Identification of a Class of Nonlinear Systems Under Stationary Non-Gaussian Excitation
Jonathon C. Ralston, Member, IEEE, Abdelhak M. Zoubir, Member, IEEE, and Boualem Boashash, Senior Member, IEEE

Abstract—This paper provides new solutions to the nonlinear system identification problem when the input to the system is a stationary non-Gaussian process. We propose the use of a model called the Hammerstein series, which leads to significant reductions in both the computational requirements and the mathematical tractability of the nonlinear system identification problem. We show that unlike the Volterra series, one can obtain closed-form expressions for the Hammerstein series kernels and the quadratic coherence function in the non-Gaussian case. Estimation of the kernels and quadratic coherence function is discussed. A comparison with a nonlinear system identification approach that uses the Volterra series is provided. An automotive engineering application illustrates the usefulness of the proposed method.

I. INTRODUCTION

SYSTEM identification is concerned with characterizing an unknown system using observations of the system’s input and output signals. Traditional approaches to the system identification problem have predominantly assumed linear models. However, the use of nonlinear models is often more appropriate and leads to a more accurate characterization of the system. This is evidenced by their widespread application in fields including biomedicine, prediction, control, equalization, and detection [1]–[5].

The Volterra series is a model that is often used for characterizing nonlinear systems, where it is assumed that the relationship between a discrete time random input \( X(t) \) and output \( Y(t) \), \( t = 0, \pm 1, \cdots \), of a time-invariant nonlinear system may be expressed by a series of the form

\[
Y(t) = \sum_{\tau_1 = -\infty}^{\infty} h_1(\tau_1)X(t - \tau_1) + \sum_{\tau_1 = -\infty}^{\infty} \sum_{\tau_2 = -\infty}^{\infty} h_2(\tau_1, \tau_2)X(t - \tau_1)X(t - \tau_2) + \cdots
\]

where the set of functions \( \{ h_n(\tau_1, \cdots, \tau_n) \} \), \( n = 1, 2, \cdots \), \( \tau_i \in \mathbb{Z}, i = 1, 2, \cdots, n \), are the time-invariant Volterra kernels [6]. The Volterra kernels characterize the linear, quadratic, and higher order interactions of the system. Once the Volterra kernels are known, they can be used to obtain the system output for a given arbitrary input.

Closed-form expressions for the Volterra kernels have been obtained for the case where the input is a stationary Gaussian process [7], [8]. However, in many practical system identification problems, the input cannot be assumed to be a Gaussian process. This is problematic because it is extremely difficult to find closed-form expressions for the Volterra kernels in the non-Gaussian case, even for weakly nonlinear systems [4], [9]. As a result, we are often forced to resort to regression-based [4], [10] or iterative [11] approaches in order to find a solution. These methods have high computational requirements. There is also a general problem associated with the use of the Volterra series in practice due to the large number of coefficients required in modeling nonlinear systems. We attempt to address these problems by developing a simple identification procedure that is capable of characterizing nonlinear systems with minimal computational requirements when the input is a non-Gaussian process.

The outline of the paper is as follows. In Section II, we review the underlying analytical difficulties associated with nonlinear system identification in the non-Gaussian case. We briefly survey other nonlinear and non-Gaussian system identification strategies and motivate the use of a particular nonlinear model called the Hammerstein series, which is considered in detail in Section III. Estimation issues are discussed in Section IV. Examples of the identification procedure with Gaussian and non-Gaussian inputs are presented in Section V. An application to a problem in automotive engineering is presented in Section VI to demonstrate the usefulness of the approach. Section VII contains our conclusion.

II. THE NONLINEAR AND NON-GAUSSIAN SYSTEM IDENTIFICATION PROBLEM

A. Assumptions and Notation

We first establish the notation used throughout this paper and indicate the assumptions necessary for analysis. Consider a time-invariant nonlinear system represented by an \( n \)th-order Volterra series as in (1). The generalized transfer functions [6] are defined as the multidimensional Fourier transforms of the Volterra kernels \( h_n(\tau_1, \cdots, \tau_n) \), \( n = 1, 2, \cdots \)

\[
H_n(\omega_1, \cdots, \omega_n) = \sum_{\tau_1 = -\infty}^{\infty} \cdots \sum_{\tau_n = -\infty}^{\infty} h_n(\tau_1, \cdots, \tau_n) e^{-j(\omega_1 \tau_1 + \cdots + \omega_n \tau_n)}
\]
that exist, provided that the Volterra kernels are absolutely summable over all arguments [12]. We introduce Cramér’s spectral representation [13] as it facilitates the derivation of the solution in the frequency domain. We assume that $X(t)$ is a real, zero-mean stationary non-Gaussian process admitting Cramér’s spectral representation

$$X(t) = \int_{-\pi}^{\pi} e^{j\omega t} dZ_X(\omega)$$

(2)

where $Z_X(\omega)$ is a complex-valued stochastic process with orthogonal increments. The $\eta$th-order polyspectrum of $X(t)$ is defined as [13]

$$\text{cum} \{dZ_X(\omega_1), \ldots, dZ_X(\omega_n)\} = C^\eta_X(\omega_1, \ldots, \omega_{n-1}) \eta(\omega_1 + \cdots + \omega_n) d\omega_1 \cdots d\omega_n$$

(3)

where $\text{cum}$ is the cumulant operator, and $\eta(\omega)$ is Dirac’s comb

$$\eta(\omega) = \sum_{k=-\infty}^{\infty} \delta(\omega + 2\pi k)$$

and $\delta(\omega)$ is Dirac’s delta function. The relationship between the $\eta$th-order polyspectrum and the $n$th-order cumulant sequence is given by the Fourier transform [13]

$$C^\eta_X(\omega_1, \ldots, \omega_n) = \left(\frac{1}{2\pi}\right)^n \sum_{\tau_1=-\infty}^{\infty} \cdots \sum_{\tau_n=-\infty}^{\infty} e^{j(\omega_1 \tau_1 + \cdots + \omega_n \tau_n)}$$

where the $\eta$th-order cumulant sequence is given by

$$\text{cum} \{X(t), X(t - \tau_1), \ldots, X(t - \tau_n)\} = C^\eta_X(\tau_1, \ldots, \tau_n).$$

(4)

A sufficient condition for the existence of the polyspectrum is that the cumulant sequence is absolutely summable over all arguments. If $X(t)$ is a stationary process and the system is time-invariant and stable, $Y(t)$ is also stationary. The cross polyspectra and cross cumulant sequence of $X(t)$ and $Y(t)$ are defined similarly to (3) and (4), respectively. Later, we will consider the quadratic system identification problem in detail and require that the cross cumulant sequences of $X(t)$ and $Y(t)$ are bounded up to fourth order only.

