Y/U = 6$.

By a hyperbolic basis of the symplectic space $(V, f)$ we will mean a vector space basis $x_i, y_i, 1 \leq i \leq n$ such that $f(x_i, y_i) = 1$ and $f(x_i, x_j) = f(x_i, y_j) = f(y_i, y_j) = 0$ for all $i \neq j$. The existence of a hyperbolic basis can be shown by an easy induction on the Witt index $\dim V$.

When the characteristic of $F$ is not two, by a hyperbolic generating set of the polar space $(P, L)$, we will mean a collection of projective points $p_i, q_i, 1 \leq i \leq n$ such that $p_i, q_i$ are non-collinear (non-orthogonal with respect to $f$) while the pairs $\{p_i, p_j\}, \{p_i, q_j\}, \{q_i, q_j\}$ are collinear (orthogonal with respect to $f$) for all $i \neq j$. Such a collection of points is referred to as a hyperbolic generating set because such a set generates the polar geometry. Any hyperbolic basis of the symplectic space $(V, f)$ gives rise to a hyperbolic generating set.

1.3. Subspaces of symplectic Grassmannians

We continue with the notation from (1.2) so that $(V, f)$ is a non-degenerate symplectic space of dimension $2n$ and $1 \leq l \leq n - 1$. Assume $\Gamma = (P, L) \cong C_{n,l}(F)$ with $P = I_l$. Suppose $S = (P_S, L_S)$ is a subspace of $\Gamma$. We will denote by $\Sigma(S)$ the subspace of $V$ that is spanned by all $x \in P_S$ and by $I(S)$ the intersection of all $x \in P_S$.

Let $A \in I_{l-k}$ with $k \geq 1$ and $B \subset A \perp$, $A \subset B$ such that the quotient space $B/A$ is non-degenerate of dimension $2m$. In this situation the collection of isotropic subspaces $T_l(B, A)$ is a subspace of the incidence geometry $(P, L)$ and is isomorphic to a symplectic Grassmannian space $C_{m,k}(F)$. These are the “natural” examples but there are three other examples which only exist when char $F = 2$, which we now describe. In (i), (ii) and (iii) assume the field $F$ is perfect and has characteristic two:

(i) The symplectic polar space $C_{m,1}(F)$ is isomorphic to the orthogonal polar space $B_{m,1}(F)$. The latter in its natural representation has a radical of dimension one. This identification gives rise to the following class of embeddings of $C_{m,1}(F)$ in $C_{n,l}(F)$:

Let $A$ be isotropic of dimension $l - 1$ and $B \subset A \perp$ contain $A$ such that $B/A$ has dimension $2m + 1$ and a radical of dimension one. Of course, for $A \subset C \subset B$ with $C/A$ nondegenerate of dimension $2m$ the subspace $T_l(C, A)$ is isomorphic to $C_{m,1}(F)$, however, there is another class of subspaces of $T_l(B, A)$ isomorphic to $C_{m,1}(F)$ which are not of this form. We will refer to these as the “exceptional polar examples”.

(ii) The generalized quadrangle $C_{2,1}(F)$ is isomorphic to its dual, which is well known to be $B_{2,1}(F) \cong Q(5, F)$. This latter geometry is a geometric hyperplane of $Q^+(6, F) \cong D_{3,1}(F) \cong G_{3,2}(F)$. Suppose now that $\dim V \geq 8$ and $l = 2$. Let $M$ be a totally isotropic 4-subspace of $V$. The subspace $T_2(M, 0)$ is a convex subspace and is isomorphic to a polar space $D_{3,1}(F) \cong Q^+(6, F)$. Then $T_2(M, 0)$ contains a geometric hyperplane isomorphic to $C_{2,1}(F)$.

We will refer to this as the “exceptional inclusion of $C_{2,1}(F)$ in $C_{4,2}(F)$”.

(iii) Let $V$ be an eight dimensional orthogonal space with maximal Witt index, so that the geometry of singular points and totally singular lines is $D_{4,1}(F)$. From triality it follows that the geometry $D_{4,2}(F)$ contains three classes of subspaces $B_{3,2}(F)$. Let $S_i, i = 1, 2, 3$ be representatives of these classes. For one of the classes, say $S_3$, $\Sigma(S_3)$ is a hyperplane of $V$ and for the other two $\Sigma(S_1) \Sigma(S_2) = V$. 
Now let the characteristic of $\mathbb{F} = 2$ and assume that $\mathbb{F}$ is perfect so that $B_{4,1}(\mathbb{F}) \cong C_{4,1}(\mathbb{F})$ and let the points of $C_{4,1}(\mathbb{F}) = PG(V)$. In this way we obtain two classes of subspaces $S_i, i = 1, 2$ with $S_i \cong C_{3,2}(\mathbb{F}) \cong B_{3,2}(\mathbb{F}) = (P, L)$ and such that $\Sigma(S_i) = V$. We refer to these two classes as the “exceptional inclusions of $C_{3,2}(\mathbb{F})$ in $C_{4,2}(\mathbb{F})$”.

Here we will not deal with the case that $\text{char}(\mathbb{F}) = 2$ which will be handled in a subsequent paper. Thus, our main result is as follows:

Main Theorem. Assume $\text{char}(\mathbb{F}) \neq 2$ and let $S$ be a subspace of $C_{n,1}(\mathbb{F}), S \cong C_{m,k}(\mathbb{F})$. Then there exists a totally isotropic subspace $A$ of dimension $l - k$ and a subspace $B$ with $A \subset B \subset A^\perp$ with $B/A$ non-degenerate and such that $S = T_l(B, A)$.

Before proceeding to the proof we introduce some notation:

Notation. Since we will generate all kinds of subspaces, of the symplectic space $V$, of the geometry $\Gamma = (\mathcal{P}, \mathcal{L})$, etc. we need to distinguish between these. When $X$ is some collection of subspaces or vectors from $V$ we will denote the subspace of $V$ spanned by $X$ by $\langle X \rangle_\mathbb{F}$. When $X$ is a subset of $\mathcal{P}$ we will denote the subspace of $\Gamma = (\mathcal{P}, \mathcal{L})$ generated by $X$ by $\langle X \rangle_\Gamma$.

For a point $p \in \mathcal{P}$ we will denote by $\Delta^\Gamma(p)$ the collection of all points of $\mathcal{P}$ which are collinear with $p$ in $(\mathcal{P}, \mathcal{L})$ (including $p$).

2. General properties of symplectic Grassmannians

In this short section we review some properties of symplectic Grassmannians. We omit the proofs of most because these propositions are either well known or easy to prove.

Lemma 2.1. (i) The symplectic Grassmannian space $(\mathcal{P}, \mathcal{L}) \cong C_{n,1}(\mathbb{F})$ has two classes of maximal singular subspaces with representatives $T_l(B, 0)$ where $B$ is a totally isotropic subspace of $V$, $\dim B = l + 1$, and $T_l(C, A)$ where $A$ and $C$ are incident totally isotropic subspaces of $W$, where $\dim A = l - 1, \dim C = n$. In the former case $T_l(B, 0) \cong PG_{l}(\mathbb{F})$ and in the latter $T_l(C, A) \cong PG_{n-l}(\mathbb{F})$. We refer to the first class as a type one maximal singular subspaces and the latter as type two singular subspaces.

(ii) If $M_1$ and $M_2$ are maximal singular subspaces of different types then either $M_1 \cap M_2$ is empty or a line.

(iii) If $M_1$ and $M_2$ are distinct type one maximal singular subspaces then $M_1 \cap M_2$ is either empty or a point.

