Spectral algebras for strong equational classes of partial algebras

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Abstract. We present here a construction of an object which can be used as a free object for partial algebras and strong equations. Spectral algebras are useful in finding an algebraic characterization of classes of partial algebras definable by a set of strong equations.

1. Introduction

As it is well known, the generalization of universal-algebraic notions to partial algebras yields different kinds of homomorphisms, subalgebras, validity of equations etc. Three types of equations: existential, weak and strong, have been investigated by many algebraists with interesting results on algebraic characterization of equationally definable classes of partial algebras.

One of the most significant papers in this area of research is the work by H. Andréka and I. Németi [1] where the category-theoretical point of view is applied to the problem of HSP-characterization. In effect, the authors have given an algebraic description of equational classes definable by many different kinds of equations and generalized equations (i.e., equational implications). On the other hand, they have introduced tools which turned out to be very useful for searching “good” characterizations of partial varieties, e.g. the universal solution, a free object in some sense. However, the results given in [1] do not include the strong equation case, which is our subject of research. For an example see Section 3.

There are other interesting results concerning partial varieties. P. Burmeister [4] characterized existential varieties i.e. classes of partial algebras definable by

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existential equations. A restricted algebraic characterization of weak varieties was given by L. Rudak in [11] and previously by H. Höft [8]. Höft’s characterization was not purely algebraic, though. Note that H. Höft and J. Slomiński [14] used the notions of strong equation and equation for existential and strong equation in P. Burmeister’s sense, respectively [3].

In the sixties, J. Slomiński [14], when dealing with extensions of partial algebras (of quasi-algebras, in his terminology ), has obtained interesting results on weak, strong, and existential equations, as well as some algebraic constructions corresponding to them. There have been no further results concerning algebraic characterization of strong equational classes of partial algebras since then.

The results presented in this paper form part of our research for a Birkhoff-like theorem for classes of partial algebras definable by sets of strong equations. The main tool in Birkhoff’s proof for the case of total algebras is the notion of a free algebra. We present here a construction of an algebra (called spectral) which is ”almost free” for a class of partial algebras defined by strong equations. The notion of a spectral algebra for regular strong equations was introduced in [15], which contains a complete algebraic characterization of strong regular varieties and also a completeness theorem for the strong regular equational logic. In the second section we give some useful definitions and theorems. The third contains the definition and the construction of a spectral algebra, and a precise explanation of why our results cannot be derived as conclusions from [1] of H. Andréka and I. Némethi. In the last section we show how the classical Birkhoff theorem (for total algebras) and an algebraic characterization of regular varieties (in the total case) can be deduced as special cases of our results on spectral algebras.

We hope the notion of a spectral algebra will be helpful for a complete algebraic characterization of strong equational classes of partial algebras.

2. Preliminaries

We recall some basic notions and facts concerning partial algebras, which will be useful in the sequel. All notions undefined here can be found e.g. in [3] or [15].

A signature is a pair \( \langle F, n \rangle \) where \( F \) is a set and \( n: F \rightarrow \omega \) is the arity function. A pair \( \langle A, (f^A)_{f \in F} \rangle \) is a partial algebra of signature \( \langle F, n \rangle \) iff for each \( f \in F, f^A \) is an \( n(f) \)-ary partial operation in the set \( A \) (with a domain denoted by \( \text{dom } f^A \)).

We assume that a signature \( \langle F, n \rangle \) is fixed, thus in the sequel ”algebra” will be used in the sense of ”partial algebra of signature \( \langle F, n \rangle \) ”. \( \_A \) usually denotes an algebra the carrier set of which is \( A \). A class of algebras is always a nonempty class of partial algebras.
An equivalence relation Θ on \( A \) is a *congruence* in an algebra \( A \) iff for any \( f \in F \) and for all \( a_1, \ldots, a_{n(f)}, b_1, \ldots, b_{n(f)} \in A \), the conditions

\[
(a_i, b_i) \in \Theta \quad \text{for} \quad i = 1, \ldots, n(f) \quad \text{and} \quad (a_1, \ldots, a_{n(f)}), (b_1, \ldots, b_{n(f)}) \in \text{dom } f^A
\]
imply \((f^A(a_1, \ldots, a_{n(f)})), (f^A(b_1, \ldots, b_{n(f)})) \in \Theta\). A congruence is *closed* iff

\[
(a_i, b_i) \in \Theta \quad \text{for} \quad i = 1, \ldots, n(f) \quad \text{and} \quad (a_1, \ldots, a_{n(f)}) \in \text{dom } f^A
\]
imply \((b_1, \ldots, b_{n(f)}) \in \text{dom } f^A\).