B. The Analytical Problem

We begin by examining the general relationships that exist between the generalized transfer functions and the polynomials of non-Gaussian processes. Given (2), we can rewrite (1) in the frequency domain as

$$dZ_Y(\omega) = H_1(\omega)dZ_X(\omega) + \int_{-\pi}^{\pi} H_2(\omega, \lambda) dZ_X(\omega + \lambda) dZ_X(\lambda)$$

$$+ \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} H_3(\lambda_1, \lambda_2) dZ_X(\lambda_1 - \lambda_2) dZ_X(\omega - \lambda_2)$$

$$\times dZ_X(\lambda_1) dZ_X(\lambda_2 - \lambda_1) dZ_X(\omega - \lambda_2) + \cdots$$

(5)

The usual procedure to solve for the generalized transfer functions is to evaluate cross polyspectra of (5) and attempt to solve the resulting set of equations for a given nonlinear order [5], [7]. In the general case, this leads to

$$C_{YX}(\omega) = H_1(\omega)C_{XX}(\omega)$$

$$+ \int_{-\pi}^{\pi} H_2(\omega - \lambda, \lambda) C_{XX}(\omega - \lambda, \lambda) d\lambda$$

$$+ \cdots$$

$$C_{XX}(\omega_1, \omega_2) = H_1(\omega_1 + \omega_2) C_{XX}(\omega_1, \omega_2)$$

$$+ \int_{-\pi}^{\pi} H_2(\omega_1 + \omega_2 - \lambda, \lambda)$$

$$\times C_{XX}(\omega_1, \omega_2) d\lambda$$

$$+ 2H_2(\omega_1, \omega_2) C_{XX}(\omega_1) C_{XX}(\omega_2) + \cdots$$

which highlight the underlying problem associated with nonlinear system identification in the non-Gaussian case. We see that the generalized transfer functions cannot be easily separated from the integral–polyspectral terms, and as a result, closed-form solutions cannot be obtained in the general case. Several different approaches have been proposed in an attempt to find a solution to this problem, which we now briefly review.

1) Model Assumptions:

1) Memoryless systems: The Volterra kernels of a memoryless nonlinear system are constants $h_\eta(\tau_1, \ldots, \tau_n) \equiv h_\eta$, which is a polynomial function [19]. Closed-form expressions can be found when the input is a stationary non-Gaussian process, but the model is very restrictive.

2) Homogeneous $n$th-order system: In this case, only a single term of the Volterra series expansion is considered. A closed-form expression for the $n$th-order generalized transfer function exists for a Gaussian input process [6]. While the result is theoretically valid, the model does not find broad applicability.

3) Reparameterized Volterra series: A Volterra series can be reparameterised and expressed in the form of a general linear model. A regression is subsequently solved in the discrete time [4] or frequency domain [10]. The main advantage of this approach is that the input can be non-Gaussian. However, the approach generally has high computational requirements. Additionally, the matrix inversion involved with this procedure may introduce severe numerical problems, particularly if a large number of parameters need to be estimated [14].

2) Input Assumptions:

1) Deterministic input signals: These include impulse, multistep, and sinusoidal inputs (e.g., [15]). This is the easiest approach but, perhaps, the least general since the excitation cannot always be controlled in a manner that suits the experimenter.

2) Gaussian input process: This is the most common assumption for nonlinear system identification. The generalized transfer functions of a quadratically nonlinear Volterra model can be found in closed form for a Gaussian input [7], [8]. Generalizations have been made for
an $n$th-order Volterra series with a Gaussian input [16].

The estimation of various block-based approximations have also been considered under the Gaussian assumption [2], [11]. However, the Gaussian input assumption cannot always be used in a practical identification scenario—despite the analytic simplifications it imparts to the identification problem.

3) Special non-Gaussian input process: A special class of non-Gaussian input process is considered in [5] and [9] in an attempt to more fully characterize the non-Gaussian nature of the input. Optimal closed-form expressions for the first- and second-order Volterra kernels have been derived under the assumption that the input process has a zero fourth-order cumulant. However, the special condition placed on the input process may prove restrictive in practice.

The various problems and limitations associated with the approaches listed above motivate an alternative approach. In the following subsection, we propose the use of a particular model structure that enables closed-form expressions to be determined when the input is a stationary non-Gaussian process. This approach will prove to be both simple in formulation and computationally attractive when compared with existing methods.

C. An Alternative Model: The Hammerstein Series

Consider representing a time-invariant nonlinear system using the model

$$Y(t) = \sum_{\tau=-\infty}^{\infty} g_1(\tau)X(t-\tau) + \sum_{\tau=-\infty}^{\infty} g_2(\tau)X(t-\tau)^2 + \sum_{\tau=-\infty}^{\infty} g_3(\tau)X(t-\tau)^3 + \cdots \tag{6}$$

where the functions $\{g_n(\tau)\}, n = 1, 2, \cdots$, characterize the linear, quadratic, and higher order responses of the system. We call the model in (6) the Hammerstein series (cf. Hammerstein model) by analogy with the Volterra kernels and call the functions $\{g_n(\tau)\}, n = 1, 2, \cdots$, the Hammerstein kernels.

The Hammerstein model has been widely applied in fields including control, signal processing, and engineering [17], [3], [11], [18]. However, the Hammerstein series as a nonlinear model has not been previously formalized in this way (e.g., [19]). In addition, the utility of the Hammerstein series topology in the non-Gaussian input case has not been recognized. The Hammerstein series is more general than the Hammerstein model in terms of modeling a nonlinear system’s dynamic (memory) as each nonlinear term has a separate function associated with it to characterize the dynamics. A schematic of an $n$th-order Hammerstein series is shown in Fig. 1.

In order to determine the relationship between the Hammerstein series and the Volterra series, we equate like nonlinear orders between (6) and (1). For the $n$th-order term, we find that

$$h_n(\tau_1, \cdots, \tau_n) \equiv g_n(\tau_1)Y(\tau_1 - \tau_2) \cdots \delta(\tau_1 - \tau_n) \tag{7}$$

where $\delta(\tau)$ in the above equation is Kronecker’s delta, i.e.

$$\delta(\tau) = \begin{cases} 1, & \tau = 0, \\ 0, & \tau \neq 0 \end{cases}$$

and thus, $g_n(\tau)$ can be seen to be equivalent to the diagonal slice of the $n$th-order Volterra kernel. By analogy with the generalized transfer functions, we consider the Fourier transform of the Hammerstein kernels (assuming the kernels are absolutely summable)

$$G_n(\omega) = \sum_{\tau=-\infty}^{\infty} g_n(\omega)e^{-j\omega\tau}$$

for $n = 0, 1, \cdots$, which we call the Hammerstein transfer functions. The relationship between the $n$th-order generalized transfer function and the $n$th-order Hammerstein transfer function is

$$H_n(\omega_1, \cdots, \omega_n) \equiv G_n(\omega_1 + \cdots + \omega_n), \tag{8}$$

The summing of the frequency variables in (8) will be shown to be the key for determining closed-form expressions in the non-Gaussian case. Using the relationship in (8), we can express (6) in the frequency domain as (see Appendix A)

$$dZ_Y(\omega) = G_1(\omega)dZ_X(\omega) + G_2(\omega)\int_{-\pi}^{\pi}\! dZ_X(\omega - \omega_2) dZ_X(\omega_2) + G_3(\omega)\int_{-\pi}^{\pi}\! dZ_X(\omega - \omega_2) dZ_X(\omega_2 - \omega_3) + \cdots$$