Definition. A symp of $(\mathcal{P}, \mathcal{L})$ is a maximal geodesically closed subspace which is isomorphic to a polar space.

Lemma 2.2. There are two classes of symps in $(\mathcal{P}, \mathcal{L})$. One class has representative $T_l(E, D)$ where $D \subset E$ are totally isotropic subspaces, $\dim D = l - 2, \dim E = l + 2$. In this case $T_l(E, D) \cong D_{3,1}(\mathbb{F})$ the polar space of a non-degenerate six dimensional orthogonal space with maximal Witt index. The second class has representative $T_l(C^\perp, C)$ where $C$ is a totally isotropic subspace, $\dim C = l - 1$. In this case $T_l(C^\perp, C)$ is isomorphic to the polar space of a non-degenerate symplectic space of dimension $2(n - l + 1)$. We refer to the former as a type one symp and the latter as a type two symp.

Lemma 2.3. There are three classes of points at distance two in $\Gamma = (\mathcal{P}, \mathcal{L})$:
(i) The pairs \( \{x, y\} \) which satisfy \( \dim(x \cap y) = l - 2 \) and \( x \perp y \). Such a pair \( \{x, y\} \) lies in a unique symp which is \( T_l(x + y, x \cap y) \). Note this only occurs if the Witt index of the symplectic space is greater than or equal to four;
(ii) The pairs \( \{x, y\} \) that satisfy \( \dim(x \cap y) = l - 1 \) and \((x + y)/(x \cap y)\) is a non-degenerate two space. This pair belongs to a unique symp which is \( T_l((x \cap y)^\perp, x \cap y) \);
(iii) The pairs \( \{x, y\} \) which satisfy \( \dim(x \cap y) = l - 2 \) and \( \dim((x + y) \cap [x + y]^\perp) = l \). There is a unique point collinear with both \( x \) and \( y \), namely \([x + y] \cap [x + y]^\perp\).

**Definition.** The first class of pairs in **Lemma 2.3** will be referred to as type one symp pairs, the second as type two symp pairs and the third type as special pairs. For a point \( x \) we will denote by \( \Delta(2,i)(x) \) all the points \( y \) such that the pair \( x, y \) is a type \( i \)-symp pair and by \( \Delta(2,x)(x) \) the points \( y \) such that \( x, y \) is a special pair.

**Lemma 2.4.** Let \( S \) be a type two symp of the incidence geometry \((\mathcal{P}, \mathcal{L}) \cong C_{n,l}(\mathbb{F}) \) and \( x \in \mathcal{P} \setminus S \). Then \( \Delta^I(x) \cap S \) is either empty or a singular subspace.

**Lemma 2.5.** Let \((\mathcal{P}, \mathcal{L}) = C_l(W) \cong C_{n,l}(\mathbb{F}) \) and let \( p \neq q \in \mathcal{I}_l(W) \).
(i) Assume \( p \perp q \) and let \( x \in T_l(p^\perp, p) \). Then one of the following occurs:
   (a) \( q \subset x \) and \( x \in T_l(q^\perp, q) \);
   (b) \( q \) is not contained in \( x \), \( x \subset q^\perp \) and \( \Delta(x) \cap T_l(q^\perp, q) = T_l(x + q, q) \) is a singular subspace isomorphic to \( \mathbb{P}G_{l-1}(\mathbb{F}) \); and
   (c) \( x \) is not contained in \( q^\perp \) and \( (x \cap q^\perp, q) \) is the unique point in \( \Delta(2,2)(x) \cap T_l(q^\perp, q) \).
(ii) Assume \( p \) and \( q \) are non-orthogonal. Then \( T_l(p^\perp, p) \cap T_l(q^\perp, q) = \emptyset \). If \( x \in T_l(p^\perp, p) \) then there is a unique \( y = (x \cap q^\perp, q)_V \) that is the unique point in \( T_l(q^\perp, q) \cap \Delta(2,2)(x) \).

### 2.1. Properties of the geometry \( C_{3,2}(\mathbb{F}) \)

The particular geometry \( C_{3,2}(\mathbb{F}) \) plays a prominent role in our proof and we use several properties of this geometry which we will make explicit here for later reference. Throughout this subsection we will let \( W \) be a non-degenerate six dimensional symplectic space over \( \mathbb{F} \) and \((\mathcal{P}, \mathcal{L}) \) will be the geometry \( C_2(W) \cong C_{3,2}(\mathbb{F}) \).

**Lemma 2.6.** The maximal singular subspaces of \( C_2(W) \) are projective planes. If \( M_1, M_2 \) are two such subspaces then \( M_1 \cap M_2 \) is either empty or a point.

**Proof.** Suppose \( x \) and \( y \) are collinear points of \( C_2(W) \). Then \( x \cap y \in \mathcal{I}_1(W) \) and \( T_2([x \cap y]^\perp, [x \cap y]) \) is a generalized quadrangle and therefore its lines are maximal singular subspaces. Therefore, if \( z \) is collinear with both \( x \) and \( y \) but does not lie on the line \( T_2(x + y, x \cap y) \) then \( z \) must lie in the totally isotropic three space \( x + y \) and \( (x, y, z)_I = T_2(x + y, 0) \) is a projective plane (dual to \( T_1(x + y, 0) \)). We have therefore shown that the maximal singular subspaces of \( C_2(W) \) are all of the form \( T_2(U, 0) \) for \( U \) a totally isotropic subspace of \( W \) of dimension three.

Now let \( M_i = T_2(U_i, 0), i = 1, 2 \) where \( U_i \) are distinct maximal totally isotropic subspaces of \( W \). Then \( \dim(U_1 \cap U_2) \leq 2 \). If \( \dim(U_1 \cap U_2) = 2 \) then \( U_1 \cap U_2 \) is the unique point in \( M_1 \cap M_2 \). Otherwise \( M_1 \cap M_2 = \emptyset \). \( \square \)

**Lemma 2.7.** Assume that \( M_1, M_2 \) are maximal singular subspaces of \( C_2(W) \) and \( M_1 \cap M_2 \) contains a point \( x \). For a point \( y \in M_1, y \neq x \) we have the following:
(i) \( \Delta(y) \cap M_2 = \{x\} \).
(ii) \( [\Delta(2,2)(y) \cap M_2] \cup \{x\} \) is the line \( T_2(M_2, x \cap y) \).
Proof. Let $M_i = T_2(U_i, 0), i = 1, 2$ so that $x = U_1 \cap U_2$. Let $y \in M_1$. Then $y \cap x = p$ is a projective point of $W$. Since $U_2$ is a maximal totally singular subspace and $y$ is not contained in $U_2$, it follows that $y^\perp \cap U_2 = x$ and so $\{x\} = \Delta(y) \cap M_2$. On the other hand, suppose $z \in M_2$ and $z \cap x = p$. Then $y \cap z = p$ and the pair $y, z$ is a type two symplectic pair. Thus, every point of the line $T_2(U_2, p)$, apart from $x$ belongs to $\Delta(2, 2)(y)$. Moreover, if $w \in M_2$ and $w \cap x = q \neq p$ then $y, w$ is a special pair. Thus, we have shown (i) and (ii). □

Lemma 2.8. Let $M_1$ and $M_2$ be maximal singular subspaces of $C_2(W)$ such that $M_1 \cap M_2 = \emptyset$. Then one of the following occurs:

(i) There are lines $m_1 \subseteq M_1, i = 1, 2$ satisfying the following: For each point $x \in m_1$, $\Delta(x) \cap M_2$ is a point and $m_2 \subseteq \Delta(x) \cap M_2$ is a line and $m_1 \cap m_2 = \emptyset$. For each point $x \in m_1$, $\Delta(x) \cap M_2$ is a line and if $y \in M_2, y \not\in \Delta(x, y) \cap M_2$ then $d(x, y) = 3$.

