NOTE

REALIZATION OF CERTAIN GENERALIZED PATHS IN TOURNAMENTS*

Brian ALSPACH
Department of Mathematics, Simon Fraser University, Burnaby, BC, Canada

Moshe ROSENFELD
Department of Mathematics, Ben Gurion University of the Negev, Beer Sheva, Israel

Received 2 July 1980

1. Introduction

A tournament $T_n$ consists of a set of $n$ vertices and a single directed edge joining every pair of distinct vertices. We denote the vertices of $T_n$ by $\{1, \ldots, n\}$. A permutation $a_1, \ldots, a_n$ of the vertices is a generalized path. The type of the path is characterized by the sequence $\sigma_{n-1} = \varepsilon_1 \cdots \varepsilon_{n-1}$ where $\varepsilon_i = +1$ if $a_i \rightarrow a_{i+1}$ and $\varepsilon_i = -1$ if $a_i \rightarrow a_{i+1}$. We also say that $\sigma_{n-1}$ is realized in $T_n$. In [5], it was conjectured that if $n \geq 8$, then every tournament $T_n$ realizes all possible $2^{n-1}$ types $\sigma_{n-1}$. Certain types are known to be always realizable. Thus, since every tournament has a Hamiltonian path the types $\sigma^+ = +1, +1, \ldots, +1$ and $\sigma^- = -1, -1, \ldots, -1$ are always realizable. Grünbaum (Harary [3, p. 211, ex. 16.26]) observed that if $n \geq 6$, then every $T_n$ has a Hamiltonian path $a_1 \rightarrow \cdots \rightarrow a_n$ with $a_n \leftrightarrow a_1$. Thus, types in which all $\varepsilon_i$ but one have the same sign are realizable. Grünbaum [2] and Rosenfeld [4] proved that every tournament with three exceptions, has an antidirected Hamiltonian path. The exceptions are the regular tournaments $T_{R_3}, T_{R_5}, T_{R_7}$, where $T_{R_7}$ is the only regular tournament on 7 vertices with no transitive sub-tournament on 4 vertices. Thus every $T_n$, $n \geq 8$ realizes the type $\sigma_{n-1} = \varepsilon_1 \cdots \varepsilon_{n-1}$, $\varepsilon_i = (-1)^i$. Finally, Forcade [1] proved that if $n = 2^k$ then every type $\sigma_{n-1}$ is realizable by $T_n$. The purpose of this note is to prove that sequences with "large blocks" are always realizable.

2. Definitions and notation

A block in a type $\sigma_{n-1} = \varepsilon_1 \cdots \varepsilon_{n-1}$ is a maximal subsequence of consecutive elements having the same sign (maximal with respect to inclusion). Thus every

* This research was partially supported by the Natural Science and Engineering Research Council of Canada through Grant A-4792.
type is uniquely determined by its blocks \( B_1 \cdots B_k \) and we write \( \sigma_{n-1} = \varepsilon_1 \cdots \varepsilon_{n-1} = B_1 \cdots B_k \). We denote by \( \text{sg} B_i \) the sign of the elements of \( B_i \).

\[
I(x) = \{ y \mid y \in T_n, y \to x \}, \quad \text{i}(x) = \text{card } I(x).
\]

The orientation of the ordered pair \((x, y)\) is +1 if \( x \to y \) and -1 if \( x \not\to y \). We say that a set of vertices \( A \) dominates the set \( B \) if every vertex of \( A \) dominates every vertex of \( B \). Finally, we denote by \( T_n(a_1 \cdots a_l) \) the subtournament spanned in \( T_n \) by the vertices \( \{a_1, \cdots, a_l\} \) and by \( T_n \setminus \{a_1, \cdots, a_l\} \) the subtournament spanned by the other vertices.