Note in the above equation that the Hammerstein transfer functions have separated from the integral-cumulant expressions as compared with the generalized transfer functions in (5). This separability result does not occur for the generalized transfer functions [4], [5], nor for any other nontrivial subset of the Volterra series in the non-Gaussian case. Note that in using this model, we do not solve for the Volterra kernels but rather for the Hammerstein kernels. Use of the Hammerstein series leads to significant reductions in computational requirements, as well as in the mathematical tractability of the nonlinear system identification problem.
III. QUADRATIC SYSTEM IDENTIFICATION

We now consider the specific case of a quadratically nonlinear Hammerstein series. The quadratic case serves to illustrate the general concepts associated with the approach, which can be readily formulated for higher order nonlinear systems. A quadratic version of the model in (6) is given by

\[
Y(t) = \sum_{\tau=-\infty}^{\infty} g_1(\tau)X(t-\tau) + \sum_{\tau=-\infty}^{\infty} g_2(\tau)X(t-\tau)^2 + N(t)
\]

where \(X(t)\) and \(Y(t)\) denote the stationary input and output signals, respectively. We have allowed for an additive zero-mean stationary noise process \(N(t)\), and we assume that \(X(t)\) and \(N(t)\) are independent. The linear and quadratic interactions of the system are characterized by \(g_1(\tau)\) and \(g_2(\tau)\), respectively. A schematic of the quadratic Hammerstein series is shown in Fig. 2.

Forming the first-order cross cumulant sequence of \(Y(t)\) given in (9) with \(X(t-v), v \in \mathbb{Z}\) leads to

\[
\text{cum} \{Y(t), X(t-v)\} = \sum_{\tau=-\infty}^{\infty} g_1(\tau)\text{cum} \{X(t-\tau), X(t-v)\} + \sum_{\tau=-\infty}^{\infty} g_2(\tau)\text{cum} \{X(t-\tau)^2, X(t-v)\}
\]

\[
c_{YX}(v) = \sum_{\tau=-\infty}^{\infty} g_1(\tau)c_{XX}(v-\tau) + \sum_{\tau=-\infty}^{\infty} g_2(\tau)c_{XX}(0,v-\tau), \tag{10}
\]

Fourier transforming (10) with respect to \(v\) leads to

\[
C_{YX}(\omega) = G_1(\omega)C_{XX}(\omega) + G_2(\omega)C_{XX}(\omega)C_{XX}(\omega) \quad \tag{11}
\]

where we again note that \(G_2(\omega)\) has separated from the integral term [20]. This separation does not occur for the generalized transfer functions of the Volterra series.

We now compute the second-order cross cumulant sequence. It has been previously shown [20], [21] that it is sufficient to only consider a particular slice of the second-order cross cumulant sequence when deriving a closed-form solution for the Hammerstein kernels. Therefore, we have

\[
\text{cum} \{Y(t), X(t-v)X(t-v)\} = \sum_{\tau=-\infty}^{\infty} g_1(\tau)\text{cum} \{X(t-\tau), X(t-v)X(t-v)\} + \sum_{\tau=-\infty}^{\infty} g_2(\tau)\text{cum} \{X(t-\tau)X(t-\tau), X(t-v)\}
\]

\[
+ \sum_{\tau=-\infty}^{\infty} g_2(\tau)c_{XX}(0,v-\tau) \quad \tag{12}
\]

Simultaneously solving (11) and (12) leads to a closed-form expression for \(G_1(\omega)\) and \(G_2(\omega)\)

\[
G_1(\omega) = \frac{\int_{-\pi}^{\pi} C_{YX}(\omega)\,d\lambda + \int_{-\pi}^{\pi} C_{XX}(\omega)\,d\lambda}{\int_{-\pi}^{\pi} C_{XX}(\omega-\lambda, \lambda)\,d\lambda + \int_{-\pi}^{\pi} C_{XX}(\omega+\lambda, \lambda)\,d\lambda}
\]

and

\[
G_2(\omega) = \frac{-\int_{-\pi}^{\pi} C_{YX}(\omega)\,d\lambda}{\int_{-\pi}^{\pi} C_{XX}(\omega-\lambda, \lambda)\,d\lambda + \int_{-\pi}^{\pi} C_{XX}(\omega+\lambda, \lambda)\,d\lambda}
\]

provided that the denominators are nonzero. These solutions can be shown to be optimal in a mean-square sense (see Appendix B).
A. Simplified Solutions

Although the above solutions for \( G_1(\omega) \) and \( G_2(\omega) \) are complete, they are complicated. We can adopt an alternative notation to simplify the solution. The Hammerstein series suggests the consideration of special slices [20], [22], [23] of the cumulant sequences. Slices of the cumulant sequences have been previously considered in our system identification problems [24]–[26]. We only require one particular slice of the cumulant sequence, namely,

\[
c_{X^mX^n}(\tau) = \text{cum} \{X(t)^m, X(t-\tau)^n\} = \text{cum} \{X(t) \cdots X(t), X(t-\tau) \cdots X(t-\tau)\}
\]

for \( m, n \geq 1 \). Since \( X(t) \) is stationary, the sliced cumulant sequence is a function of \( \tau \) only. Note that (13) is the cumulant of products of random variables, which can be expressed in terms of \((m+n)\)th and lower order cumulant sequences [13].

The polyspectrum corresponding to the Fourier transform of \( c_{X^mX^n}(\tau) \) with respect to \( \tau \) is given by

\[
C_{X^mX^n}(\omega) = \sum_{\tau=-\infty}^{\infty} c_{X^mX^n}(\tau)e^{-j\omega\tau}
\]

which we refer to as the integrated polyspectrum. The notion of an “integrated” polyspectrum arises from the fact that the 1-D Fourier transform of the sliced cumulant sequence can be expressed in terms of integrated versions of the conventional polyspectrum [22].

For the case of a quadratic Hammerstein series, the slices of the third- and fourth-order cumulant sequences and their corresponding polyspectrum are of special interest. For the third- and fourth-order cases, it can be shown that the relationship between the integrated polyspectrum and the conventional polyspectrum is given by

\[
C_{X^2X}(\omega) = \int_{-\pi}^{\pi} C_{XXX}(\omega, \lambda) d\lambda
\]

\[
C_{XX^2}(\omega) = \int_{-\pi}^{\pi} C_{XXX}(\omega, -\lambda, \lambda) d\lambda d\lambda
\]

and

\[
C_{X^2X^2}(\omega) = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} C_{XXXX}(\omega, -\lambda_1, \lambda_1, -\lambda_2, \lambda_2) d\lambda_1 d\lambda_2
\]

\[
+ 2 \int_{-\pi}^{\pi} C_{XX}(\omega, -\lambda) C_{XX}(\lambda) d\lambda
\]

Note that \( C_{X^2X}(\omega) = C_{XX^2}(\omega)^* \) by symmetry, where * denotes the complex conjugate. Using the integrated polspectrum notation as in (14), we have the equivalent representations for (11) and (12), respectively

\[
C_{YX}(\omega) = G_1(\omega)C_{XX}(\omega) + G_2(\omega)C_{X^2X}(\omega)
\]

and

\[
C_{YX^2}(\omega) = G_1(\omega)C_{XX^2}(\omega) + G_2(\omega)C_{X^2X^2}(\omega).
\]

Closed-form solutions for \( G_1(\omega) \) and \( G_2(\omega) \) can be expressed using the simplified notation as

\[
G_1(\omega) = \frac{C_{YX}(\omega)C_{YX}(\omega) - C_{XX}(\omega)C_{YX^2}(\omega)}{C_{XX}(\omega)C_{XX^2}(\omega) - C_{XX}(\omega)C_{X^2X}(\omega)},
\]

\[
G_2(\omega) = \frac{C_{YX}(\omega)C_{YX^2}(\omega) - C_{X^2X}(\omega)C_{YX}(\omega)}{C_{XX}(\omega)C_{XX^2}(\omega) - C_{XX}(\omega)C_{X^2X}(\omega)}.
\]

Thus, we have found closed-form expressions for the linear and quadratic Hammerstein transfer functions in the non-Gaussian input case. Clearly, the use of the integrated polyspectrum notation greatly simplifies the form of the solution. Herein, we use the integrated polspectrum notation.

