Common fuzzy fixed point theorems in ordered metric spaces

L. Ćirić, M. Abbas, B. Damjanović, R. Saadati

Article history:
Received 28 July 2010
Accepted 23 December 2010

Keywords:
Fuzzy mapping
Fuzzy set
Fuzzy fixed point

1. Introduction and preliminaries

Let $X$ be a space of points with generic element of $X$ denoted by $x$ and $I = [0, 1]$. A fuzzy subset of $X$ is characterized by a membership function which associates with each element in $X$ a real number in the interval $I$. Let $(X, d)$ be a metric linear space and $A$ be a fuzzy set in $X$ characterized by a membership function $A$. The $\alpha$- level set of $A$, denoted by $A_\alpha$, is defined by

\[ A_\alpha = \{ x : A(x) \geq \alpha \} \quad \text{if } \alpha \in (0, 1) \]
\[ A_0 = \{ x : A(x) > 0 \} \]

where $\bar{B}$ denotes the closure of the non fuzzy set $B$.

A fuzzy set $A$ in a metric linear space is said to be an approximate quantity if and only if $A_\alpha$ is compact and convex in $X$ for each $\alpha \in [0, 1]$ and $\sup_{x \in X} A(x) = 1$. We denote by $W(X)$, the family of all approximate quantities in $X$.

Let $A, B \in W(X)$, then $A$ is said to be more accurate than $B$, denoted by $A \leq B$, if and only if $A(x) \leq B(x)$ for each $x \in X$, where $B$ denotes the membership function of $B$. For $x \in X$, we write $\{ x \}$ the characteristic function of the ordinary subset $\{ x \}$ of $X$. We denote $W^0(X) = \{ \{ x \} : x \in X \}$.

For $\alpha \in (0, 1]$, the fuzzy point $(x)_\alpha$ of $X$ is the fuzzy set of $X$ given by $x_\alpha(x) = \alpha$ and $\alpha \neq x$.

Let $l^I$ be the collection of all fuzzy subsets in $X$ and $W(X)$ be a sub collection of all approximate quantities. For $A, B \in W(X), \alpha \in [0, 1]$, define

\[ p_\alpha(A, B) = \inf_{x \in A_\alpha, y \in B_\alpha} d(x, y), \]
\[ P(A, B) = \sup_{\alpha} p_\alpha(A, B), \]
where $H$ is the Hausdorff metric induced by the metric $d$. We note that $p_\alpha$ is a non-decreasing function of $\alpha$ and $D$ is a metric on $W(X)$.

Let $\alpha \in [0, 1]$, then the family $W_\alpha(X)$ is given by $\{ A \in \mathcal{P}^X : A_\alpha$ is non empty convex and compact $\}$. Let $X$ be an arbitrary set, $Y$ be a metric linear space. A mapping $T$ is called fuzzy mapping if $T$ is a mapping from $X$ into $W(Y)$, that is, $Tx \in W(Y)$ for each $x$ in $X$. Thus if we characterize a fuzzy set $Tx$ in a metric linear space $Y$ by a membership function $Tx$, then $Tx(y)$ is the grade of membership of $y$ in $Tx$. Therefore a fuzzy mapping $T$ is a fuzzy subset on $X \times Y$ with membership function $Tx(y)$.

A fuzzy point $x_\alpha$ in $X$ is called a fixed fuzzy point of the fuzzy mapping $T$ if $x_\alpha \subset Tx$. If $\{x\} \subset Tx$, then $x$ is a fixed point of $T$.

**Definition 1.** Let $X$ be a nonempty set. Then $(X, d, \preceq)$ is called an ordered metric space iff

1. $(X, d)$ is a metric space,
2. $(X, \preceq)$ is partial ordered.

**Definition 2.** Let $(X, \preceq)$ be a partial ordered set. $x, y \in X$ are called comparable if $x \leq y$ or $y \leq x$ holds.

Following lemmas are needed in the sequel.

