The ghost solid method for the elastic solid–solid interface

A. Kaboudian\textsuperscript{a,}\textastertilde, B.C. Khoo\textsuperscript{b,}\textasteriskcentered

\textsuperscript{a} NUS Graduate School for Integrative Sciences & Engineering, National University of Singapore, 9 Engineering Drive 1, EA-03-19S, Singapore 117576, Singapore

\textsuperscript{b} Department of Mechanical Engineering, National University of Singapore, 10 Kent Ridge Crescent, Singapore 119260, Singapore

\textbf{A B S T R A C T}

In this work, three variants of Ghost Solid Method (GSM) are proposed for application to the boundary conditions at the solid–solid interface of isotropic linearly elastic materials, in a Lagrangian framework. It is shown that, in the presence of the wave propagation through the solid–solid mediums, the original GSM \cite{Kaboudian2013} can lead to non-physical oscillations in the solution, even for first-order solvers. It is discussed and numerically shown that these oscillations will be more severe if a higher order solver is employed using the original GSM. A scheme for prediction of these non-physical oscillations at the interface is also introduced. The other two variants of GSM proposed, however, can remove the non-physical oscillations that may rise at the interface. Next, the extension to two-dimensional settings with slip and no-slip conditions at the interface is carried out. Numerous numerical examples in one- and two-dimensional settings are provided attesting to the viability and effectiveness of the GSM for treating wave propagation at the solid–solid interface.

\textcopyright 2013 Elsevier Inc. All rights reserved.

1. Introduction

Dynamics of elastic solid–solid interactions and the study of wave propagation through elastic mediums can be important in various scientific areas. These areas include non-destructive testing of solid systems \cite{Benson2000}, geophysical dynamics \cite{Sellevold2006}, and others. In mechanical systems, there are a wide range of applications for dynamics of elastic solids.

There are two major approaches for modeling elastic solid–solid interactions: using an Eulerian frame of reference, or a Lagrangian frame of reference. The Eulerian framework has the advantage that no regeneration of the mesh is required throughout the computational process. However, the challenge is that all physical boundaries should be somehow tracked through the mesh, as they may not be fixed in this frame \cite{Benson2001}. Moreover, tracking the interface needs special attention. This may require the use of level-set methods \cite{Osher2003} or any other accurate front tracking techniques \cite{Benson2002, Benson2003}. On the other hand, the Lagrangian framework does not require tracking of the boundaries through the mesh. This is due to the fact that the boundaries of the solid are usually Lagrangian points. Moreover, most of the engineering measuring devices for solids, like strain-gages, are attached to the solid considered to be in a Lagrangian framework for ease of reference and comparison. The only possible drawback for the Lagrangian framework is that the computer codes developed under this framework may regularly require mesh regeneration. However, as this work concentrates on elastic solid modeling, the necessity for mesh regeneration will be minimal, as the deformations are assumed to be reasonably limited; otherwise, the solid may undergo plastic deformations. Therefore, the Lagrangian framework has been used in this work.
This work seeks to develop the Ghost Solid Methods (GSMs) to faithfully simulate and capture the boundary conditions at the interface for linearly elastic solid–solid interaction problems. Later, this can facilitate a consistent and truly multi-medium modeling of interaction of several inter-spaced layers of fluid and solid using ghost nodes. This work is inspired by the previous works done, in the field of Fluid–Solid Interaction (FSI), specially the Ghost Fluid Method (GFM). The GFM was introduced in a pioneering work by Fedkiw et al. [1]. Subsequently, another version of the method was developed in particular for gas–water flow by Fedkiw [8]; this is nominally referred to as gas–water GFM. To facilitate discussion, the above-mentioned GFM in [1] is referred to here as the original GFM (OGFM) to distinguish it from other modified versions. The OGFm with its inherent simplicity can be easily extended to multi-dimensions and applied to physical problems of fairly complex geometry. These characteristic features of OGFm have led to development of similar methods for simulating multi-medium flow [9–12].