B. The Quadratic Coherence Function

The coherence function provides a mechanism for validating the chosen model. For a linear time-invariant system, it is well known that the linear coherence function \( L(\omega) \), which is defined as [19]

\[
L(\omega) = \frac{|C_{YY}(\omega)|^2}{C_{XX}(\omega)C_{YY}(\omega)}
\]

indicates the extent to which the input is linearly related to the output at a given frequency \( \omega \). However, the linear coherence function may not adequately describe the extent to which a nonlinear model characterizes a system since the linear coherence function only takes into account linear interactions between the input and output. Therefore, a nonlinear coherence function is of special interest for nonlinear systems.

We use the concept of system coherency [27] to formulate the coherence function for the quadratic Hammerstein series. We define the quadratic coherence function \( Q(\omega) \) as the ratio of the quadratic model output spectral density to the observed output spectral density

\[
Q(\omega) = \frac{C_{YY}(\omega) - C_{NN}(\omega)}{C_{YY}(\omega)}
\]

where \( C_{YY}(\omega) - C_{NN}(\omega) \) represents the model output spectral density. The quadratic coherence function \( Q(\omega) \) is nonnegative and bounded \( 0 \leq Q(\omega) \leq 1 \) since \( C_{NN}(\omega) \geq 0 \) and \( C_{YY}(\omega) \leq C_{YY}(\omega) \). The output spectral density of the quadratic model in (6) can be shown to be

\[
C_{YY}(\omega) = |G_1(\omega)|^2 C_{XX}(\omega) + 2\Re\{G_1(\omega)G_2(\omega)*C_{XX^2}(\omega)\}
\]

\[
+ |G_2(\omega)|^2 C_{X^2X}(\omega) + C_{NN}(\omega)
\]

where \( \Re \) denotes the real part. Since the system is nonlinear and the input is non-Gaussian, the above expression consists of three model terms that correspond to the linear, linear-quadratic, and quadratic interactions as they contribute to the output spectrum \( C_{YY}(\omega) \).

The quadratic coherence function of a Volterra system cannot generally be obtained in closed form when the input is non-Gaussian. In contrast, a closed-form expression for the quadratic coherence function for the Hammerstein series can be found by substituting the solutions for \( G_1(\omega) \) and \( G_2(\omega) \) from (15) into (18). After some manipulations, we obtain

\[
Q(\omega) = \frac{|C_{YY}(\omega)|^2 |C_{XX^2}(\omega)|^2 + 2\Re\{C_{YY}(\omega)C_{XX^2}(\omega)*C_{XX}(\omega)^*\} + |C_{XX}(\omega)|^2 C_{YY}(\omega)}{C_{YY}(\omega)C_{XX}(\omega)C_{XX^2}(\omega) - C_{XX}(\omega)C_{XX^2}(\omega)}
\]
provided that the denominator is nonzero. The importance of the above result lies in the fact that the derived quadratic coherence function is not a function of the model parameters $G_1(\omega)$ and $G_2(\omega)$. Thus, for a quadratic Hammerstein series, the quadratic coherence function in (19) is a relative measure bounded between $[0, 1]$, which indicates how well the linear and quadratic Hammerstein kernels characterize the system input-output relationship.

C. Reductions for Special Cases

We now show how the solutions for the Hammerstein transfer functions and the quadratic coherence function reduce to simpler formulations when special cases are considered.

1) Solution for Gaussian Excitation: When $X(t)$ is a zero-mean Gaussian process, the solutions for $G_1(\omega)$ and $G_2(\omega)$ simplify to become

$$G_1(\omega) = \frac{C_{YY}(\omega)}{C_{XX}(\omega)}, \quad G_2(\omega) = \frac{C_{YX}(\omega)}{C_{XX}(\omega)}$$

where $C_{XX}(\omega)$ is equivalent to

$$C_{XX}(\omega) = 2 \int_{-\pi}^{\pi} C_{XX}(\omega - \lambda)C_{XX}(\lambda) d\lambda.$$ 

The quadratic coherence function also simplifies to become

$$Q(\omega) = \frac{|C_{YX}(\omega)|^2}{C_{XX}(\omega)C_{YY}(\omega)} + \frac{|C_{YX}(\omega)|^2}{C_{XX}(\omega)C_{YY}(\omega)} = \frac{|C_{YX}(\omega)|^2}{C_{XX}(\omega)C_{YY}(\omega)}$$

which is similar to the quadratic coherence function for a quadratic Volterra series in the Gaussian input case [7]. Note that the linear and quadratic terms in the above coherence function have “decoupled,” i.e., there is no linear-quadratic interaction as was the case for a non-Gaussian input (see (19)). This separability of the coherence terms enables us to clearly assess the relative contribution of the linear and quadratic components to the output spectral density.

2) Solution for a Linear System: When the system is purely linear, then $G_2(\omega) \equiv 0$. This implies that

$$C_{YX}(\omega) = \frac{C_{XX}(\omega)C_{YX}(\omega)}{C_{XX}(\omega)}$$

from (15). This is consistent with the equations

$$C_{YX}(\omega) = G_1(\omega)C_{XX}(\omega)$$

$$C_{YX}(\omega) = G_1(\omega)C_{XX}(\omega)$$

which arise for a purely linear system, and therefore, the solution reduces to

$$G_1(\omega) = \frac{C_{YX}(\omega)}{C_{XX}(\omega)}$$

which is a familiar result. The quadratic coherence function immediately becomes

$$Q(\omega) = \frac{|C_{YX}(\omega)|^2}{C_{XX}(\omega)C_{YY}(\omega)}$$

which is equivalent to the well-known linear coherence function [19].