**Lemma 3 (Heilpern [2]).** Let $(X, d)$ be a metric space, $x, y \in X$ and $A, B \in W(X)$:

1. if $p_\alpha(x, A) = 0$, then $x_\alpha \subset A$
2. $p_\alpha(x, A) \leq d(x, y) + p_\alpha(y, B)$
3. if $x_\alpha \subset A$, then $p_\alpha(x, B) \leq D_\alpha(A, B)$.

**Lemma 4 (Lee and Cho [3]).** Let $(X, d)$ be a complete metric space, $T$ be a fuzzy mapping from $X$ into $W(X)$ and $x_0 \in X$. Then there exists a $x_1 \in X$ such that $\{x_1\} \subset Tx_0$.


The aim of this paper is to establish the existence of a common fuzzy fixed point of generalized contractive mappings without employing any commutativity condition. Our result generalize, improve and extend many known results in the comparable literature [18,20,7].

2. Main results

We begin with the following result.

**Theorem 5.** Let $X$ be a complete ordered space. Suppose that $T_1, T_2 : X \rightarrow W_\alpha(X)$ are two fuzzy mapping on $X$ satisfying

$$\varphi(D_\alpha(T_1 x, T_2 y)) \leq \varphi(d(x, y)) - \varphi(d(x, y))$$

for all comparable elements $x, y \in X$, where $\varphi : [0, \infty) \rightarrow [0, \infty)$ is a continuous and monotone nondecreasing functions with $\varphi(t) = 0$ if and only if $t = 0$ and $\varphi : [0, \infty) \rightarrow [0, \infty)$ is lower semi continuous with $\varphi(t) = 0$ if and only if $t = 0$. Suppose that if $\{y\} \subset T_1(x_0)$, then $y, x_0 \in X$ are comparable. Further, if $x, y \in X$ are comparable, then every $u \in T_1 x_\alpha$ and every $v \in T_2 y_\alpha$ are comparable. Also suppose that if a sequence $\{x_n\} \rightarrow x$ and its consecutive terms are comparable, then $x_n, x \in X$ are comparable for all $n$. Then there exists a point $x \in X$ such that $x_\alpha \subset T_1 x$ and $x_\alpha \subset T_2 x$. 


Proof. Let $x_0$ be in $X$. By Lemma 2, there exists $x_1$ in $X$ such that $\{x_1\} \subset T_1(x_0)$ which implies that

$$p_\alpha (x_1, T_1x_0) = 0 \quad \text{for each } \alpha \in [0, 1],$$

which is possible if and only if $x_1 \in (T_1x_0)_\alpha$. By assumption, $x_0$ and $x_1$ are comparable. Since $(T_2x_1)_\alpha$ is nonempty compact subset of $X$, there exists $x_2 \in (T_2x_1)_\alpha$ such that

$$d(x_1, x_2) = p_\alpha (x_1, T_2x_1) \leq D_\alpha (T_1x_0, T_2x_1).$$

Moreover, $x_1$ and $x_2$ are comparable. Continuing this process, we can construct a sequence $\{x_n\}$ in $X$ such that $x_{2n+1} \in (T_1x_{2n})_\alpha$ and $x_{2n+2} \in (T_2x_{2n+1})_\alpha$ for all $n \geq 0$, $x_{2n}$ and $x_{2n+1}$ are comparable and $d(x_{2n+1}, x_{2n+2}) \leq D_\alpha (T_1x_{2n}, T_2x_{2n+1})$. Since $\phi$ is nondecreasing, $\phi(d(x_{2n+1}, x_{2n+2})) \leq \phi(D_\alpha (T_1x_{2n}, T_2x_{2n+1}))$. Since $x_{2n}$ and $x_{2n+1}$ are comparable. Thus by taking $x_{2n}$ for $x$ and $x_{2n+1}$ for $y$ in the inequality (1), it follows that

$$\phi(d(x_{2n+1}, x_{2n+2})) \leq \phi(D_\alpha (T_1x_{2n}, T_2x_{2n+1})) \leq \phi(d(x_{2n}, x_{2n+1}) - \phi(d(x_{2n}, x_{2n+1})).$$

Similarly

$$\phi(d(x_{2n+3}, x_{2n+2})) \leq \phi(d(x_{2n+2}, x_{2n+1})) \leq \phi(d(x_{2n+2}, x_{2n+1})).$$