A function \( h: A \rightarrow B \) is called a *homomorphism* from an algebra \( A \) into \( B \) (notation: \( h: A \rightarrow B \)) iff for all \( f \in F \), if \( a \in \text{dom } f^A \) then \( h \circ a \in \text{dom } f^B \) and then \( h(f^A(a)) = f^B(h \circ a) \).

A homomorphism \( h: A \rightarrow B \) is *closed* iff for all \( f \in F \)

\[
\text{if } h \circ a \in \text{dom } f^B \text{ then } a \in \text{dom } f^A.
\]

Every homomorphism \( h: A \rightarrow B \) induces a congruence \( \ker h \) in \( A \) (the *kernel* of \( h \)). The kernel of a closed homomorphism is a closed congruence.

We say that an algebra \( A \) is *total* iff the domain \( \text{dom } f^A \) of each operation \( f^A \) is the whole set \( A^{n(f)} \). \( T_F(X) \) will denote the usual total term algebra of signature \( F \), generated by a set \( X \). \( \text{Var}(p) \) denotes the set of variables of a term \( p \).

An algebra \( B = (B, (f^B)_{f \in F}) \) is an *initial segment* in \( A \) iff

(i) \( B \) is a relative subalgebra of \( A \)

(ii) if \( a \in \text{dom } f^A \) and \( f^A(a) \in B \) then \( a \in B^{n(f)} \).

Recall that: A pair of terms \((p, q) \in T^2_F(X)\) is a *strong equation* in an algebra \( A \) \( (A \models p \approx q) \) iff for any tuple \( a \) in \( A \) (of suitable length)

(i) \( a \in \text{dom } p^A \) iff \( a \in \text{dom } q^A \)

(ii) if \( a \in \text{dom } p^A \) then \( p^A(a) = q^A(a) \),

where for a term \( t \), \( t^A \) represents the term function induced by \( t \) in \( A \).

### 3. Spectral algebra and its construction

We first quote a theorem due to J. Schmidt, which is basic for our construction.
Theorem 3.1. [13] Let $A$ be an algebra. For any function $v: X \to A$ there exists a unique closed homomorphism $\tilde{v}$ of an initial segment $\text{dom}\, \tilde{v}$ of $T_F(X)$ into $A$ such that $\tilde{v} \mid X = v$.

Observe that using the homomorphism of the theorem, we get an equivalent definition of a strong equation in $A$: $(p, q)$ is a strong equation in an algebra $A$ iff for any valuation $v: X \to A$

(i) $p \in \text{dom}\, \tilde{v}$ iff $q \in \text{dom}\, \tilde{v}$

(ii) if $p \in \text{dom}\, \tilde{v}$ then $\tilde{v}(p) = \tilde{v}(q)$.

Theorem 3.1 suggests the following generalizing definition:

Definition 3.2. [15] An algebra $S$ is spectral over a set $Y$ for a class $K$ of partial algebras iff $S$ is total and for every algebra $A$ from $K$ and for every function $h: Y \to A$ there exists a unique closed homomorphism from an initial segment of $S$, extending $h$ (we denote it by $\tilde{h}$).

It is easy to see (using Theorem 3.1) that the term algebra over a set $X$ is a spectral algebra over this set, for any class $K$ of algebras. In a rather obvious sense this is the greatest spectral algebra over $X$.

A spectral algebra is a very useful tool in investigation of strong varieties, as we’ll see later, quite similar to a free algebra in the total case. Incidentally note that a spectral algebra need not to be a universal solution in the sense of H. Andréka and I. Némethi [1].

Example. Let $F$ be a signature with one binary operation only and let $K$ be the class of all the discrete algebras of this signature i.e. partial algebras in which the operation has an empty domain. Then $\text{Seq}_K X$, the strong equational theory of $K$ (for symbols not defined here see Section 4) is the set of all equations which are not of the form $x \approx t$, i.e. a variable equal to a term which is not this variable. In this case we have $\mathcal{L} = \text{Mod}(\text{Seq}_K X) = K \cup K'$, where $K' = \{\frac{A}{\bot} : A \in K\} \cup$ one-element total algebras. So, the universal solution for $\mathcal{L}$ over a set $X$ is the discrete algebra (just the set $X$), but the corresponding spectral algebra is total, namely, $X\bot$. Note also that for any operators $H \in \{H_c, H_r, H_w\}$ and $S \in \{S_c, S_r, S_w\}$ we have $\mathcal{L} \neq HSP(K)$ and $H_c S_c P(K) = K \cup$ one-element total algebras.