IV. ESTIMATION

In this section, we discuss the estimation of the third- and fourth-order integrated polyspectra. Estimates of the sample spectra are computed using an averaged periodogram-based approach [13]. A similar approach for estimating integrated polyspectra is used in [28]. For the third-order case, we have

$$c_{X^2X}(\tau) \triangleq \text{cum} \{X(t)X(t), X(t-\tau)\}$$

$$= \text{cum} \{X(t), X(t), X(t-\tau)\}$$

which arises from the relationship that exists between cumulants involving products [13] and assuming that cum\{X(t)\} = 0. Using third-order cumulant-moment relationships, we obtain

$$c_{X^2X}(\tau) = E \{X(t)X(t)X(t-\tau)\} = E \{X(t)^2X(t-\tau)\}.$$ 

Similar formulations follow for $c_{XX}(\tau)$ and $c_{YX}(\tau)$. For the fourth-order case, we have

$$c_{X^2X}(\tau) \triangleq \text{cum} \{X(t)X(t), X(t-\tau)X(t-\tau)\}$$

which again involves cumulant sequences of products of $X(t)$. This can be expressed in terms of fourth- and second-order cumulant sequences as [13]

$$\text{cum} \{X(t)X(t), X(t-\tau)X(t-\tau)\}$$

$$= \text{cum} \{X(t), X(t), X(t-\tau), X(t-\tau)\}$$

$$+ 2(\text{cum} \{X(t), X(t-\tau)\})^2.$$
Using fourth-order cumulant-moment relationships, we then obtain

\[ c_{X^2X^2}(\tau) = E\{X(t)^2X(t-\tau)^2\} - (E\{X(t)^2\})^2. \]

Given the input–output sequences \( X(t) \) and \( Y(t) \), \( t = 0, \ldots, T-1 \), we segment them into \( M \) stretches, each of length \( N \), which are denoted by \( X(t, m) \) and \( Y(t, m) \), respectively, for \( t = 0, \ldots, N-1 \) and \( m = 1, \ldots, M \) such that \( T = MN \). We formulate the cross periodogram based estimator for \( C_{X^2Y^2}(\omega_k), \omega_k = 2\pi k/N, k \in \mathbb{Z} \), namely,

\[ \hat{C}_{X^2Y^2}(\omega_k) = \frac{1}{M} \sum_{m=1}^{M} I_{X^2Y^2}^{(N)}(\omega_k, m) \]

where \( I_{X^2Y^2}^{(N)}(\omega_k, m) \) is given by

\[
I_{X^2Y^2}^{(N)}(\omega_k, m) = \frac{1}{2\pi N} \left( \sum_{t=0}^{N-1} (X(t, m) - \hat{c}_X^{(N)}(m))^2 e^{-j\omega_k t} \right) \times \left( \sum_{t=0}^{N-1} (X(t, m) - \hat{c}_X^{(N)}(m))^2 e^{j\omega_k t} \right)^* \]

and where the sample mean of \( X(t, m) \) is given by

\[ \hat{c}_X^{(N)}(m) = \frac{1}{N} \sum_{t=0}^{N-1} X(t, m) = \bar{X}(m). \]

A similar procedure is used to estimate \( C_{X^2X^2}(\omega) \) and \( C_{Y^2X^2}(\omega) \).

We estimate the fourth-order integrated polyspectrum \( C_{X^2X^2}(\omega_k) \) using the estimator

\[ \hat{C}_{X^2X^2}(\omega_k) = \frac{1}{M} \sum_{m=1}^{M} I_{X^2X^2}^{(N)}(\omega_k, m) \]

\[ - (\hat{c}_X^{(N)}(\omega_k))^2 \delta(\omega_k) \]

where the sample mean of \( X(t, m)^2 \) is found by

\[ \hat{c}_X^{(N)}(m)^2 = \frac{1}{N} \sum_{t=0}^{N-1} (X(t, m) - \hat{c}_X^{(N)}(m))^2 \]

\( \delta(\omega_k) \) represents Kronecker’s delta, and \( I_{X^2X^2}^{(N)}(\omega_k, m) \) is given by

\[ I_{X^2X^2}^{(N)}(\omega_k, m) = \frac{1}{2\pi N} \left| \sum_{t=0}^{N-1} (X(t, m) - \hat{c}_X^{(N)}(m))^2 e^{-j\omega_k t} \right|^2. \]

Estimates of \( C_{X^2X^2}(\omega), C_{Y^2X^2}(\omega) \) and \( C_{Y^2X^2}(\omega) \) are found in a similar manner. The large sample properties of this class of estimate are discussed in [13] and [22]. These estimates are substituted into (15) and (19) to yield estimates of the linear and quadratic Hammerstein transfer functions and the quadratic coherence function, respectively.

Closed-form solutions for higher order nonlinear systems can be resolved in a similar manner to the quadratic case. This
is possible since the separability property of the Hammerstein transfer functions persists for higher order (see Appendix A). Identification of an $n$th-order Hammerstein series system requires estimates of polyspectra up to and including $(2n+1)$th order. This may prove difficult in practice for higher order nonlinear systems, particularly if the amount of data available is limited.

V. Model Evaluation and Comparison

This section compares the Hammerstein series approach with the Volterra series approach in [4], which is one of the few frequency domain nonlinear system identification methods that exists for the non-Gaussian input case. Tick’s [7] Gaussian–Volterra series approach is also given in order to demonstrate the deleterious effects of an inaccurate input assumption. Modeling performance and computational aspects are discussed, as well as the effect of data length on the respective techniques. Since the Volterra and Hammerstein series models differ in the way they describe nonlinear systems, the respective quadratic coherence functions for the two models are used for validation and comparison.

For the following simulations in Sections V-A and V-B, 300 input–output records, each of length 256, are used to estimate the linear and respective quadratic coherence functions of the Hammerstein and Volterra identification approaches [7], [4]. A white zero-mean Gaussian (noise) process is added such that the SNR is 15 dB. Both Gaussian and non-Gaussian inputs are considered in turn for two different nonlinear systems. The non-Gaussian input is generated by the square of a uniformity distributed random process, which has a nonsymmetrical density function.

A. Simulated System I

Consider the dynamic nonlinear system given by

$$Y(t) = -0.64X(t) + X(t-2) + 0.9X(t)^2 + X(t-1)^2 + N(t)$$

(20)

which is the same as the Kim and Powers example [4], except that an additive noise process $N(t)$ is included for increased realism. It is assumed that $N(t)$ and $X(t)$ are independent. The first- and second-order generalized transfer functions of this system are given by

$$H_1(\omega) = -0.64 + e^{-j2\omega}$$

and

$$H_2(\omega_1, \omega_2) = 0.9 + e^{-j(\omega_1 + \omega_2)}$$

respectively, and are shown in Fig. 3.
Fig. 6. (a) Linear transfer and (b) quadratic transfer functions.

1) Gaussian Excitation: Fig. 4 shows estimates of the linear coherence function, the Volterra quadratic coherence functions of [7] and [4], and the Hammerstein quadratic coherence function for the system in (20). The linear coherence function in Fig. 4(a) indicates that a linear model provides a poor system characterization. The quadratic component clearly improves modeling performance, as is evident by the closeness of the quadratic coherence functions to unity in Fig. 4(b)–(d). Unlike the Kim and Powers [4] approach, Tick’s [7] method can be used to obtain closed-form expressions (cf. matrix inversions) for the quadratic coherence function since the input is a stationary Gaussian process.

Note the similarity between Fig. 4(b) and Fig. 4(c), which indicates that both the Hammerstein and Volterra series can characterize the system. The Volterra series does not provide improvement in system characterization, despite the fact that it is more general than the Hammerstein series. The Hammerstein quadratic coherence function is also computed in a closed-form manner and is therefore computationally efficient. Note also that the quadratic coherence in Fig. 4(c) exceeds unity at some frequencies due to estimation considerations.