Proof. (i) Let $M_i = T_2(U_i, 0), i = 1, 2$ where $U_1 \cap U_2 = p$ is a projective point. Set $m_i = T_2(U_i, p), i = 1, 2$ lines of $M_1, M_2$ respectively. Suppose $x \in m_1$. Let $y = U_2 \cap x^\perp$. Then $y \in m_2$ and it is the unique point of $M_2$ collinear with $x$. For any other point $y' \in m_2, \dim(x \cap y') = 1$ and therefore $x, y'$ is a type two symplectic pair. For any other $z \in M_2, x \cap z = 0$. However, $x^\perp \cap z \neq 0$ since $p$ is contained in $z^\perp \cap x$ and therefore $x, z$ is a special pair.

(ii) Now let $M_i = T_2(U_i, 0), i = 1, 2$ where $U_1 \cap U_2 = 0$. Then for each $x \in M_1, y \in M_2$, $x \cap y = 0$ and $x, y$ cannot be collinear or a type two symplectic pair and so either $x, y$ is a special pair or $d(x, y) = 3$. However, for $x \in M_1, x^\perp \cap U_2 = p$ is a projective point of $W$ and all the points of the line $T_2(U_2, p)$ are in $\Delta(2, s)(x)$. □

Notation. If $M_1, M_2$ are maximal singular subspaces of $C_2(W)$ we will write $M_1 \sim M_2$ if $M_1 \cap M_2$ is a point and $M_1 \ast M_2$ if $M_1, M_2$ are as in Lemma 2.8 part (i).

Lemma 2.9. Let $\mathcal{M}$ be the collection of all maximal singular subspaces of $C_2(W)$. Then

(i) The graph $(\mathcal{M}, \sim)$ is connected.
(ii) The graph $(\mathcal{M}, \ast)$ is connected.

Proof. (i) The graph $(\mathcal{M}, \sim)$ is the collinearity graph of the dual polar space $C_{3,3}(\mathbb{F}) = DW(5, \mathbb{F}) = DSp(6, \mathbb{F})$ which is known to be connected.

(iii) In light of (i) it suffices to prove that if $M_1 \sim M_2$ then there exists a path from $M_1$ to $M_2$. Suppose $M_i = T_2(U_i, 0), i = 1, 2$ where $U_1 \cap U_2 \in \mathcal{I}(W)$. Let $v_1, v_2$ be a basis for $U_1 \cap U_2$. Extend this to a basis $v_1, v_2, v_3$ for $U_1$ and $v_1, v_2, w_3$ for $U_2$. Now $v_3$ and $w_3$ are non-orthogonal. Then $(v_3 + w_3)\perp$ is a non-degenerate four dimensional subspace of $W$ which contains $v_1 + v_2$. Extend this to a base $v_1, v_2, w_1, w_2$ where $v_i \perp w_j$ for $i \neq j$ and $v_1 \perp w_2$. Now let $M_3 = \langle v_1, w_2, v_3 + w_3 \rangle$. Then $M_1 \ast M_3 \ast M_2$. □

3. Proof of the main theorem

In this section we prove our main theorem. Let $(V, f)$ be a non-degenerate symplectic space of dimension $2n$ over $\mathbb{F}$ and $(W, g)$ a non-degenerate symplectic space of dimension $2m$ over $\mathbb{F}$. When necessary, we will distinguish orthogonality in $V$ by writing $\perp_V$ and in $W$ by $\perp_W$. Before proceeding to the proof we introduce some notation: When $A, B$ are subspaces of $V$ and $l$ is an positive integer we will denote by $T_{(V, l)}(B, A)$ the collection $l$-dimensional totally isotropic subspaces of $V$ which satisfy $A \subseteq C \subseteq B$ and in a similar fashion we define $T_{(W, l)}(E, D)$.
Fix an \( l, 1 \leq l \leq n - 1 \) and let \( \Gamma' = (\mathcal{P}, \mathcal{L}) \) where \( \mathcal{P} = \mathcal{I}_l(V) \) and \( \mathcal{L} \) consists of all sets \( \lambda(B, A) = T_{(V, l)}(B, A) \) where \( A \subset B \subset B^\perp \) are subspaces of \( V \), \( \dim A = l - 1 \) and \( \dim B = l + 1 \).

Now fix \( k, 1 \leq k \leq m - 1 \) and set \( \mathcal{P}' = \mathcal{I}_k(W) \) and set \( \mathcal{L}' \) equal to the collection of all sets \( \lambda(B', A') = T_{(W, k)}(B', A') \) where \( A' \subset B' \subset (B')^\perp \) are subspaces of \( W \), \( \dim A' = k - 1 \) and \( \dim B' = k + 1 \) so that \( \Gamma'' = (\mathcal{P}', \mathcal{L}') \cong C_{m,k}(F) \). Now assume that \( \mathcal{S} \) is a subspace of \( \Gamma', S = (\mathcal{P}_S, \mathcal{L}_S) \cong (\mathcal{P}', \mathcal{L}') \). Let \( \sigma : \Gamma'' \to \mathcal{S} \) be an isomorphism. For a totally isotropic subspace \( U \in \mathcal{I}_l(W) \), \( 1 \leq l \leq m \), we will denote by \( S_U \) the image under \( \sigma \) of \( T_{(W, k)}(U^\perp, U) \).

We will show the conclusions of our main theorem hold in a sequence of lemmas. Our proof is by induction on \( N = n + l + m + k \).

**Lemma 3.1.** Let \( x, y \in \mathcal{P} \) be collinear and \( z \) on the line \( xy \). Then \( x \cap y \subset z \subset x + y \).

**Proof.** This is an immediate consequence of the definition of collinearity in \( C_{n,l} \) and of a line. \( \square \)

**Corollary 3.2.** Let \( \mathcal{S} \) be a subspace of \( \Gamma = C_1(V) \cong C_{n,l}(F) \) and \( X \) a generating set of \( \mathcal{S} \), that is, a subset \( X \) of \( \mathcal{S} \) such that \( \langle X \rangle \cap \Gamma \neq \emptyset \). Then \( \Sigma(\mathcal{S}) = \Sigma(X) \).

**Proof.** We define a sequence of sets \( P_n(X) \subset \mathcal{P}, n \geq 0 \) inductively as follows: \( P_0(X) = X \) and \( P_{n+1}(X) = P_n(X) \cup \bigcup_{\lambda \in \mathcal{L} \cap \{ \lambda \cap P_n(X) \} \geq 2} \lambda \)

and set \( P(X) = \bigcup_{n \geq 0} P_n(X) \). Note that \( P_{n+1}(X) \supset P_n(X) \). We claim that \( P(X) \) is a subspace of \( \Gamma \). For suppose that \( \lambda \) is a line and \( x \neq y \in \lambda \cap P(X) \). Then there are natural numbers \( m, n \) such that \( x \in P_m(X), y \in P_n(X) \). If \( n' = \max\{m, n\} \) then \( x, y \in P_{n'}(X) \) and then \( \lambda \subset P_{n'+1}(X) \). This proves that \( P(X) \) is a subspace.