In the sequel we use $S$ instead of $S_c$ for the subalgebra operator.

Now we give a construction of an algebra which is the least spectral algebra over a fixed set for a given class of algebras. Let $K$ be a class of algebras, $A$ an algebra in $K$, $h: X \to A$ any function and $\tilde{h}: \text{dom}\, \tilde{h} \to A$ the unique closed
extension of \( h \) to an initial segment \( \text{dom} \tilde{h} \) in \( T_F(X) \) [see Theorem 3.1]. Let \( \Theta_h = \ker \tilde{h} \cup \left(T_F(X) \setminus \text{dom} \tilde{h}\right)^2 \), where \( \ker \tilde{h} \) is the closed congruence induced by \( \tilde{h} \) on its domain \( \text{dom} \tilde{h} \) (see [14]). It is easy to check that \( \Theta_h \) is a congruence in \( T_F(X) \).

Let now \( \Theta_A = \bigcap \{ \Theta_h : h: X \rightarrow A \} \) and \( \Theta_K = \bigcap \{ \Theta_A : A \in K \} \). Then the following theorem holds:

**Theorem 3.3.** \( T_F(X)/\Theta_K \) is a spectral algebra over \( X/\Theta_K = \{ [x]_{\Theta_K} : x \in X \} \) for any class \( K \) of algebras.

**Proof.** Quite similar a proof is presented in our paper [15]. We show here only the main facts, on which the proof is based.

Let \( A = (A, \{(f^A)_{f \in F}\}) \) be an algebra. Then we define its one-point extension \( A^\perp \) as follows: \( A^\perp = (A \cup \{\perp\}, \{(f^A)^\perp\}_{f \in F}) \), where \( \perp \notin A \) and

\[
(f^A)^\perp(a) = \begin{cases} f^A(a) & \text{if } a \in \text{dom} f^A \\ \perp & \text{otherwise.} \end{cases}
\]

![Diagram](image)

**Figure 1**

The proof uses the properties based on the diagram of Figure 1, where \( g \) is an arbitrary function, \( G = \text{dom} (g \circ \text{nat}_X) \) and \( \text{nat}_G = \text{nat} | G \) and, where \( \hookrightarrow \) represents the inclusion homomorphism.

(i) Theorem 3.1.

(ii) If we have a closed homomorphism \( h: I \rightarrow A \) of an initial segment \( I \) in an algebra \( S \) then there is a unique homomorphism \( \mathcal{H}: S \rightarrow A^\perp \) such that \( \mathcal{H} | I = h \) and \( \mathcal{H} \) is defined as:

\[
\mathcal{H}(a) = \begin{cases} h(a) & \text{if } a \in I, \\ \perp & \text{otherwise.} \end{cases}
\]
(iii) \( A \) is an initial segment of \( A^\perp \).

(iv) The Diagram Completion Lemma for full homomorphisms (see e.g. [3]).

**Theorem 3.4.** Let \( h: A \rightarrow B \) be a surjective full homomorphism, and \( g: A \rightarrow C \) a homomorphism.

\[
\begin{array}{c}
\text{\[ A \xrightarrow{h} B \]

\text{\[ A \xleftarrow{g} C \]

\text{\[ j \]

\text{\[ B \]}}}
\end{array}
\]

\textbf{Figure 2}

Then \( \ker h \subseteq \ker g \) iff there exists a unique homomorphism \( j: B \rightarrow C \) such that \( j \circ h = g \). Moreover, if \( g \) is closed then \( j \) also is closed too.

(v) A closed homomorphic preimage under \( h: A \rightarrow B \) of an initial segment of \( B \) is an initial segment of \( A \).

(vi) If \( B \) is an initial segment of \( A \) then \( A \setminus B \) is a subalgebra of \( A \).

Theorem 3.4 yields the immediate corollary:

**Corollary 3.5.** For every algebra \( A \) in \( K \) there exists a set \( X \) such that \( A \in H_c\text{In}(T_F(X)/\Theta_K) \), where \( H_c(K) \) and \( \text{In}(K) \) represent the classes of all closed homomorphic images and of all algebras isomorphic to initial segments of algebras in \( K \), respectively.

This is an analog of the property, that each algebra from a class is a homomorphic image of a certain algebra which is free for this class.