2) Non-Gaussian Excitation: Fig. 5 shows estimates of the linear coherence function, the quadratic coherence function of the Tick [7] and Kim and Powers [4] methods, and the quadratic coherence function of the Hammerstein method for the system in (20) with a white non-Gaussian input. Again, the linear coherence indicates that a purely linear model does not lead to satisfactory modeling performance. The improvement, however, in the linear coherence function (see Fig. 4(a)) is observed because the non-Gaussian input has a nonzero third-order cumulant, which gives rise to joint linear-quadratic interactions.

Comparing Fig. 4(c) and (d) with Fig. 5(c) and (d), it is seen that the quadratic coherence functions associated with both the Hammerstein and Volterra models are quite similar, despite the fact that the input is now non-Gaussian. This demonstrates the validity of the Hammerstein series approach in both the Gaussian and non-Gaussian cases.

However, Tick’s [7] quadratic coherence function in Fig. 5(b) exceeds unity (note the scale) because the Gaussian assumption is violated. From the general definition of the quadratic coherence functions in (17), it is clear that the only way that the coherence can exceed unity is for the model to introduce additional noise to the output spectral density. Thus, incorrectly assuming the Gaussianity of the input results in meaningless parameter estimates.

B. Simulated System II

Consider the quadratically nonlinear Volterra series system with linear and quadratic transfer functions given by

\[ H_1(\omega) = -0.64 + e^{-j2\omega} \]

and

\[
H_2(\omega_1, \omega_2) = \begin{bmatrix}
1 \\
\frac{1}{e^{-j\omega_1}} \\
0.6 & 0.7 & 0.5 \\
0.8 & 0.5 & 0.4
\end{bmatrix} \begin{bmatrix}
1 \\
e^{-j\omega_1} \\
e^{-j\omega_2} \\
e^{-j2\omega_2}
\end{bmatrix}
\]

as shown in Fig. 6. This example is taken from [9] and represents a more complicated quadratic nonlinearity than the system shown in Section V-A. Consequently, it would be expected that the Volterra approach of [4] would provide a better characterization of the system than the Hammerstein series at the expense of computation. This aspect is now explored for both Gaussian and non-Gaussian inputs in turn.

1) Gaussian Excitation: Fig. 7 shows estimates of the linear coherence function, the quadratic coherence functions of the Tick [7] and Kim and Powers [4] methods and the quadratic coherence function of the Hammerstein method for the system in (21) driven by a white Gaussian input. The linear coherence function in Fig. 7(a) reveals that a linear model only characterizes the system at higher frequencies as the system is nonlinear. The Hammerstein quadratic coherence function shown in Fig. 7(d) is not as close to unity as the Volterra quadratic coherence function in Fig. 7(b) and (c) in the lower frequency range, where quadratic cross interactions are most significant. The Hammerstein series still accounts for some quadratic interaction, despite the complicated form of the second-order generalized transfer function for this simulation. It also provides improvement over the use of the linear model alone.
If the Gaussianity of the input was assured \textit{a priori}, then the Tick [7] closed-form approach would be particularly useful for this quadratic Volterra identification problem. However, in practice, it is not always possible to assume the Gaussianity of the input. The computational expense associated with computing the Volterra quadratic coherence function in Fig. 7(c) is far greater than that of the Hammerstein quadratic coherence function in Fig. 7(d). Note that matrix regularization was also required with the Kim and Powers [4] approach in order to avoid numerical problems because the input spectral matrix was ill-conditioned at some frequencies.

2) Non-Gaussian Excitation: Fig. 8 shows estimates of the linear coherence function, the quadratic coherence functions of the Tick [7] and Kim and Powers [4] methods, and the quadratic coherence function of the Hammerstein method for the system in (21) driven by a white non-Gaussian input. As would be expected, the linear coherence function shown in Fig. 8(a) is not significantly different from the linear coherence function shown in Fig. 7(a). Tick’s [7] quadratic coherence function again \textit{exceeds} unity (note the scale in Fig. 8(b)), as the method is \textit{invalid} for non-Gaussian inputs. However, the closed-form solutions for the Hammerstein series are valid for both Gaussian and non-Gaussian inputs. The Kim and Powers [4] Volterra approach in Fig. 8(c) again performs well, as in the Gaussian case.

A comparison of the Hammerstein quadratic coherence functions in Figs. 7(d) and 8(d) suggests that the modeling performance of the Hammerstein series may improve slightly in the non-Gaussian input case. This is possibly due to nonzero linear-quadratic interactions. The quadratic Hammerstein series does not characterize the quadratic Volterra series in (21) at the lower frequencies where quadratic cross-kernel interactions are dominant.

3) Cubic Hammerstein Series: It is possible to improve modeling performance by fitting a \textit{cubic} Hammerstein series to the \textit{quadratic} Volterra model in (21). The cubic model can be easily realized because of the simplicity of the Hammerstein series. Fig. 9 shows the quadratic and cubic coherence functions of the Hammerstein series fitted to the quadratic Volterra model in (21) for the non-Gaussian input. The addition of the cubic component is seen to \textit{improve} modeling performance over the quadratic Hammerstein series. Although the coherence function is not as good as the quadratic Volterra coherence function in Fig. 8(c), it still provides reasonably good modeling performance. The improvement in modeling performance incurs a relatively \textit{small} increase in computational cost over the quadratic Hammerstein series. Thus, the validity of the Hammerstein series as a nonlinear model for a given identification task can be readily established by computing the associated nonlinear coherence function.

C. Estimation Error Versus Realizations

In a practical identification scenario, the amount of input–output data available may be limited. It is therefore of interest to ascertain, even qualitatively, the performance of the Hammerstein and Volterra series identification techniques.
Fig. 8. Coherences for a non-Gaussian input. (a) Linear coherence. (b) Tick’s quadratic coherence using the Volterra series. (c) Kim and Powers’ quadratic coherence using the Volterra series. (d) Quadratic coherence using the Hammerstein series.

when the data length is relatively small. Since the models are different, the normalized mean-square prediction error is used in order to make a fair comparison, i.e.,

$$q = \frac{\sum_{t=0}^{N-1} (Y(t) - \hat{Y}(t))^2}{\sum_{t=0}^{N-1} Y(t)^2}$$

where \(Y(t)\) and \(\hat{Y}(t)\) are the observed and predicted output signals, respectively, from the quadratic Hammerstein series and the quadratic Volterra series.

Estimates of the Hammerstein and generalized transfer functions were made using the same input–output data set over a varying number of realizations \(R = 20, 21, \ldots, 50\) with a data length of \(N = 64\) points each. These estimates were then used to predict the output signal \(\hat{Y}(t)\). The prediction errors were then averaged for the given number of realizations \(M = 50\). Fig. 10(a) shows the normalized mean-square prediction error (log scale) for the two models over the numbers of realizations in the noise-free case. The Hammerstein series clearly demonstrates improved modeling performance over the Volterra approach. The improvement is particularly noticeable in the case where a small number of realizations is used.