Since \( X \subset P(X) \) and \( X \) generates \( \mathcal{S} \) we can conclude that \( \mathcal{S} \subset P(X) \). On the other hand, a simple induction implies that \( P_n(X) \subset \mathcal{S} \) for each \( n \geq 0 \), whence \( P(X) \subset \mathcal{S} \) and consequently, \( P(X) = \mathcal{S} \).

We next claim that \( \Sigma(P_n(X)) \subset \Sigma(X) \) for all \( n \geq 0 \). The proof is by induction on \( n \). Since \( P_0(X) = X \) the base case is clear.

Now assume that \( \Sigma(P_n(X)) \subset \Sigma(X) \) and let \( z \in P_{n+1}(X) \backslash P_n(X) \). Then there is a line \( \lambda \) containing \( z \) with \( |\lambda \cap P_n(X)| \geq 2 \). Let \( x \neq y \in \lambda \cap P_n(X) \). By the inductive hypothesis, \( x, y \in \Sigma(X) \) and then by **Lemma 3.1** it follows that \( z \in \Sigma(X) \).

Since \( P_n(X) \subset P_{n+1}(X) \) it follows that \( \Sigma(P_n(X)) \subset \Sigma(P_{n+1}(X)) \) and consequently, \( \bigcup_{n \geq 0} \Sigma(P_n(X)) \) is a subspace and equal to \( \Sigma(\bigcap_{n \geq 0} P_n(X)) \). We can then conclude that \( \Sigma(X) \supset \bigcup_{n \geq 0} \Sigma(P_n(X)) = \Sigma(\bigcup_{n \geq 0} P_n(X)) = \Sigma(S) \). \( \square \)

**Lemma 3.3.** If \( \mathcal{S} \cong C_{m,1}(F) \) then there exists a totally isotropic subspace \( D \) of dimension \( l - 1 \), a subspace \( E \) satisfying \( D \subset E \subset D^\perp \) with \( E/D \) non-degenerate of dimension \( 2m \) and \( \mathcal{S} = T_{(V, l)}(E, D) \).

**Proof.** The subspace \( S \) is a polar space and therefore contained in one of the two types of symps because any polar space is the convex hull of any two of its points at distance two. Suppose \( S \) is contained in a type two symp, \( T_{(V, l)}(D^\perp, D) \) where \( D \) is totally isotropic, \( \dim D = l - 1 \). Since \( \text{char}(F) \neq 2, S \) has a hyperbolic generating set, that is, a collection of points, \( p_i, q_i, 1 \leq i \leq m \) such that the pairs \( \{p_i, p_j\}, \{q_i, q_j\} \) and \( \{p_i, q_j\} \) are collinear for \( i \neq j \) and the pairs, \( \{p_i, q_i\} \) are not colinear. Since the points \( p_i, q_i \in T_{(V, l)}(D^\perp, D) \), the points \( p_i, q_i \) are \( l \) spaces containing
such that $p_i^⊥ \cap q_i = D$ and $p_i \perp p_j, p_i \perp q_i$ and $q_i \perp q_j$ for $i \neq j$. Then the space $E = \sum_{i=1}^m (p_i + q_i)$ has dimension $2m + (l - 1)$ and $E/D$ is non-degenerate. Since $\{p_i, q_i : 1 \leq i \leq m\}$ generates $S$ it follows from Corollary 3.2 that $\Sigma(S) = E$. So in this case the conclusion of the theorem holds.

Now assume that $S$ is contained in a type one symp, $T(V,l)(B, A)$ with $A \subset B \subset A^⊥_V$, subspaces of $V$ with dim $A = l - 2$ and dim $B = l + 2$. Since the symp $T(V,l)(B, A)$ has an embedding of dimension six, it must be the case that $m = 2$. However, there is an inclusion of the generalized quadrangle $C_{2,1}(F)$ in the polar space $D_{3,1}(F)$ if and only if $\text{char}(F) = 2$ which has been excluded. □

We will next be treating the case that $m \geq 3$ and $S \cong C_{m,2}(F)$ is a subspace of $\Gamma \cong C_{n,l}(F)$. For each point $p \in I_1(W), S_p \cong C_{m-1,1}$ and by Lemma 3.3 there is a totally isotropic subspace $A_p$ of dimension $l - 1$ and $B_p \subset A_p^⊥$ with $B_p/A_p$ non-degenerate of dimension $2(m - 1)$ such that $S_p = T(V,l)(B_p, A_p)$.

**Lemma 3.4.** For $p \neq q \in I_1(W), A_p \neq A_q$.

**Proof.** Suppose to the contrary that $A_p = A_q$ for some pair $p \neq q \in I_1(W)$. Set $U = A_p = A_q$. Then $S_p, S_q$ are both subspaces of $T(V,l)(U^⊥_V, U)$ which is a type two symp. By Lemma 2.4, and since $S_p$ is a type two symp of $\Gamma'$, for any point $y \in S_q \setminus S_p, \Delta_F(y) \cap S_p$ is either empty or a singular subspace. In particular, $S_p$ is not contained in $\Delta_F(y)$.

Let $x$ be a point in $S_p$ which is not collinear with $y$ and let $w, z$ be points of $S_p$ which are non-collinear but are both collinear with $x$. Since $S_p, y$ are contained in the symp $T(V,l)(U^⊥_V, U)$, $y$ is collinear with a point $w' \neq x$ on the line $xw$ and a point $z' \neq x$ on the line $xz$. However, the points $w'$ and $z'$ are non-collinear and this contradicts the fact that $S_p \cap \Delta_F(y)$ is empty or a singular subspace. Thus, $A_p \neq A_q$ for $p \neq q \in I_1(W)$. □

We shall now deal with the case $k = l = 2$.

**Lemma 3.5.** Assume that $S \cong C_{m,2}(F)$ and $\Gamma \cong C_{n,2}(F)$. Then there is a non-degenerate 2m-dimensional subspace $B$ of $W$ such that $S = I_1(B) = T(V,l)(B, 0)$.

**Proof.** For $p \in I_1(W), S_p \cong C_{m-1,1}(F)$, is a symp of $S$. By Lemma 3.3 there is a point $A_p$ of $V$ and a subspace $B_p \subset A_p^⊥$ with $B_p/A_p$ non-degenerate of dimension $2(m - 1)$ such that $S_p = T(V,2)(B_p, A_p)$. We have seen for $p \neq q \in I_1(W)$ that $A_p \neq A_q$. Thus the map $p \to A_p$ of points of $I_1(W)$ to $I_1(V)$ is injective.

Next note that if $p \perp_{W} q$ then $T(W,2)(p^⊥_{W}, p) \cap T(W,2)(q^⊥_{W}, q) = \{(p, q)_{W}\}$. If $x = \sigma((p, q)_{W})$ then $A_p$ and $A_q$ must be contained in $x$. Then they are distinct hyperplanes of $x$ and consequently, $x = (A_p, A_q)^{V}$. In particular, $A_p + A_q = (A_p, A_q)^{V}$ is totally isotropic.

Next suppose $r \neq p$ is a point of $I_1((p, q)_{W})$. Then $(p, q)_{W} = (p, r)_{W}$ from which it follows that $A_p + A_r = A_p + A_q$ which implies that $A_r \in T(V,2)(A_p + A_q, 0);$ since, for $l = 2, A_p \cap A_q = 0$.