Therefore, for all $n$

$$\phi(d(x_n, x_{n+1})) \leq \phi(D_\alpha (T_1x_{n-1}, T_2x_{n})) \leq \phi(d(x_{n-1}, x_n)).$$

Hence $\phi(d(x_n, x_{n+1})) \leq \phi(d(x_{n-1}, x_n))$. Thus, we have

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n),$$

which shows that $\{d(x_n, x_{n+1})\}$ is non-increasing sequence of positive real numbers which is bounded below by 0. Therefore there is a real number $r \geq 0$ such that

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = r.$$

Suppose that $r > 0$, then

$$0 < \phi(r) \leq \phi(d(x_n, x_{n+1})) \leq \phi(D_\alpha (T_1x_{n-1}, T_2x_n)) \leq \phi(d(x_n, x_{n-1})) - \phi(d(x_{n-1}, x_n)).$$

Now by continuity of $\phi$ and lower semicontinuity of $\phi$ we get

$$\phi(r) \leq \limsup_{n \to \infty} \phi(d(x_n, x_{n-1})) - \liminf_{n \to \infty} \phi(d(x_{n-1}, x_n))$$

and hence $\phi(r) \leq \phi(r) - \phi(r) < \phi(r)$, a contradiction. Therefore $r = 0$ and so

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = 0.$$

Following the similar arguments to those given in [21], it can be shown that $\{x_n\}$ is a Cauchy sequence in $X$. It follows from the completeness of $X$ that $x_n \to x \in X$. Since consecutive terms of $\{x_n\}$ are comparable and $x_n \leq x$. Now, we claim that $p_\alpha (x, T_2x) = 0$ for each $\alpha \in [0, 1]$. From

$$|p_\alpha (x, T_2x) - d(x, x_{2n+1})| \leq p_\alpha (x_{2n+1}, T_2x) \leq D_\alpha (T_1x_{2n}, T_2x)$$

and (1) we get

$$\phi(|p_\alpha (x, T_2x) - d(x, x_{2n+1})|) \leq \phi(D_\alpha (T_1x_{2n}, T_2x)) \leq \phi(d(x_{2n}, x)) - \phi(d(x_{2n}, x)).$$

Hence we obtain, as $\phi$ is continuous and $\phi$ is lower semicontinuous,

$$\phi(p_\alpha (x, T_2x)) \leq \phi(0) - \phi(0) = 0,$$

that is, $\phi(p_\alpha (x, T_2x)) = 0$. Hence $p_\alpha (x, T_2x) = 0$. Therefore $x = T_2x$. Similarly, $x \subset T_2x$. □

Define a class of functions $g = \{g : \mathbb{R}^5_+ \to \mathbb{R}_+\}$ satisfying the following conditions:

$(g_1)$ $g$ is nondecreasing in the first and 5th variables.

$(g_2)$ If $u, v \in \mathbb{R}^5$ are such that $g(u, v, u, u, u + v) \leq 0$, or $g(u, v, u, v, u + v) \leq 0$, then $u \leq hv$, where $0 < h < 1$ is a constant.

$(g_3)$ If $u \in \mathbb{R}^5$ is such that $g(u, 0, 0, u, u) \leq 0$, or $g(u, 0, u, u, u) \leq 0$, then $u = 0$. 

Theorem 6. Let $X$ be a complete ordered space. Suppose that $T_1, T_2 : X \rightarrow W_a(X)$ are two fuzzy mappings on $X$ satisfying
\[
g(D_a(T_1x, T_2y), d(x, y), p_a(x, T_1x), p_a(y, T_2y), p_a(x, T_2y) + p_a(y, T_1x)) \leq 0
\] (2)
for all comparable elements $x, y \in X$ and for some $g \in \mathcal{G}$. Suppose that for any $y$ in $X$ with $\{y\} \cap T_1(x_0)$ implies that $y, x_0 \in X$ are comparable and for comparable $x, y \in X$ with $u \in (T_1x_0)_{\alpha}$ and $v \in (T_2y)_{\alpha}$ imply $u, v \in X$ are comparable. Further, suppose that if a sequence $\{x_n\} \rightarrow x$ and its consecutive terms are comparable, then $x_n, x \in X$ are comparable for all $n$. Then there exists a point $x$ in $X$ such that $x_n \subset T_1x$ and $x_n \subset T_2x$.