The experiment was then repeated in the noisy case, where white Gaussian noise was added to the output such that the SNR was 15 dB. Fig. 10(b) shows the plot of the average normalized mean-square prediction error for the two models over the number of realizations. The Volterra approach severely breaks down in the noisy case, and the normalized prediction error exceeds unity. This is mainly due to the ill-conditioning of the matrices. The Hammerstein series approach has a lower normalized mean-square prediction error and does not exhibit the extremes of variability seen in the Volterra approach. In addition, a significant difference in the magnitude of the prediction error is seen for the Volterra approach in the noise-free and noisy cases. This is of concern, given that noisy measurements are more likely in a practical system identification problem. A simple examination of Fig. 10 shows that the Hammerstein series approach does not suffer to the same extent as the Volterra approach in the noisy case.

The practical implication of this simple comparison is that while the Hammerstein series may not be as general as the Volterra series, it is more robust in the small data case. Although this result is demonstrated on a specific example, it still indicates the improvement gained by the Hammerstein series approach. When a large number of realizations was used in estimation, the normalized mean-square prediction error associated with the two methods was essentially the same. Note also that the computational burden associated with the Volterra approach of [4] greatly exceeds the Hammerstein series approach.

D. Parameterization and Computational Issues

1) Parameterization Issues: The Hammerstein series largely overcomes the parameterization problem associated
with nonlinear modeling [14], [29]. Since the number of parameters required by the Volterra series increases exponentially with increasing nonlinear order, its implementation and interpretation quickly become prohibitively complicated [19]. In contrast, the number of parameters required by the Hammerstein series for the same system memory increases linearly with increasing nonlinear order. The Hammerstein series can thus economically characterize dynamic nonlinear systems in a simple manner.

2) Computational Issues: Solutions for the Hammerstein series can be obtained in a closed-form manner for any stationary input. As a result, the Hammerstein transfer functions can be obtained in a computationally efficient manner. Solutions for the Hammerstein series can be also computed using spectral 1-D forms of the polyspectrum, which also leads to reduced computation requirements over conventional (multidimensional) polyspectral-based techniques. In contrast, solutions for the Volterra series cannot generally be found in closed form for non-Gaussian inputs, and thus, matrix inversions are often required. Most algorithms for real matrix inversion are of order $P^3$ [30], where the number of unknown parameters $P$ is of order $K^n$, where $K$ is the discrete-time memory, and $n$ is the nonlinear order.

For the frequency domain regression identification method [4], the frequency resolution effectively determines the size of the matrices that need to be inverted in order to obtain a solution. Each discrete frequency requires a separate matrix inversion, and therefore, this can represent a large computational task. A similar situation exists for the time-domain approaches in [31].

VI. ENGINE TRANSMISSION MODELLING

In this section, the Hammerstein series approach is applied to the problem of modeling the transmission characteristics of a combustion engine operating in a knocking condition [5], [32], [33]. Note that we do not attempt to solve the engine knock problem in this paper, but rather focus on the application of the Hammerstein series as a nonlinear model.

A. Background

An effective means for lowering fuel consumption and improving the general efficiency of a combustion engine is to increase the compression ratio [34]. However, this may also increase the occurrence of an abnormal combustion phenomenon called knock. Knock needs to be avoided as it results in an excessively noisy, overheated, and inefficient engine and can lead to premature mechanical failure. The knocking condition can be especially severe when the engine is operating at high speed [33].

The rapid combustion of knocking cycles generates damped acoustical oscillations, which are often heard as a knocking or ringing sound. If the knocking condition can be detected, then it can be adaptively controlled without adversely effecting the overall efficiency of the engine. This knocking signal can be
transduced by placing a sensor on the housing of the engine. Previously, a quadratically nonlinear Volterra processor was used to characterize the relationship between the cylinder pressure and the engine vibration [5]. Following this approach, we propose the use of a quadratic Hammerstein series to model the transmission characteristics of the engine housing.

B. The Identification Process

The cylinder pressure and the engine housing vibration signals were treated as the respective input and output signals. The data was collected from a 1.8-liter, four-cylinder engine operating under strong knocking conditions running at full load and high speed. The data was preprocessed to remove general nonstationary trends [5]. A Gaussianity test was performed on the knock data, and the Gaussianity hypothesis was rejected in all cases [33]. Typical input–output knock data is shown in Fig. 11. One hundred fifty input–output cycles of length 128 each were used in forming estimates of the integrated polyspectra, and the remaining 36 cycles were used for validation. Estimates of the input and output spectral densities are shown in Fig. 12.

Estimates of the first- and second-order Hammerstein transfer functions were determined using the method described in this paper and are shown in Fig. 13. Fig. 14 shows the estimated linear and quadratic coherence functions. Comparing the linear and quadratic component provides modeling improvement around specific resonance frequencies of interest, namely, $f = 0.13, 0.26,$ and $0.43$ (normalized frequency). Modeling improvement due to the quadratic term is also observed in the low-frequency range. A more pronounced difference is seen in the time-domain predictions of the vibration signal.

In order to validate the use of the quadratic Hammerstein series for this application, estimates of the first- and second-order Hammerstein transfer functions were used to predict the vibration signal on 36 records that were not used in estimation. A comparison between a pure linear filter and a quadratic Hammerstein series was made in terms of the normalized mean-square prediction error.

Fig. 15 shows the best case predictions using a linear model and quadratic Hammerstein series, respectively. For the best linear case, we have $q = 0.4333$, and for the best quadratic Hammerstein series, we have $q = 0.2730$. Hence, improvement is obtained with the use of the quadratic Hammerstein series over the linear model alone. Fig. 16 shows the worst-case vibration signal predictions for the linear model $q = 0.7203$ and the quadratic Hammerstein series $q = 0.6903$, respectively. We notice that even in the worst-case prediction, the quadratic model still shows some improvement over the linear case, as most of the major signal features are still present, which is important for knock detection schemes [33].

Given that a lower relative mean-square prediction error $q$ corresponds to an improved characterization of the engine
block (as would intuitively be the case), then the quadratic Hammerstein series is superior to a linear model. The results from this experiment indicate the usefulness of the Hammerstein series for the nonlinear and non-Gaussian system identification problem. The identification of time-varying Hammerstein series in the non-Gaussian case has been considered elsewhere [35], [36].

VII. CONCLUSIONS

We have proposed a new method for identifying time-invariant nonlinear systems when the input is a stationary non-Gaussian process. Optimal closed-form expressions for a quadratic Hammerstein series model have been derived. Special forms of integrated polyspectra were used in the formulation. Since the solutions are in closed form, the approach represents significant computational advantages over existing methods. The results were applied to both simulated and real data, which indicated the potential of the approach for the nonlinear and non-Gaussian system identification problem.