Finally, suppose that $p, q \in I_1(W), p$ and $q$ non-orthogonal. We claim that $A_p$ and $A_q$ are non-orthogonal. Suppose to the contrary that $A_p \perp A_q$. Let $r \in I_1(W)$ with $p \perp r \perp q$ so that $(p, r)_{W}, (q, r)_{W}$ are two points of $C_2(W)$ which are non-collinear. Then $\sigma((p, r)_{W}) = A_p + A_r$ and $\sigma((q, r)_{W}) = A_q + A_r$ are not collinear. However, since $A_p + A_q + A_r \in I_3(V)$ and $(A_p + A_r) \cap (A_q + A_r) = A_r \neq 0$ these are collinear points, a contradiction.

Assume now that $A \in PG(A_p + A_q)$. We next claim that there exists an $r \in PG(p + q)$ such that $A_r = A$. Towards that end, let $s_1, s_2$ be non-collinear points of $W$ $p \perp s_i \perp q$ for $i = 1, 2$.
The totally isotropic lines $p + s_i, q + s_i$ meet at $s_i$ and the join, $p + q + s_i$ is totally isotropic and therefore they are collinear in $C_2(W)$. Now set $x_i = \sigma(p + s_i), y_i = \sigma(q + s_i), i = 1, 2$. Now $x_i \in T_{(V,2)}(A_p^\perp, A_p)$ and $y_i \in T_{(V,2)}(A_q^\perp, A_q)$ are collinear. It follows that there is a unique point $z_i$ on the line $T_{(V,2)}(x_i + y_i, A_{s_i})$ contained in $T_{(V,2)}(A^\perp, A)$. Since $S$ is a subspace, $z_i \in S$. Since $\sigma$ is an isomorphism of $C_2(W)$ onto $S$ there are points $u_i \in C_2(W)$ such that $\sigma(u_i) = z_i, i = 1, 2$. In fact, $u_i$ belongs to the line $T_{(W,2)}(p + q + s_i, s_i)$. Also, since $z_1, z_2$ are contained in the type two symp $T_{(V,2)}(A^\perp, A)$ it also follows that $u_1 \cap u_2$ is a point $r \in I_2(W)$ which belongs to $(p + q + s_1) \cap (p + q + s_2) = p + q$. It now follows that $S_r \subset T_{(V,2)}(A^\perp, A)$ and consequently that $A_r = A$.

We can now conclude that the injective map $p \rightarrow A_p$ defines an isomorphism of the polar space $C_1(W) \cong C_{m,1}(\mathbb{F})$ into $C_1(V)$. Since char$(\mathbb{F}) \neq 2$ it follows that the image of this map is a non-degenerate $2m$-dimensional subspace $B$ of $V$. We claim that $S = T_{(V,2)}(B, 0)$.

If $x = \sigma((p, q)_W)$ then $x = A_p + A_q \subset B$ and consequently, $S \subset I_2(B)$. Since $S \cong T_{(V,2)}(B, 0)$ it follows that $S = T_{(V,2)}(B, 0)$ as claimed. \(\square\)

**Lemma 3.6.** Assume that $S \cong C_{3,2}(\mathbb{F})$ and $\Gamma \cong C_{n,1}(\mathbb{F})$ with $l > 2$. Then there is a totally isotropic subspace $A, \dim A = l - 2$ a subspace $B \subset A^\perp$ such that $B/A$ is a six-dimensional non-degenerate space and $S = T_{(V,2)}(B, A)$.

**Proof.** Let $U \in I_3(W)$. We set $M(U) = \sigma(T_{(W,2)}(U, 0))$ which is a singular plane of $S$. There are two possibilities for $M$: (i) $M = T_{(V,3)}(D, C)$ with $C \subset D$ totally singular subspaces, \(\dim C = l - 1, \dim D = l + 2\); or (ii) $m = T_{(V,3)}(D, C)$ with $C \subset D$ totally singular subspaces, \(\dim C = l - 2, \dim D = l + 1\). We want to show that the first case cannot occur. Toward that end we first show that it is not possible for two different planes to occur in $S$.

By **Lemma 2.9** the graph on $I_3(W)$ given by $U_1 \ast U_2$ if $U_1 \cap U_2 \in I_1(W)$ is connected. Consequently, it suffices to show for any such pair that $M(U_1)$ and $M(U_2)$ have the same type. So, let $U_1, U_2 \in I_3(W)$ with $U_1 \cap U_2 \in I_1(W)$ and set $M_i = M(U_i), i = 1, 2$ and suppose $M_i = T_{(V,3)}(D_i, C_i)$ where $\dim C_i = l - 1, \dim D_i = l - 2, \dim D_i = l + 2, \dim D_i = l + 1$.

By **Lemma 2.8** there are lines $m_i \subset M_i, i = 1, 2$ such that if $x \in m_1$ then $M \cap \Delta^F(x) = m_2 \cap \Delta^F(x)$ is a point, $x'$, and for $y \in m_2 \setminus \{x'\}$ the pair $x, y'$ is a type two symp pair. Therefore for $m_1, m_2, \dim(x \cap y) = l - 1$. Since $m_1$ is a line contained in $M_2$ it has the form $T_{(V,3)}(E_1, C_1)$ for some $(l + 1)$-dimensional subspace of $D_1$ which contains $C_1$ and $\cap_{x \in m_1} x \subset C_1$. Similarly, $\cap_{y \in m_2} y$ is an $(l - 1)$-dimensional subspace of $E_2$ which contains $C_2$.

Fix $x \in m_1$ and suppose there are $y_1, y_2 \in m_2$ such that $x \cap y_1 \neq x \cap y_2$. Then $x \cap y_1$ and $x \cap y_2$ are distinct hyperplanes of $x$ and then $x = x \cap y_1 + x \cap y_2 \subset D_1 \cap D_2$. Then $x$ is a hyperplane of $D_2$ and this implies that $M \subset \Delta^F(x)$, a contradiction.

Thus, $x \cap y_1 = x \cap y_2$ for each $y_1, y_2 \in m_2$. Since $x \cap y$ has dimension $l - 1$ for $y \in m_2$ and $I(m_2)$ has dimension $l - 1$ we conclude that $x \subset I(m_2)$. Since $x$ is arbitrary, it then follows that $C_1 = I(m_1) \subset I(m_2)$. Since $\dim I(m_2) = l - 1, I(m_1) = I(m_2) = C_1$. Since $I(m_2) \subset C_2$ it is also the case that $C_2 \subset C_1$. However, if $y \in T_{(V,3)}(D_2, C_1)$ then $y \in M_2$. Since $y \cap D_1 = C_1$ is a hyperplane of $y$ contained in $y^\perp \cap U_1$ it follows that the line $T_{(V,3)}(y^\perp \cap U_1, C_1) \subset \Delta^F(y)$ which contradicts the fact that $\Delta^F(y) \cap M$ is a point. Thus, for each $U \in I_3(W)$, $M(U)$ are all of type (i) or type (ii). We show that, in fact, type (i) do not occur.

Suppose to the contrary that all the planes of $S$ are of type (i). Now let $U_1, U_2 \in I_3(W)$ with $U_1 \cap U_2 \in I_2(W)$ and set $M_i = M(U_i) = T_{(V,3)}(D_i, C_i), i = 1, 2$ and let $x = \sigma(U_1 \cap U_2)$. Since $M_1 \cap M_2 = \{x\}$ either $C_1 + C_2 = x$ or $C_1 = C_2$ and $D_1 \cap D_2 = x$. Suppose $C_1 + C_2 = x$. Let $y \in M_1, y \neq x$ so that $C_2$ is not contained in $y$. By pulling back to $C_2(W)$ and using
the isomorphism $\sigma$ we can conclude that there is a line $\lambda_y \subset M_2$ containing $x$ such that if $y' \in \lambda_y$, $y' \neq x$ then $y, y'$ is a symp pair and therefore $\dim(y \cap y') = l - 1$. Now the line $\lambda_y$ must be of the form $T_{(V,I)}(D, C_2)$ for some subspace $D$ of $D_2$. Since $\dim D = l + 1$. But then $I(\lambda_y) = C_2$.