**4. Properties**

In this section we give other properties of a spectral algebra which are quite similar to those of a free one in the total case.
Theorem 4.1. \( T_F(X)/\Theta_K \) is the least spectral algebra over \( X/\Theta_K \) for a class \( K \). More precisely, if \( S \) is a spectral algebra over \( X/\Theta_K \) for this class then there exists a surjective homomorphism of \( S \) onto \( T_F(X)/\Theta_K \) extending the identity mapping \( \text{id}: X/\Theta_K \rightarrow X/\Theta_K \).

Proof. We use facts (i)–(vi) from the proof of Theorem 3.3. Let \( S \) be any spectral algebra over \( X/\Theta_K \) for \( K \). We have to show the existence of a surjective homomorphism \( j \) of \( S \) onto \( T_F(X)/\Theta_K \) such that \( j | X/\Theta_K \) is the identity mapping.

Step 1. Let \( g: X \rightarrow X/\Theta_K \) be the natural mapping, i.e. \( g(x) = [x]_{\Theta_K} \). Then there exists exactly one homomorphism \( G: T_F(X) \rightarrow S \) extending \( g \) and it is clearly surjective.

Step 2. Let \( A \in K \) and \( h: X/\Theta_K \rightarrow A \) be any function. Then there exists an initial segment \( I \) in \( S \) and a unique closed extension \( \overline{h} \) of \( h \) to \( I \). We also have a homomorphism \( H: S \rightarrow A^\perp \) – the unique extension of \( \overline{h} \) such that the following diagram commutes:

\[ \begin{array}{ccc}
S & \xrightarrow{H} & S \\
\downarrow & & \downarrow \\
L & = & L \\
\downarrow & & \downarrow \\
X/\Theta_K & \xrightarrow{h} & A \\
\xrightarrow{id_A} & & \xrightarrow{id_A} \\
& & A^\perp \\
\end{array} \]

Figure 3

Step 3. \( h \circ g: X \rightarrow A \) is a valuation, hence by Theorem 3.1, there exists an initial segment \( \text{dom}(h \circ g) \) in \( T_F(X) \) and a closed homomorphism

\[ (h \circ g): \text{dom}(h \circ g) \rightarrow A \]

which extends \( h \circ g \). We can now extend \( (h \circ g) \) to \( J_h: T_F(X) \rightarrow A^\perp \) so that the diagram on Figure 4 commutes and

\[ J_h(t) = \begin{cases} 
\bot & \text{if } t \notin \text{dom}(h \circ g) \\
(h \circ g)(t) & \text{otherwise.}
\end{cases} \]
Step 4. \( \mathcal{H} \circ \mathcal{G} = \mathcal{J}_h \) because they are equal on the set \( X \) of generators of \( T_F(X) \). So we have that the diagram on Figure 5 commutes and the (total) homomorphism theorem states that \( \ker \mathcal{G} \subseteq \ker \mathcal{J}_h \).

From the definition of \( \Theta_K \) we have:

\[
\ker \mathcal{J}_h = \ker (h \circ g)^\sim \cup (T_F(X) \setminus \text{dom} (h \circ g)^\sim)^2 \supseteq \Theta_K
\]

and \( h \) and \( A \) being arbitrary, we get that

\[
\Theta_K = \bigcap \{ \ker \mathcal{J}_h : h: X/\Theta_K \rightarrow A, A \in K \} \supseteq \ker \mathcal{G}.
\]

Step 5. Since \( \Theta_K \supseteq \ker \mathcal{G} \), by the (total) homomorphism theorem there exists a (unique) surjective homomorphism \( j \) such that \( j \circ \mathcal{G} = \text{nat}_{\Theta_K} \).
The following corollary exhibits the connection between initial segments of a spectral algebra for a given class and initial segments of the term algebra.

**Corollary 4.2.** For any \( h: X/\Theta_K \to A \) for \( A \in K \) and \( g, G \) and \( I \) as in Theorem 4.1, \( G(\text{dom}(h \circ g)^\sim) = I \) and \( \text{dom}(h \circ g)^\sim = G^{-1}(I) \), where \( G^{-1} \) denotes the preimage of \( I \), and the following diagram commutes:

\[
\begin{array}{ccc}
\text{dom}(h \circ g)^\sim & \xrightarrow{G} & I \\
\downarrow & & \downarrow \\
(h \circ g)^\sim & \xrightarrow{\bar{h}} & A \\
\end{array}
\]

**Proof.** Consider the diagram on Figure 8.

\[
\begin{array}{ccc}
T_F(X) & \xrightarrow{G} & S \\
\downarrow & & \downarrow \\
\text{dom}(h \circ g)^\sim & \xrightarrow{I_h} & I \\
\downarrow & & \downarrow \\
(h \circ g)^\sim & \xrightarrow{\bar{h}} & A \\
\end{array}
\]

Note that the upper triangle of the diagram is commutative, i.e. \( \mathcal{H} \circ G = J_h \) and \( \ker G \subseteq \ker J_h \).