Apart from the simplicity, the OGFm appears to be rather problem-related. It has been shown that the OGFm is not quite suitable for extreme conditions like the case of high speed jet impact problems [13]. This is largely attributed to the fact that the OGFm essentially does not take into account the effect of wave interaction at the interface and the material properties. To overcome the limitation, Liu et al. [14] proposed the modified GFM (MGFM) algorithm. Subsequent to that, the real GFM (RGFM) was developed by Wang et al. [15] as a variant of MGFM-based algorithm. The latter two MGFM have been successfully applied to different extreme cases of gas–gas, gas–water, and even fluid–structure problems [13–19]. It is fairly clear that the MGFM are much less problem-related and can be used more widely.

Despite the apparent success in the application to the various multi-medium problems, there appears no attempt to explore the applicability to solid–solid interaction. This mostly provides the motivation of the present work.

2. 1D waves and solid–solid interaction

In this section, the one-dimensional elastic solid–solid interaction is investigated. We start with the Original Ghost Solid Method (OGSM), to be followed by two variants in the form of the Modified Ghost Solid Method (MGSM) and the Double Riemann Ghost Solid Method (DRGSM). The advantages and possible disadvantages of each of these methods are discussed and compared. These numerical methods are then validated and compared using numerical experiments.

2.1. Governing equation

The Cauchy equation of motion at any point inside a solid can be written in tensor notation as:

$$\rho b_i + \sigma_{j,i} - \rho a_i = 0$$

(1)

where $\rho$ is density of the material, $\vec{b}$ is the body force, $\sigma$ is the stress tensor, and $\vec{a}$ is the acceleration. Considering the body forces are negligible, Eq. (1) can be simplified to

$$\sigma_{j,i} - \rho a_i = 0.$$  

(2)

For the case of pure shear, in a one-dimensional setting, one can further simplify Eq. (2) to

$$\frac{\partial \sigma}{\partial x} - \rho \frac{\partial u}{\partial t} = 0.$$  

(3)

In this work, for the closure of the system, Hooke’s law is used as the constitutive equation. For 1D isotropic linearly elastic solid, it can be written as

$$\sigma = E \frac{\partial \varepsilon}{\partial x}.$$  

(4)

where $\varepsilon$ is the displacement in the $x$ direction, and $E$ is the modulus of elasticity. If Eq. (4) is differentiated with respect to time, this leads to:

$$\frac{\partial \sigma}{\partial t} = E \frac{\partial}{\partial t} \left( \frac{\partial \varepsilon}{\partial x} \right) = E \frac{\partial}{\partial x} \left( \frac{\partial \varepsilon}{\partial t} \right) = E \frac{\partial u}{\partial x}.$$  

(5)

since

$$u = \frac{\partial \varepsilon}{\partial t}.$$  

By using Eqs. (3) and (5), the governing equation for pressure wave propagation, in a semi-infinite isotropic linearly elastic solid, can be formulated as

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0.$$  

(6)
in conservative form, or the following non-conservative form given as
\[
\frac{\partial U}{\partial t} + A \frac{\partial U}{\partial x} = 0
\]
(7)
where
\[
U = \begin{bmatrix} u \\ p \end{bmatrix}, \quad F = \begin{bmatrix} p/\rho \\ E u \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1/\rho \\ E & 0 \end{bmatrix}.
\]
Here, \(u\) is the velocity in the \(x\) direction, \(p = -\sigma\), \(\rho\) is the density of the solid, and \(E\) is the modulus of elasticity. It is noticed that \(c = \sqrt{E/\rho}\) is the speed of sound in the elastic solid [20].

2.2. The Riemann problem for the linearly elastic solid–solid interface

The Riemann problem is given as
\[
\frac{\partial U}{\partial t} + A \frac{\partial U}{\partial x} = 0, \quad U(x, 0) = \begin{cases} U_L & x < x_I, \\ U_R & x > x_I, \end{cases}
\]
(8)
where \(x_I\) is a reference length for the problem. For the solid–solid interaction problem, \(x_I\) can be considered as the location of the interface. The subscripts \(L\) and \(R\) refer to the values on the left and right side of the interface, respectively. The subscript \(I\) refers to the interfacial values.