Now let $U_1, U_2 \in \mathcal{I}_3(W)$ with $U = U_1 \cap U_2 \in \mathcal{I}_2(W)$ and set $M_i = M(U_i) = T_{(V,I)}(D_i, C_i), i = 1, 2$ and $x = \sigma(U) \in M_1 \cap M_2$. Then $x \in D_1 \cap D_2$ and $C_1 + C_2 \subset x$. If $D_1 \cap D_2 \neq x$ and $C_1 + C_2 \neq x$ then $T_{(V,I)}(D_i \cap D_2, C_1 + C_2)$ is contained in $M_1 \cap M_2$ has points in addition to $x$, a contradiction. We claim that $C_1 = C_2$. Suppose to the contrary that $C_1 \neq C_2$. As in the above, for $y \in M_1$ we will denote by $\lambda_y$ a line in $M_2$ containing $x$ such that for $x \neq y' \in \lambda_y$ the pair $y, y'$ is a symp pair and therefore $\dim(y \cap y') = l - 1$. And, as shown above, $I(\lambda_y) = C_2$.

We have $I(M_1) = C_1 \neq C_2 = I(M_2)$. Since $\cap_{z \in \lambda_y}(y \cap z) \subset C_2$ has dimension $l - 2$ and $\dim(y \cap z) = l - 1$ for $z \in \lambda_y$ there must be $z_1, z_2 \in \lambda_y$ with $y \cap z_1 \neq y \cap z_2$. Then $y \cap z_1, y \cap z_2$ are distinct hyperplanes of $y$ and $y = (y \cap z_1) + (y \cap z_2) \subset D_1 \cap D_2$. Since $y$ is arbitrary, $D_1 = \Sigma(M_1) \subset D_2$ and therefore $D_1 = D_2$. But any two hyperplanes of $D_1 = D_2$ are then collinear, whence every point of $M_1$ with every point of $M_2$, a contradiction. Thus, $C_1 = C_2$.

As argued previously, this implies there is a fixed $(l - 2)$-dimensional subspace $C$ such that $M(U) = T_{(V,I)}(D, C)$ for all $U \in \mathcal{I}_3(W)$. But then $S$ is contained in $T_{(V,I)}(C^{\perp}, C) \cong C_{l-(m-2),2}(\mathbb{F})$ and we are done by Lemma 3.5.

Lemma 3.7. Assume that $S \cong C_{m,2}(\mathbb{F})$ and $\Gamma \cong C_{n,1}(\mathbb{F})$ with $l > 2$. Then there is a totally isotropic subspace $A$, $\dim A = l - 2$ a subspace $B \subset A^{\perp \gamma}$ such that $B/A$ is a $2$-dimensional non-degenerate space and $S = T_{(V,I)}(B, A)$.

Proof. For a point $p \in \mathcal{I}_1(W)$ we let $S_p = \sigma(T_{(W,Z)}(p^{\perp}, p)) \cong C_{m-1,1}(\mathbb{F})$. By Lemma 3.3, $S_p = T_{(V,I)}(B_p, A_p)$ where $A_p$ is a totally isotropic space of dimension $l - 1$, $B_p \subset A_p^{\perp \gamma}$ and $B_p/A_p$ is non-degenerate of dimension $2(m - 1)$. From Lemma 3.4 the map $p \rightarrow A_p$ is injective. Now suppose $q_1, q_2$ are two points of $W$ such that $q_1 \perp W p \perp W q_2$. We claim that $A_p \cap A_{q_1} = A_p \cap A_{q_2}$.

Let $W'$ be the non-degenerate six-dimensional subspace of $W$ which contains $p + q_1 + q_2$ and let $S' = \sigma(T_{(W,Z)}(W', 0)) \cong C_{3,2}(\mathbb{F})$. By Lemma 3.6 it follows that $S' = T_{(V,I)}(D, A)$ where $A$ is a totally isotropic subspace, $\dim A = l - 2$, $D \subset A^{\perp \gamma}, D/A$ is non-degenerate of dimension six. For a point $y \in \mathcal{I}_1(W')$ set $S'_y = S_y \cap S'$. Then $S'_y \cong C_{2,1}(\mathbb{F})$ and $S'_y = T_{(V,I)}(B'_y \cap D, A_y)$. Now for all $y \in S', A_y \supset A$. On the other hand, if $y, z \in S'$ with $A_y \neq A_z$ then $A_y \cap A_z = A$. In particular, $A_p \cap A_{q_1} = A_p \cap A_{q_2}$. Now the graph whose vertices consist of those of pairs $(p, q)$ in $\mathcal{I}_1(W)$ with $p \perp W q$ given by $y \sim \beta$ if and only if $[\alpha \cap \beta] = 1$ is connected. From this it follows that $I(S) = A$ and $S \subset T_{(V,I)}(A^{\perp}, A)$. Applying Lemma 3.5 completes the result.

We treat the case of $S \cong C_{m,l}(\mathbb{F})$ be a subspace of $C_l(V) \cong C_{n,l}(\mathbb{F})$. We will make use of our inductive hypothesis: if $S' \cong C_{m',l'}(\mathbb{F})$ is a subspace of $C_l(V)$ with $m' + l' < m + l$ then
the conclusion of our theorem holds: there is a totally isotropic subspace $A$ of dimension $l - l'$ and a subspace $B$ with $A \subset B \subset A^\perp$ such that $B/A$ is totally isotropic of dimension $2m'$ with $S' = T_{V,l}(B, A)$.

Before proceeding to the proof we prove a lemma about “large” subspaces of symplectic spaces which will be used in the succeeding result.

**Lemma 3.8.** Let $(W, f)$ be a non-degenerate symplectic space of dimension $2m$ and let $1 < l \leq m$. Let $X$ be a proper subspace of $W$ and assume for every element of $x \in \mathcal{I}_l(W)$ that $x \subset X$ or $x \cap X$ is a hyperplane of $x$. Then $X$ is a hyperplane of $W$.

**Proof.** Assume to the contrary that $X$ is not a hyperplane of $W$. Let $Y$ be a complement to $X$ in $W$. Suppose $Y$ contains a totally isotropic subspace $Z$ of dimension two. Let $x \in T_l(Z^\perp, Z)$. Then $(x \cap X) \cap Z = 0$ and therefore $x \cap X$ cannot contain a hyperplane of $x$. It follows from this that $\dim Y = 2$ and therefore $\dim X = 2m - 2$.

Suppose $X \cap X^\perp \neq 0$. Then, since $\dim X$ is even it follows that $\dim (X \cap X^\perp) \geq 2$. Let $x_1, x_2$ be independent vectors in $X \cap X^\perp$. We can choose vectors $y_1, y_2$ such that $x_i \perp y_j$ for $\{i, j\} = \{1, 2\}, y_1 \perp y_2$ and $f(x_i, y_j) = 1$. Now if $x \in T_l(\langle y_1, y_2 \rangle^\perp, \langle y_1, y_2 \rangle)$ it is again the case that $x \cap X$ cannot contain a hyperplane of $x$. Thus, it must be the case that $X$ is non-degenerate.