The homomorphisms \( \mathcal{H} \) and \( J_h \) are determined as follows:

\[
\mathcal{H}(a) = \begin{cases} 
\bar{h}(a) & \text{if } a \in I \\
\bot & \text{otherwise.}
\end{cases}
\]

\[
J_h(t) = \begin{cases} 
(h \circ g)^\sim(t) & \text{if } t \in \text{dom}(h \circ g)^\sim \\
\bot & \text{otherwise.}
\end{cases}
\]

Using these facts, it is easy to see that \( G | \text{dom}(h \circ g)^\sim \) is a closed surjective homomorphism onto \( G(\text{dom}(h \circ g)^\sim) \) with \( G(\text{dom}(h \circ g)^\sim) = I \) and \( G^{-1}(I) = \text{dom}(h \circ g)^\sim \). For more details see [15].
Since
\[ \ker(G | \text{dom}(h \circ g)^*) \subseteq \ker(J_h | \text{dom}(h \circ g)^*) = \ker(h \circ g)^* , \]
the lower triangle of the diagram is also commutative.

The next two theorems show another property of our construction – the correspondence between the least spectral algebra for a class and the strong equational theory of this class, just like for free algebras in the total case.

We write:
\[ K \models p \approx q \text{ iff for every } A \in K, A \models p \approx q \]
and
\[ \text{Seq}_K X = \left\{ (p, q) \in T^2_F(X) : K \models p \approx q \right\} . \]

J. Slomiński [14] has already shown that the set of all strong equations valid in a given algebra (or a class of algebras) forms a congruence on the term algebra.

**Theorem 4.3.** For a given class \( K \) of algebras, a set \( X \) and a pair \( (p, q) \in T^2_F(X) \) the following conditions are equivalent:

(i) \( K \models p \approx q \),
(ii) \( (p, q) \in \Theta_K \),
(iii) for all \( z_1, \ldots, z_n \in X/\Theta_K \),
\[ p_{T^2_F(X)/\Theta_K} (z_1, \ldots, z_n) = q_{T^2_F(X)/\Theta_K} (z_1, \ldots, z_n) \]

**Proof.** (i) is equivalent to (ii): \( K \models p \approx q \) iff for every algebra \( A \in K, A \models p \approx q \) iff for every \( A \in K \) and for every valuation \( v: X \rightarrow A \), \( (p, q) \in \Theta_v \) (by the definition of a strong equation) iff for every \( A \in K, (p, q) \in \Theta_A \) iff \( (p, q) \in \Theta_K \).

(i) \( \rightarrow \) (iii): Take \( z_1, \ldots, z_n \in X/\Theta_K \). Let \( z_1 = [x_1]_{\Theta_K} \ldots z_n = [x_n]_{\Theta_K} \).
Then
\[ p_{T^2_F(X)/\Theta_K} (z_1, \ldots, z_n) = p_{T^2_F(X)/\Theta_K} ([x_1]_{\Theta_K}, \ldots, [x_n]_{\Theta_K}) = [p_{T^2_F(X)}(x_1, \ldots, x_n)]_{\Theta_K} , \]
also
\[ q_{T^2_F(X)/\Theta_K} (z_1, \ldots, z_n) = [q_{T^2_F(X)}(x_1, \ldots, x_n)]_{\Theta_K} . \]
Since \( K \models p \approx q \), then as we have already proved,
\[ \left( p_{T^2_F(X)}(x_1, \ldots, x_n), q_{T^2_F(X)}(x_1, \ldots, x_n) \right) \in \Theta_K \]
(observe that $\Theta_K$ is invariant under any substitution of variables). Thus
\[ p^{T_F(X)/\Theta_K}(z_1, \ldots, z_n) = q^{T_F(X)/\Theta_K}(z_1, \ldots, z_n). \]