**Lemma 1.** If \( n > 3 \), \( \sigma_{n-1} = B_1 B_2 \), then every tournament \( T_n \) realizes \( \sigma_{n-1} \).

**Proof.** Observe first that if \( |B_i| = 1 \) (\( i = 1 \) or \( i = 2 \)), then \( \sigma_{n-1} \) is realizable if \( n \neq 5 \) (Grünbaum's observation), the case \( n = 5 \) can be easily verified. We prove the lemma by induction. When \( n = 4 \) by Forcade's theorem, every \( T_4 \) realizes every sequence \( \sigma_3 \). Without loss of generality, we may assume that \( |B_1| \geq |B_2| \). (Otherwise, we realize \(- (B_2 B_1)\) and write it backwards). We may obviously assume that \( \text{sg} B_1 = +1 \). If \( |B_1| > \frac{1}{2} (n-1) \), let \( x \in T_n \) be a vertex with \( \text{i}(x) = k < |B_1| \). Let \( a_1 \cdots a_k \) be a Hamiltonian path in \( I(x) = T_n(a_1, \ldots, a_k) \). By the induction hypothesis, \( T_n \setminus \{a_1, \ldots, a_k, x\} \) realizes \( \sigma = B_1 B_2 \) where \( |B_1| = |B_1| - k - 1 \), \( \text{sg}(B_1) = +1 \). Obviously, \( a_1 \cdots a_k x_1 \cdots x_k \) (where \( x_1 \cdots x_k \) is the realization of \( \sigma \)) is a realization of \( \sigma_{n-1} \) in \( T_n \). If \( |B_1| = \frac{1}{2} (n-1) \) and \( \text{i}(x) = \frac{1}{2} (n-1) \), then, if \( n = 5 \), \( T_5 \) is the regular tournament with 5 vertices. It is a simple matter to check that it realizes \( \sigma_4 \). (The only sequence that this tournament does not realize is \( \pm (+1 - 1 + 1 + 1) \)). If \( n > 5 \), then \( T_n \) is strong and thus contains a cycle of length \( \frac{1}{2} (n-1) \). The same technique as in the first part of the proof of Theorem 1 applies.

**Theorem 1.** Let \( \sigma_{n-1} = B_1 \cdots B_k, \ |B_i| \geq i + 1 \), then every tournament \( T_n \) realizes \( \sigma_{n-1} \).

**Proof.** By induction on \( k \). Since every \( T_n \) has a Hamiltonian path, the theorem is obvious if \( k = 1 \). The case \( k = 2 \) was treated in Lemma 1. Let \( \sigma_{n-1} = B_1 \cdots B_{k+1}, \ k \geq 2 \). Let \( |B_{k+1}| = r \). Assume first that \( T_n \) contains a cycle \( a_0 \to a_1 \cdots \to a_{r-1} \to a_0 \) of length \( r \). By the induction hypothesis, \( \sigma = B_1 \cdots B_k \) is realizable in \( T_n \setminus \{a_0, \ldots, a_{r-1}\} \). Let

\[
x(1, 1) \cdots x(1, i_1) x(2, 1) \cdots x(k, 1) \cdots x(k, i_k)
\]

be such a realization, i.e.

\[
i_1 = |B_1| + 1, \quad i_j = |B_j|
\]

and

\[
\text{sg}(x(m, j), x(m, j+1)) = \text{sg} B_m, \quad \text{sg}(x(m, i_m), x(m+1, 1)) = \text{sg} B_{m+1}.
\]
We may assume that $s_B = -1$ (otherwise, we realize $-\sigma_{n-1}$ in $\bar{T}_m$, the complement of $T_n$). Hence the orientation of $(x(m, j), x(m, j+1))$ is $(-1)^{k-m+1}$ and the orientation of $(x(m, i_m), x(m+1, 1))$ is $(-1)^{k-m}$.

**Definition.** A vertex $x(m, j)$ is said to be properly oriented if for all $0 \leq i \leq r - 1$ the orientation of $(x(n, j), a_i)$ is $(-1)^{k-m}$ where $j < i_m$ and $(-1)^{k-m+1}$ when $j = i_m$.

Assume that there exists a non-properly oriented vertex. Let $x(s, l)$ be such a vertex which is furthest; i.e. $x(m, j)$ is properly oriented if $m > s$ or if $m = s$ and $j > l$. If $x(s, l) = x(k, i_k)$, we may assume that $x(k, i_k) \rightarrow a_0$. (We can always renumber the vertices of the cycle $a_0 \cdots a_{r-1}$.) It is easily seen that $x(1, 1) \cdots x(k, i_k)a_0 \cdots a_{r-1}$ realizes $\sigma_{n-1}$.