Now choose a complement $Y$ to $X$ in $W$ with $Y \cap X^\perp = 0$ and let $y_1, y_2$ be a for $Y$. Since $Y \cap X^\perp = 0$ there must exist a two space $Z$ containing $y_1$ with $Z \cap X = 0$. Again, if $x \in T_l(Z^\perp, Z)$ then $(x \cap X) \cap Z = 0$ and $X$ cannot contain a hyperplane of $x$. □

**Lemma 3.9.** Let $S \cong C_{m,l}(\mathbb{F})$ be a subspace of $C_l(V) \cong C_{n,l}(\mathbb{F})$. Then there is a non-degenerate subspace $B$ of dimension $2m$ such that $S = T_{V,l}(B, 0)$.

**Proof.** The proof of this closely follows the proof of Lemma 3.5 but differs in enough of its details to warrant its inclusion. In light of Lemma 3.5 we may assume that $m > l \geq 2$.

As previously defined, for a point $p \in \mathcal{I}_l(W)$ we let $S_p = \sigma(T_{l}(W, k)(p^\perp, p)) \cong C_{m-1,l-1}(\mathbb{F})$. By our inductive hypothesis there is an isotropic point $A_p$ and a subspace $B_p$ satisfying $A_p \subset B_p \subset A_p^\perp$ with $B_p/A_p$ non-degenerate of dimension $2(m-1)$ and $S_p = T_{V,l}(B_p, A_p)$. We first show that the map $p \mapsto A_p$ from $\mathcal{I}_l(W)$ to $\mathcal{I}_l(V)$ is injective.

Suppose first that $p \neq q \in \mathcal{I}_l(W)$ are orthogonal and $A_p = A_q$. Set $A = A_p = A_q$. Note that $S_p \cap S_q = \sigma(T_{l}(W, k)(\langle p, q \rangle^\perp, \langle p, q \rangle)) = T_{V,l}(B_p \cap B_q, A) \cong C_{m-2,l-2}(\mathbb{F})$.

By Lemma 2.5 if $x \in S_p$ then either $x \in S_q$, $\Delta^I(x) \cap S_q$ is a singular subspace, $\mathbb{F}G_{l-1}(\mathbb{F})$ or there is a unique $y \in \Delta_{(2,2)}(x) \cap S_q$.

In the first case $x \in B_q$. In the second case, if $y \in S_q \cap \Delta^I(x)$ then $x \cap y$ is a hyperplane of $x$ and therefore we can conclude that $B_q \cap x$ contains a hyperplane of $x$. Finally, in the third case if $y \in \Delta_{(2,2)}(x) \cap S_q$ then we can conclude that $B_q \cap x$ contains a hyperplane of $x$.

It therefore follows that $(B_p \cap B_q)/A$ meets every element of $T_{V,l}(B_p, A)$ in a hyperplane and since $l > 2$, Lemma 3.8 applies and $(B_p \cap B_q)/A$ is a hyperplane of $B_p/A$. In particular, $\dim(B_p \cap B_q) = 2(m-1)$. Since $A \subset B_p \cap B_q \subset Z = A^\perp$ it follows that $\dim [B_p \cap B_q] \cap [B_p \cap B_q]^\perp = 2$. We will get a contradiction.

Let $A' = \text{Rad}(B_p \cap B_q) = [B_p \cap B_q] \cap [B_p \cap B_q]^\perp$ and let $C$ be a complement to $A'$ in $B_p \cap B_q$. Now $S_p \cap S_q \cong C_{m-2,l-2}$ contains $T_{V,l}(A + C, A) \cong C_{m-2,l-1}(\mathbb{F})$. Clearly, if $m > l + 1$ then this cannot occur (in particular, because of the incompatibility of symps). Therefore, we may assume that $m = l + 1$. 


Let $U \in \mathcal{I}_{l-1}(V)$ with $A \subset U \subset A + C$ and let $x_1, x_2 \in T(V,l)(B_p \cap B_q, U)$. Then $x_1, x_2$ are in the type two symm $T(V,l)(B_p, U)$ of $S_p$ (and the symm $T(V,l)(B_q, U)$ of $S_q$). It then must be the case that the symm $T(V,l)(B_p, U)$ is contained in $S_p \cap S_q$. In particular, it must be the case that there are distinct points $y_1, y_2$ in $S_p \cap S_q$ which are collinear with both $x_1$ and $x_2$. However, the only point of $T(V,l)(B_p \cap B_q, A)$ which is collinear with both $x_1$ and $x_2$ is $U + A'$, a contradiction. Thus, we cannot have $p \perp_w q$ and $A_p = A_q$.

Now assume that $p$ and $q$ are non-orthogonal points of $W$ and that $A_p = A_q = A$. Note that $S_p \cap S_q = \emptyset$. Let $x \in S_p = T(V,l)(B_p, A_p) = T(V,l)(B_p, A)$. Then it cannot be the case that $x \notin B_q$ because otherwise we would have $x \in T(V,l)(B_q, A) = S_q$ contradicting $S_p \cap S_q = \emptyset$. On the other hand, there is a unique point $y \in \Delta_2(x) \cap S_q$. Then $x \cap y$ is a hyperplane of $x$ contained in $B_q \cap x$. It therefore follows that for every $x \in T(V,l)(B_p, A)$, $(B_p \cap B_q) / A$ meets $x/A$ in a hyperplane. By Lemma 3.8 it follows that $(B_p \cap B_q) / A$ is a hyperplane of $B_p / A$. But then $B_p \cap B_q$ must contain an member of $T(V,l)(B_p, A)$ contradicting $S_p \cap S_q = \emptyset$. Thus, the map from $\mathcal{I}_1(W)$ to $\mathcal{I}_1(V)$, $p \mapsto A_p$ is injective.

When $p \neq q \in \mathcal{I}_1(W)$ and $p \perp_w q$ then $S_p \cap S_q \neq \emptyset$ from which it follows that $A_p \perp_V A_q$. On the other hand, suppose $p, q \in \mathcal{I}_1(W)$ and are non-orthogonal. We claim that $A_p$ and $A_q$ are non-orthogonal. Suppose to the contrary that $A_p \perp_V A_q$. We will get a contradiction.

We first show that either $A_p \subset B_q$ or $A_q \subset B_p$. Let $U \in \mathcal{I}_{l-1}(W)$ be contained in $p^{\perp_w} \cap q^{\perp_w}$ and set $X = (U, p)_W, Y = (U, q)_W$. Then $X, Y \in C_l(W)$ and belong to the type two symm $T_l(U^\perp, U)$. Set $x = \sigma(X), y = \sigma(Y)$. Then $(x, y) \in \Delta_2$ and so $x \cap y \in \mathcal{I}_{l-1}(V)$. Suppose neither $A_p$ nor $A_q$ is contained in $x \cap y$. Then $x + y = (x \cap y, A_p, A_q)_V$ is totally isotropic which means that $x$ and $y$ are collinear, a contradiction. This proves our assertion. Without loss of generality we can assume that $A_p \subset B_q$.

By Lemma 2.5, for each $x' \in S_p$ there is a unique point $y' \in S_q$ with $(x', y') \in \Delta_2$. Then $x' \cap y' \subset B_q$ is a hyperplane. By Lemma 3.8, $(B_p \cap B_q) / A_p$ is a hyperplane of $B_p / A_p$ and consequently, $B_p \cap B_q$ is a hyperplane of $B_p$. It then follows that $T(V,l)(B_p \cap B_q, A_p) \neq \emptyset$. Let $x' \in T(V,l)(B_p \cap B_q, A_p)$ and $y' \in S_p \cap S_q$, a contradiction. However, it now follows that $(x', A_q)_V \subset B_q$ and that $T(V,l)((x', A_q)_V, A_q) \subset \Delta^l(x') \cap S_q$, a contradiction. Thus, if $p, q \in \mathcal{I}_1(W)$ are non-orthogonal then the points $A_p$ and $A_q$ in $V$ are non-orthogonal.