(iii) $\rightarrow$ (ii): Let $p = p(x_1, \ldots, x_n)$, $q = q(x_1, \ldots, x_n)$ be terms such that $\text{Var}(p), \text{Var}(q) \in \{x_1, \ldots, x_n\}$. Let $z_1 = [x_1]_{\Theta_K}$, ..., $z_n = [x_n]_{\Theta_K}$; then
\[ [p(x_1, \ldots, x_n)]_{\Theta_K} = p^{T_F(X)/\Theta_K}(z_1, \ldots, z_n) = q^{T_F(X)/\Theta_K}(z_1, \ldots, z_n) = [q(x_1, \ldots, x_n)]_{\Theta_K}. \]

As an easy conclusion from Theorem 4.3 we get the following corollary:

**Corollary 4.4.** $\Theta_K = \text{Seq}_K X$.

Theorem 4.3 asserts that the generators of the least spectral algebra for a class satisfy all the strong equations valid in the class. Nevertheless, in general these equations need not be valid in the algebra itself. The reason is that $\Theta_K$ is not always a fully invariant congruence in $T_F(X)$. In other words, the substitution rule does not preserve strong equations. For a complete system of inference rules for strong equational logic see [10].

**Theorem 4.5.** An algebra $A$ satisfies the strong equational theory of a class $K$ iff for a countable set $X$, $T_F(X)/\Theta_K$ is spectral for $\{A\}$.

**Proof.** If $A$ satisfies $\text{Seq}_K X$ then $A \in \text{Mod}(\text{Seq}_K X) = \{B : B \models \text{Seq}_K X\}$ and we use the following simple fact that $\text{Seq}_K X = \text{Seq}_{\text{Mod}(\text{Seq}_K X)} X$.

To show that $A \models \text{Seq}_K X$, let $(p, q) \in \text{Seq}_K X$; then $[p]_{\Theta_K} = [q]_{\Theta_K}$. Let $(a_1, \ldots, a_n) \in \text{dom} \ p^A$. Then we take a valuation $v : X/\Theta_K \rightarrow A$ such that $v([x_i]_{\Theta_K}) = a_i$ for $i = 1, \ldots, n$ and there is a closed extension $\mathfrak{v}$ of $v$ from an initial segment $I$ of $T_F(X)/\Theta_K$ into $A$ such that $p \in I$ and, obviously, $q \in I$. Hence
\[ p^A(a_1, \ldots, a_n) = \mathfrak{v}([p(x_1, \ldots, x_n)]_{\Theta_K}) = \mathfrak{v}([q(x_1, \ldots, x_n)]_{\Theta_K}) = q^A(a_1, \ldots, a_n). \]

So, using this theorem we have a tool adequate for checking whether a given algebra belongs to an equational class $K$ and to extend the class adding algebras...
which satisfy the equations of the class. Note that the spectral algebra is not generally spectral for itself.

5. Applications

We shall present here some applications of our construction to the case of a fully invariant congruence \( \Theta_K \) in \( T_F(X) \).

**Theorem 5.1.** For any class \( K \), a given set \( X \), and a pair \((p, q)\) \( \in T_F^2(X) \), if \( \Theta_K \) is a fully invariant congruence in \( T_F(X) \) then the following conditions are equivalent:

(i) \( K \models p \approx q \)
(ii) \( (p, q) \in \Theta_K \)
(iii) for any \( z_1, \ldots, z_n \in X/\Theta_K \),

\[ pT_F(X)/\Theta_K(z_1, \ldots, z_n) = qT_F(X)/\Theta_K(z_1, \ldots, z_n) \]

(iv) \( T_F(X)/\Theta_K \models p \approx q \).

**Proof.** By Theorem 4.3 items (i)–(iii) are equivalent. If \( T_F(X)/\Theta_K \models p \approx q \), then \((p, q)\) satisfies (iii). If \( K \models p \approx q \), then \((p, q) \in \Theta_K \) and \( \Theta_K \) is a fully invariant congruence in \( T_F(X) \). Thus by a well known theorem of Birkhoff for total algebras [2], \( T_F(X)/\Theta_K \models p \approx q \).

**Corollary 5.2.** \( T_F(X)/\Theta_K \models \text{Seq}_K X \) iff \( \Theta_K \) is a fully invariant congruence in \( T_F(X) \).

**Proof.** If \( \Theta_K \) is fully invariant, then \( T_F(X)/\Theta_K \models \Theta_K \) and by the above theorem and by Corollary 4.4, \( T_F(X)/\Theta_K \models \text{Seq}_K X \).