Proof. Let $x_0$ be in $X$. By Lemma 2, there exists $x_1$ in $X$ such that $\{x_1\} \subset T_1(x_0)$ which implies that
\[p_a(x_1, T_1x_0) = 0\]
for each $\alpha \in [0, 1]$. This is possible if and only if $x_1 \in (T_1x_0)_{\alpha}$. By given assumption $x_0$ and $x_1$ are comparable. Since $(T_2x_1)_{\alpha}$ is nonempty compact subset of $X$, therefore there exists $x_2 \in (T_2x_1)_{\alpha}$ such that
\[d(x_1, x_2) = p_a(x_1, T_2x_1) \leq D_a(T_1x_0, T_2x_1).
\]
Also, $x_1$ and $x_2$ are comparable. Since $x_0$ and $x_1$ are comparable, then
\[g(D_a(T_1x_0, T_2x_1), d(x_0, x_1), p_a(x_0, T_1x_0), p_a(x_1, T_2x_1), p_a(x_0, T_2x_1) + p_a(x_1, T_1x_0)) \leq 0.
\]
Since $p_a(x_1, T_1x_0) = 0$ and $p_a(x_0, T_2x_1) \leq d(x_0, x_1) + p_a(x_1, T_2x_1)$, then
\[g(d(x_1, x_2), d(x_0, x_1), d(x_0, x_1), d(x_1, x_2) + d(x_0, x_1)) \leq 0.
\]
Hence, as $g \in \mathcal{G}$,
\[d(x_1, x_2) \leq hd(x_0, x_1).
\]
Similarly, we obtain that $x_2$ and $x_3$ are comparable and
\[d(x_2, x_3) \leq hd(x_1, x_2) \leq h^2d(x_0, x_1).
\]
Continuing this process, we can construct a sequence $\{x_n\}$ in $X$ such that $x_{2n+1} \in (T_1(x_{2n}))_{\alpha}$ and $x_{2n+2} \in (T_2(x_{2n+1}))_{\alpha}$ for all $n \geq 0, x_{2n}$ and $x_{2n+1}$ are comparable and $d(x_{2n+1}, x_{2n+2}) \leq hd(x_{2n}, x_{2n+1})$. Thus, by induction we have
\[d(x_n, x_{n+1}) \leq h^n d(x_0, x_1).
\]
From the proceeding inequality we conclude that $\{x_n\}$ is a Cauchy sequence in $X$. It follows from the completeness of $X$ that $x_n \rightarrow x \in X$. Since consecutive terms of $\{x_n\}$ are comparable, then $x_n, x \in X$ are comparable for all $n$. Also, note that $x \in \lim_{n \rightarrow \infty}(T_1x_{2n})_{\alpha}$ and $x \in \lim_{n \rightarrow \infty}(T_2x_{2n+1})_{\alpha}$.

Now, we claim that $p_a(x, T_2x) = 0$ for each $\alpha \in [0, 1]$. From
\[|p_a(x, T_2x) - d(x, x_{2n+1})| \leq p_a(x_{2n+1}, T_2x) \leq D_a(T_1x_{2n}, T_2x)
\]
and (2) we get, as $g \in \mathcal{G}$,
\[g(p_a(x_{2n+1}, T_2x), d(x_{2n+1}, x), p_a(x_{2n+1}, T_1x_{2n}), p_a(x, T_2x), p_a(x_{2n+1}, T_2x) + p_a(x, T_1x_{2n})) \leq 0.
\]
Hence we obtain
\[p_a(x, T_2x) = 0.
\]
Therefore $x_\alpha \subset T_2x$. Similarly, $x_\alpha \subset T_1x$. □

Acknowledgements

The first and third authors accomplished research results on the project IO 174025, and resources for its implementation have been provided by Ministry of Science and Technological Development of Republic Serbia.

The fourth author is grateful to the Young research Club, Islamic Azad University—Ayatollah Amoli Branch, Amol, Iran.

References