We next show that of $X \in \mathcal{I}_2(W)$ then $\{A_p : p \in \mathcal{P}(X)\}$ is contained in a totally singular line of $V$. Let $p \neq q \in \mathcal{I}_1(W), p \perp w q$ and let $r \in \mathcal{P}((p, q)_w), r \neq p$. Then $S_p \cap S_q = S_p \cap S_r$. Therefore, $T(V,l)(B_p \cap B_q, A_p + A_q) = T(V,l)(B_p \cap B_r, A_p + A_r)$. In particular, $A_r \in \mathcal{P}(A_p + A_q)$.

Finally, we prove that if $p \neq q \in \mathcal{I}_1(W), p \perp q$ then the collection $\{A_r : r \in \mathcal{P}((p, q)_w)\} = \mathcal{P}((A_p, A_q)_V)$. Let $U \in \mathcal{I}_{l+1}(W)$ with $\langle p, q \rangle_W \subset U$. Then $T(V,k)(U, 0)$ is a type one maximal singular subspace of $C_l(W)$ and isomorphic to $\mathcal{P}((p, q)_w)$. Let $X = \sigma(T(U,k)(U, 0))$ a singular subspace of $C_l(V)$ (here we are making use of the assumption that $S$ is a subspace, not just a subgeometry, of $C_l(V)$). Note that $X \cap S_p$ is a type one maximal singular subspace of $T(V,l)(B_p, A_p)$ and consequently must be of the form $T(V,l)(U', A_p)$ for $U' \in \mathcal{I}_{l+1}(V)$. It follows that $X = T(U,l)(U', 0)$. Now suppose that $A \in \mathcal{P}(A_p + A_q)$. Then $X_A = X \cap T(V,l)(A^{\perp}, A)$ is a hyperplane of $X$ and so, $\sigma^{-1}(X_A)$ is a hyperplane of $T(V,k)(U, 0)$ and therefore there must be a point $r \in \mathcal{P}((p, q)_w)$ such that $\sigma^{-1}(X_A) = T(U,k)(U, r)$. Then $X_A \subset S_r = T(V,l)(B_r, A_r)$. Note that $I(X_A) = A$ and consequently, $A = A_r$ completing the assertion.

We can now say that the map $p \rightarrow A_p$ of $\mathcal{I}_1(W)$ into $\mathcal{I}_1(V)$ is a full embedding of the polar space $(\mathcal{I}_1(W), \mathcal{I}_2(W))$ into the polar space $(\mathcal{I}_1(V), \mathcal{I}_2(V))$. Since the characteristic of $\mathbb{F}$ is not
two, this implies that \( B = \{ A_p : p \in I_1(W) \} \) is a non-degenerate \( 2m \)-dimensional space of \( V \).

This completes the lemma. \( \square \)

We now complete our main result. We can assume that \( S \cong C_{m,k}(F) \) is a subspace of \( C_l(V) \cong C_{n,l}(F) \) with \( k < l \). We will show that there is a totally isotropic subspace \( A \) of dimension \( l - k \) such that \( S \subset T_{(V,l)}(A^T, A) \) and then the result will follow from Lemma 3.9.

Let \( U \) be a non-degenerate subspace of \( W \) of dimension \( 2(k + 1) \), \( Y \) a maximal totally singular subspace of \( U \) and \( X \) a subspace of \( Y \) of dimension \( k - 2 \). Set \( M = M(Y) = \sigma(T_{(W,k)}(Y,0)) \) a singular subspace of \( S \) isomorphic to \( PG(k, F) \). Also, set \( S(U) = \sigma(T_{(W,k)}(U,0)) \cong C_{k+1,k}(F) \) and \( S' = \sigma(T_{(W,k)}(U \cap X^{\perp,w}, X)) \cong C_{3,2} \) and \( M' = S' \cap M \). We have seen in Lemma 3.6 that \( M' = T_{(V,l)}(D,C) \) for totally isotropic subspaces \( C \subset D \) with \( \dim C = l - 2 \), \( \dim D = l + 1 \).

The unique maximal singular subspace of \( \Gamma \) containing \( M' \). Since \( M' \subset M \) it follows that \( M \subset T_{(V,l)}(D,0) \) and consequently, \( M = T_{(V,l)}(D_Y, A_Y) \) where \( \dim A_Y = l - k \) and \( \dim D_Y = l + 1 \). Note that since \( D_Y \subset D \) and \( \dim D_Y = \dim D \) it follows that \( D_Y = D \).

We next claim that if \( Y_1, Y_2 \) are totally isotropic subspaces of \( W \) of dimension \( k + 1 \) which satisfy \( \dim(Y_1 \cap Y_2) = k \) and \( Y_1, Y_2 \) are not orthogonal then \( A_{Y_1} = A_{Y_2} \). For convenience set \( D_{Y_i} = D_i, A_{Y_i} = A_i, i = 1, 2 \). The singular subspaces \( M_i = T_{(V,l)}(D_i, A_i) \) have a common point \( x \). Moreover, there is a one-to-one correspondence between the lines on \( x \) in \( M_1 \) to the lines on \( x \) in \( M_2 \) such that if \( \lambda \) is a line on \( x \) in \( M_1 \) and \( \lambda' \) is the corresponding line in \( M_2 \) then for \( x \neq y \in \lambda \) and \( x \neq \lambda \) then \( y \in \lambda' \) and \( y \in \lambda' \).

Fix \( y \in M_1, y \neq x \) and let \( \lambda_y \) be the line on \( x \) and \( y \). Suppose there are \( z_1, z_2 \in \lambda_y \) such that \( y \cap z_1 \neq y \cap z_2 \) then \( y \cap z_1 \) and \( y \cap z_2 \) are distinct hyperplanes of \( y \) and then \( y = y \cap z_1 + y \cap z_2 \subset D_1 \cap D_2 \). Then \( y \) is a hyperplane of \( D_2 \) in which case \( M_2 \subset \Delta^F(y) \), a contradiction. Thus, \( y \cap z_1 = y \cap z_2 \) for any points \( z_1, z_2 \subset \lambda_y \). By reversing the argument we can conclude that \( \cap_{w \in \lambda_y} w = \cap_{z \in \lambda_y} z \) has dimension \( l - 1 \). From this it follows that \( A_1 = I(M_1) = \cap_{y \in M_1} y = \cap_{z \in M_2} z = I(M_2) = A_2 \) as claimed.

Finally, since the graph on \( I_{k+1}(W) \) given by \( Y_1 \sim Y_2 \) if and only if \( \dim(Y_1 \cap Y_2) = k \), \( Y_1 \) and \( Y_2 \) non-orthogonal is connected, it follows for any two \( Y_1, Y_2 \in I_{k+1}(W) \) that \( A_{Y_1} = A_{Y_2} \). Let \( A = A_Y \) for some totally isotropic subspace of dimension \( k + 1 \) in \( W \). Since every point \( x \) of \( S \) belongs to a singular subspace \( M(Y) \) of some totally isotropic subspace \( Y \) of \( W \) of dimension \( k + 1 \), it follows that \( x \in T_{(V,l)}(A^T, A) \) and the proof of the main result is complete. \( \square \)

References