If \( T_F(X)/\Theta_K \models \text{Seq}_K X \), then \( \Theta_{T_F(X)/\Theta_K} \supseteq \text{Seq}_K X \) and since \( T_F(X)/\Theta_K \) is a total algebra we get that \( \Theta_{T_F(X)/\Theta_K} \) is fully invariant (see again [2]).

Let now \((p, q) \in \Theta_{T_F(X)/\Theta_K} \), i.e. \( T_F(X)/\Theta_K \models p \approx q \); then \( T_F(X)/\Theta_K \) satisfies \((p, q)\) and hence by Theorem 4.3 and Corollary 4.4, \((p, q) \in \text{Seq}_K X \). Thus \( \text{Seq}_K X = \Theta_{T_F(X)/\Theta_K} \) - a fully invariant congruence in \( T_F(X) \).

In [15] we have shown a construction of a spectral algebra for classes definable by a set of strong regular equations and an application of this construction to the proof of a Birkhoff-like theorem characterizing algebraically these classes.
We say that a pair of terms \((p, q)\) is a *strong regular equation* in an algebra \(A\) iff \(A \models p \approx q\) and \(\text{Var}(p) = \text{Var}(q)\). We shall consider the following algebraic operators on classes:

- \(H_c\) the operator of closed homomorphic images,
- \(In\) the operator of initial segments,
- \(S\) the operator of subalgebras (\(S_c\) in [1]),
- \(P\) the operator of direct products,
- \(\perp\) the operator of iterated ”pins”, where \(\perp(K)\) is defined as follows: \(A^{\perp_0} = A\), \(A^{\perp_{n+1}} = (A^{\perp_n})^{\perp}\) and \(\perp(A) = \{A^{\perp_n} : n \in \mathbb{N}\}\), \(\perp(K) = \{B : B \in \perp(A), A \in K\}\).

We have shown in [15] the following theorems:

**Theorem 5.3.** Let \(X\) be any set and \(K\) be a class of algebras, then the set of all strong regular equations valid in \(K\) is a fully invariant congruence in \(T_F(X)\).

**Theorem 5.4.** A class of algebras \(K\) is definable by a set of strong regular equations (i.e. \(K\) is a strong regular variety) iff \(K = H_c In SP \perp(K)\).

Let us turn to the total world i.e. consider classes of total algebras only. In this world the operator \(H_c\) becomes the usual operator of homomorphic images, \(In\) yields subalgebras only (each total initial segment is a subalgebra), so \(InS\) is the usual operator of subalgebras, \(P\) is the operator of direct products, while \(\perp\) remains unchanged.

Thus we get a new algebraic characterization of regular varieties considered by many algebraists e.g. by J. Płonka [9] and E. Graczyńska [6].

**Corollary 5.5.** A class \(K\) of total algebras is a regular variety iff \(K = HSP \perp(K)\).

The second example shows that the classic Birkhoff theorem can be derived from our results on partial algebras.

Let \(T\) be an operator such that

\[
T(A) = \begin{cases} 
\{A\} & \text{if } A \text{ is total,} \\
\{A^{\perp}, A\} & \text{otherwise.}
\end{cases}
\]

We call it the *operator of totalization* of an algebra \(A\).

Let \(T(K) = \{B : B \in T(A), A \in K\}\). Observe that \(T\) is an algebraic closure operator and the sequence of operators \(H_c In SP T\) is an algebraic closure operator also (comp. [15]). As we have shown in [15], the operators \(H_c, In, S, P\) preserve...
all strong equations; we shall prove here that \( T \) preserves equations of a special kind. An equation is \textit{discrete} in a partial algebra \( \underline{A} \) iff either \( \underline{A} \) is total or \( p \) and \( q \) induce empty operations in \( \underline{A} \). It will be shown below that the totalization operator preserves strong regular equations and discrete equations.

Moreover, these are the only equations preserved by \( T \).

**Theorem 5.6.** Let \( G_A = \{ (p, q) \in \Theta_A : (p, q) \text{ is regular or discrete} \} \). Then

(i) \( \Theta_{T(A)} = G_A \)

(ii) \( G_A \) is the greatest fully invariant congruence included in \( \Theta_A \)

(iii) if \( K \) is a class of algebras, then \( G_K = \bigcap \{ G_A : A \in K \} \) is the greatest fully invariant congruence included in \( \Theta_K \).