If $l = i_s$, $s < k$, assume again that $(x(s, i_s), a_0)$ is the non properly oriented edge. We first construct a realization of $B_1 \cdots B_{k-1}$ as follows:

$$x(1, 1) \cdots x(s, i_s)a_0x(s + 1, 1) \cdots x(s + 1, i_{s+1} - 1)a_1x(s + 2, 1) \cdots x(s + t, i_{s+t} - 1)a_x(s + t + 1, 1) \cdots x(k - 1, 1) \cdots x(k - 1, i_{k-1} - 1) \quad (1)$$

The vertices $x(m, i_m)$, $s < m \leq k - 1$, missing in (1), are all properly oriented by our assumption. They can be expressed as the disjoint union $A_0 \cup A_1$ where

$$A_i = \{x(m, i_m) \mid m > s, k - m \equiv i \text{ (mod 2)}\}.$$

Observe that $A_1$ dominates $\{a_0, \ldots, a_{r-1}\}$ and $A_0$ is dominated by this set. Let $A_1 = \{y_1, \ldots, y_o\}$. Observe that $a \leq \frac{1}{2}k$. Let $y_1' \rightarrow y_2' \rightarrow \cdots \rightarrow y_k'$ be a Hamiltonian path in the tournament spanned by $A_1 \cup \{x(k, 1), \ldots, x(k, i_k - a)\}$. By our assumption this set dominates $\{a_0, \ldots, a_{r-1}\}$. The continuation of (1) given by

$$x(1, 1) \cdots x(k - 1, i_{k-1} - 1)a_{k-s-1}y_1' \cdots y_k' \quad (2)$$

is easily seen to realize $B_1 \cdots B_k$.

Let $A_0 = \{z_1, \ldots, z_w\}$. $A_0$ is dominated by $\{a_0, \ldots, a_r\}$. Let $z_1 \rightarrow z_2 \rightarrow \cdots \rightarrow z_w$. The following continuation of the sequence (2) is a realization of $\sigma_{n-1}$ in $T_n$:

$$x(1, 1) \cdots x(k - 1, i_{k-1} - 1)a_{k-s-1}y_1' \cdots y_k'a_{k-s}x(k, i_k) \cdots x(k, i_k - a + 1)a_{k-s+1} \cdots a_{r-1}z_1 \cdots z_w. \quad (3)$$

The sequence (3) is a realization since $s \geq 1$ and $r - 1 \geq k + 1$. If $l < i_s$, the construction of the realization of $\sigma_{n-1}$ is very similar to the above. We sketch it. First the realization of $B_1 \cdots B_{k-1}$ will be

$$x(1, 1) \cdots x(s, l)a_0x(s, l + 1) \cdots x(s, i_s - 1)a_1x(s + 1, 1) \cdots x(s + t, i_{s+t} - 1)a_{i_s}x(s + t + 1, 1) \cdots x(k - 1, i_{k-1} - 1). \quad (4)$$

Define the sets $A_0, A_1$ as before, and it is easily checked that the continuation of (4) as was done in (2) and (3) yields a realization of $\sigma_{n-1}$. (If $s = k$, then

$$x(1, 1) \cdots x(k, 1) \cdots x(i_k, l)a_0x(k, l + 1) \cdots x(k, i_k - 1)a_1 \cdots a_{r-1}x(k, i_k)$$
is a realization of $\sigma_{n-1}$. This construction obviously can be used to establish Lemma 1.)

We may therefore assume that all vertices are properly oriented. The sequence:

$$a_0 x(1, 1) \cdots x(1, i_1-1) a_1 x(2, 1) \cdots x(j, i_j-1) a_j x(j+1, 1) \cdots x(k-1, i_{k-1}-1)$$

(5)

is a realization of $B_1 \cdots B_{k-1}$. Construct the sets $A_0$, $A_1$ as before. The continuation of (5) given by

$$a_0 x(1, 1) \cdots x(k-1, i_{k-1}) a_{k-1} y'_1 \cdots y'_q a_k x(k, i_k) \cdots x(k, i_k - a + 1) a_{k-1} \cdots a_{r-1} z_1 \cdots z_w$$

(6)

is obviously a realization of $\sigma_{n-1}$. Observe that this construction is possible since $r-1 \geq k + 1$.

Finally, if $T_n$ does not admit an $r$-cycle, then $T_n$ is not strong. Hence there exists a set $A$, with less than $r$ vertices that dominates all the other vertices. Let $A = \{a_1 \cdots a_r\}$ where $a_1 \rightarrow a_2 \rightarrow \cdots \rightarrow a$. Let $b_1 \cdots b_m$ be a realization of $B_1 \cdots B_k$ in the subtournament spanned by these vertices. Such a realization exists by the induction hypothesis. Let $c_1, \ldots, c_l$ be the remaining vertices and let $c'_0, \ldots, c'_l$ be a directed path in the subtournament $T_n(b_m, c_1, \ldots, c_l)$. Then

$$b_1 \cdots b_{m-1} a_1 \cdots a_c c'_0 \cdots c'$$

is easily seen to be a realization of $\sigma_{n-1}$.

References