APPENDIX A

In this appendix, we show the separability result associated with the Hammerstein series in the frequency domain. Since the Hammerstein series consists of a sum of homogeneous terms, it suffices to demonstrate the separability relationship...
using a single term. Let $Y_n(t)$ be the output associated with the $n$th-order term of the Hammerstein model, i.e.

$$Y_n(t) = \sum_{r=-\infty}^{\infty} g_n(\tau)X(t-\tau)^n.$$

Noting that the above expression can be rewritten as

$$Y_n(t) = \prod_{\tau_n=-\infty}^{\infty} \cdots \prod_{\tau_1=-\infty}^{\infty} g_n(\tau_1)\delta(\tau_1 - \tau_2) \cdots \\
\times \delta(\tau_1 - \tau_n)X(t-\tau_1) \cdots X(t-\tau_n)$$

where $\delta(\tau)$ is Kronecker’s delta, we substitute (2) into the above to give

$$Y_n(t) = \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} e^{i(\omega_1 + \cdots + \omega_n)t} \\
\times G_n(\omega_1 + \cdots + \omega_n) dZ_X(\omega_1) \cdots dZ_X(\omega_n).$$

Setting $\omega = \omega_1 + \cdots + \omega_n$ and $\omega_n = \omega - \omega_1 - \cdots - \omega_{n-1}$ leads to

$$Y_n(t) = \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} e^{i\omega t} G_n(\omega) \\
\times dZ_X(\omega_1) \cdots dZ_X(\omega_1 - \cdots - \omega_{n-1}).$$

Expressing $Y_n(t)$ using Cramér’s spectral representation in (2) results in

$$dY_n(\omega) = G_n(\omega) \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} dZ_X(\omega_1) \\
\times dZ_X(\omega_2) \cdots dZ_X(\omega_1 - \cdots - \omega_{n-1})$$

where the above expression is an $(n-1)$ dimensional integration. Note that the Hammerstein transfer function $G_n(\omega)$ has separated from the integral expression. It is this separability result that is used to lead to closed-form solutions for the Hammerstein transfer functions.

### APPENDIX B

In this appendix, we show that the solutions for the Hammerstein transfer functions are optimal in a minimum-mean-square error sense. The quadratic system model is given by

$$Y(t) = \sum_{\tau=\infty}^{\infty} g_1(\tau)X(t-\tau) + \sum_{\tau=\infty}^{\infty} g_2(\tau)X(t-\tau)^2 \\
+ N(t).$$

We seek to minimize

$$E\{N(t)^2\} = E\left\{ \left( Y(t) - \sum_{\tau=\infty}^{\infty} g_1(\tau)X(t-\tau) \\
- \sum_{\tau=\infty}^{\infty} g_2(\tau)X(t-\tau)^2 \right)^2 \right\}$$

with respect to $g_1(\tau)$ and $g_2(\tau)$. Taking the partial derivative of (22) with respect to $g_1(\omega)$ for $v = 0, \pm 1, \cdots$

$$\frac{\partial E\{N(t)^2\}}{\partial g_1(\omega)} = -2E\left\{ X(t-v)\left( Y(t) - \sum_{\tau=\infty}^{\infty} g_1(\tau)X(t-\tau) \\
- \sum_{\tau=\infty}^{\infty} g_2(\tau)X(t-\tau)^2 \right) \right\}$$

and equating to zero leads to the equation

$$c_{YX}(v) = \sum_{\tau=\infty}^{\infty} g_1(\tau)c_{XX}(v-\tau) \\
+ \sum_{\tau=\infty}^{\infty} g_2(\tau)c_{XX}(v-\tau)^2$$

for $v = 0, \pm 1, \cdots$. Similarly, we take the partial derivative of (22) with respect to $g_2(\omega)$

$$\frac{\partial E\{N(t)^2\}}{\partial g_2(\omega)} = -2E\left\{ X(t-v)^2\left( Y(t) - \sum_{\tau=\infty}^{\infty} g_1(\tau)X(t-\tau) \\
- \sum_{\tau=\infty}^{\infty} g_2(\tau)X(t-\tau)^2 \right) \right\}$$
and equate to zero, which leads to
\[ c_{12}(v) = \sum_{\tau = -\infty}^{\infty} g_1(\tau) c_{12}(v - \tau) + \sum_{\tau = -\infty}^{\infty} g_2(\tau) c_{12}(v - \tau) . \] (24)

We take the Fourier transform (23) and (24) with respect to \( v \), and solve the resulting simultaneous equation in the frequency domain to yield closed-form expressions for \( G_1(\omega) \) and \( G_2(\omega) \), which are given in (15). Similar formulations follow for higher order.

**ACKNOWLEDGMENT**

The authors would like to thank colleagues from the Signal Processing Research Centre for their useful comments. They would also like to thank Professor J. F. Böhme from the Signal Theory Division at the Ruhr University Bochum and Volkswagen AG, Wolfsburg, Germany, for kindly providing the knock data used in this paper.

**REFERENCES**


Jonathon C. Ralston (M’92) received the Bachelor of Engineering degree in electronic and computer engineering (Honours I) in 1992 and the Ph.D degree in electrical engineering in 1996, both from the Queensland University of Technology, Brisbane, Australia. He is currently working as a research engineer in the Division of Exploration and Mining at the CSIRO, Brisbane. His research interests include nonlinear system identification, statistical signal processing, time-varying parameter estimation, higher order spectral analysis, and real-time control with application to analysis, guidance, and automation in the mining industry.
Abdelhak M. Zoubir (M’92) received the Dipl.-Ing. degree from Fachhochschule Niederrhein, Germany, in 1983 and the Dipl.-Ing. and Dr.-Ing. degrees from Ruhr University Bochum, Bochum, Germany, in 1987 and 1992, all in electrical engineering.

From August to December 1983, he was with Klöckner-Moeller, Krefeld, Germany. He then joined the Control Division at Siempelkamp AG, Krefeld, where he was a consultant until March 1987. From April 1987 to March 1992, he was a teaching and research assistant (associate lecturer) with the Signal Theory Division of Ruhr University Bochum. In June 1992, he joined the Signal Processing Research Centre at the Queensland University of Technology, Brisbane, Australia, as a lecturer, where he became a senior lecturer in 1994. His current research interests include statistical methods for signal processing with applications in sonar, radar, biomedical engineering, and the automotive industry.

Boualem Boashash (SM’89) received the Diplôme d’ingénieur-Physique-Électronique from the ICPI University Lyon, Lyon, France, in 1978, the M.S. degree from the Institut National Polytechnique de Grenoble, Grenoble, France in 1979, and Doctorate (Docteur-Ingénieur) degree from the same university in May 1982.

In 1979, he joined Elf-Aquitaine Geophysical Research Centre, Pau, France. In May 1982, he joined the Institut National des Sciences Appliquées de Lyon, Lyon, France, where he was a Maitre-Assistant associé. In January 1984, he joined the Electrical Engineering Department of the University of Queensland, Brisbane, Australia, as a lecturer, then became senior lecturer in 1986, and reader in 1989. In 1990, he joined Bond University, Graduate School of Science and Technology, as Professor of Electronics. In 1991, he joined the Queensland University of Technology, Brisbane, as the Foundation Professor of Signal Processing and Director of the Signal Processing Research Centre. His research interests are time-frequency signal analysis, spectral estimation, signal detection and classification, and higher order spectra. He is also interested in wider issues such as the effect of engineering developments on society.

Dr. Boashash was technical chairman of ICASSP 1994, which is the premium conference in signal processing. He is the editor of two books, has written over 200 technical publications, and has supervised 20 Ph.D. students and five Masters students. He is a Fellow of IE Australia and a Fellow of the IREE.