**Proof.** (i) If \( A \) is total, then \( T(A) = \{ A \} \) and \( G_A = \Theta_A = \Theta_{T(A)} \). Let \( A \) be a nontotal algebra. If \( (p, q) \) is a discrete equation in \( A \) i.e. \( \text{dom } p_A = \text{dom } q_A = \emptyset \), then for any appropriate tuple \( a \) of elements of \( A^\perp \), \( p_A^\perp(a) = q_A^\perp(a) = \bot \) and hence \( A^\perp \models p \approx q \). If \( (p, q) \) is regular, then \( A^\perp \models p \approx q \) (see [15]). Thus \( G_A \subseteq \Theta_{T(A)} \).

Let now \( T(A) \models p \not\approx q \). If \( (p, q) \) is not regular nor discrete in \( A \) then let us assume that \( \text{Var}(q), \text{Var}(p) \subseteq \{ x_1, \ldots, x_n \} \) and \( x_i \in \text{Var}(p) \setminus \text{Var}(q) \). Because \( q \) is not discrete, there exists \( (a_1, \ldots, a_n) \in A^n \) such that \( (a_1, \ldots, a_n) \in \text{dom } q_A^\perp \). Now substitute \( a_j \) for \( x_j \) for \( j \neq i \) and \( \bot \) for \( x_i \). Then

\[
q_A^\perp(a_1, \ldots, a_n) = q_A(a_1, \ldots, a_n) \in A
\]

and

\[
p_A^\perp(a_1, \ldots, a_{i-1}, \bot, a_{i+1}, \ldots, a_n) = \bot \notin A,
\]

a contradiction.

(ii) \( \Theta_{T(A)} \) is the equational theory of the total algebra from \( T(A) \) (recall that \( A \) is an initial segment of \( A^\perp \) and \( In \) preserves strong equations), hence \( G_A = \Theta_{T(A)} \) is a fully invariant congruence in \( T_F(X) \), the inclusion \( G_A \subseteq \Theta_A \) is obvious. We have to show that \( G_A \) is the greatest such congruence.

The total case is clear. Let \( A \) be nontotal and \( (p, q) \in \Theta_A \setminus G_A \). This means that \( (p, q) \) is neither regular nor discrete. Then, as in (i), there exists \( (a_1, \ldots, a_n) \in A^n \) (if \( p, q \) are \( n \)-ary) such that \( (a_1, \ldots, a_n) \in \text{dom } q_A^\perp \). Assume that \( x_i \in \text{Var}(p) \setminus \text{Var}(q) \) and, since \( A \) is nontotal, let \( t \) be an \( m \)-ary term such that there is \( (b_1, \ldots, b_m) \in A^m \) such that \( (b_1, \ldots, b_m) \notin \text{dom } t_A \).

If \( (p, q) \) is in a fully invariant congruence \( \Theta \) then

\[
(p(x_1, \ldots, x_{i-1}, t(y_1, \ldots, y_m), x_{i+1}, \ldots, x_n), q(x_1, \ldots, x_n)) \in \Theta.
\]
But it is not in $\Theta_A$ because we can substitute $a_j$ for $x_j$ for $j \neq i$ and $b_i$ for $y_i$.

(iii) $G_K$ is a fully invariant congruence included in $\Theta_K$ as an intersection of fully invariant congruences $G_A$ included in $\Theta_A$, for every algebra $A$ in $K$. It is the greatest one because if $\Theta$ is a fully invariant congruence such that $\Theta \subseteq \Theta_K$ then for every $A$ in $K$, $\Theta \subseteq \Theta_A$ and $G_A \subseteq \Theta_A$. But $G_A$ is the greatest fully invariant congruence in $\Theta_A$. Hence $\Theta \subseteq G_A$ for every $A$ in $K$ and consequently $\Theta \subseteq G_K$.

**Corollary 5.7.** For any class $K$ of algebras,

$$G_K(X) = \text{Seq}_{T(K)} X = \text{Seq}_{HcInSP(T(K))} X.$$ 

**Corollary 5.8.** For any class $K$ of algebras, the algebra $T_F(X)/G_K(X)$ is in $HcInSP(K)$ and it is spectral for this class.

Using these facts we can prove the theorem characterizing classes definable by sets of strong regular and discrete equations.

**Theorem 5.9.** A class $K$ is definable by a set of strong regular equations and discrete equations iff $K = HcInSP(K)$.

If we restrict our consideration to the total world we get the classic Birkhoff theorem:

**Corollary 5.10.** A class $K$ is equationally definable iff $K = HSP(K)$.

**Proof.** Discrete equations in a total algebra are simply equations. The operator $T$ does not change anything. The remaining operators are described before Corollary 5.2.

**References**


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