The objective is to find the values of \(U_I = U(x_I, 0)\). One can now solve the stated Riemann problem as illustrated in Fig. 1. In the leftward wave region, the information propagates along the characteristic \(x/t = c_L\). Therefore, in this region, the following characteristic equation holds true:
\[
\frac{d}{dt} \left( p + \frac{E_L}{c_L} u \right) = 0 \Rightarrow dp + \frac{E_L}{c_L} du = 0.
\]
(9)
Similarly, in the rightward wave region, the information propagates alongside the characteristic line \(x/t = -c_R\). Therefore, in this region, the following characteristic equation can be considered. That is,
\[
\frac{d}{dt} \left( p - \frac{E_R}{c_R} u \right) = 0 \Rightarrow dp - \frac{E_R}{c_R} du = 0.
\]
(10)
Now, one can integrate (9) as
\[
\int_{p_L}^{p_L} dp + \frac{E_L}{c_L} \int_{u_L}^{u_I} du = 0 \Rightarrow (p_L - p_L) + \frac{E_L}{c_L} (u_I - u_L) = 0
\]
(11)
and (10) as
\[
\int_{p_R}^{p_R} dp - \frac{E_R}{c_R} \int_{u_R}^{u_I} du = 0 \Rightarrow (p_R - p_R) - \frac{E_R}{c_R} (u_I - u_R) = 0.
\]
(12)
The subscripts \(I_L\) and \(I_R\) refer to the interfacial values when integrated on the leftward and rightward wave regions, respectively. With continuity and balance of force, we have
\[
u_{IL} = u_{IR} = u_I, \quad p_{IL} = p_{IR} = p_I.
\]
(13)
Using (11), (12), and (13), one can derive at the interfacial values $u_I$ and $u_L$ as
\begin{equation}
    u_I = \frac{c_I c_R (p_L - p_R) + E_I c_R u_L + E_R c_L u_R}{E_I c_R + E_R c_L}
\end{equation}
and
\begin{equation}
    p_I = p_L - \frac{E_I}{c_L} \left( \frac{c_I c_R (p_L - p_R) + E_I c_R (u_R - u_L)}{E_I c_R + E_R c_L} \right).
\end{equation}

It is worthwhile to mention, in the derivations from (13) to (15), since the values of $U_L$ and $U_R$ are assumed to be uniform, on the left and right side of the interface, one can choose appropriate locations on the left and right side of the interface, such that the characteristics originating from these points meet at the interface. Therefore, in these derivations, it is reasonable to assume the characteristics meet at the interface.

This Riemann solver is used below in conjunction with the proposed numerical methods. One key issue is the assignment of the appropriate real values for the leftward and rightward regions, and selection of the (numerical) interface.

2.3. GSM-based algorithms

2.3.1. Outline of various ghost solid methods

For the Ghost Solid Method (GSM)-based algorithms, we shall assume that the solution at time $t = t^n$ is known. In the implementation, usually a band of 2 to 5 grid points is defined as ghost nodes, in the neighborhood of the solid–solid interface. Note that the minimum number of required ghost nodes, which in a particular application may be only one ghost node or even more than 5 ghost nodes, depends on the computational cell of the single medium solver which is employed. At each ghost node, ghost solid and real solid are both present. In the GSM-based algorithm for a multi-medium interaction problem, one has to solve for two separate 1-medium Riemann problems at each time step. One is for the left solid medium with the following initial conditions
\begin{equation}
    \frac{\partial U}{\partial t} + A \frac{\partial U}{\partial x} = 0, \quad U(x, 0) = \begin{cases} U^n_L & x < x_I, \\ U^n_R & x > x_I. \end{cases}
\end{equation}
Here, the '*' sign is used to represent the ghost status at $t = t^n$. Eq. (16) solves the Riemann problem from the first grid node on the left to the ghost node on the right of the interface. Similarly, the other Riemann problem for the right solid medium is given as
\begin{equation}
    \frac{\partial U}{\partial t} + A \frac{\partial U}{\partial x} = 0, \quad U(x, 0) = \begin{cases} U^n_L & x < x_I, \\ U^n_R & x > x_I. \end{cases}
\end{equation}
It solves from the ghost node on the left of the interface to the last node on the right. To solve the above-mentioned (16) and (17) Riemann problems, it is essential to assign the values of $U^n_L$ and $U^n_R$ on their respective ghost nodes properly. Once these ghost values are defined, at the time step $t = t^n$, any desired solid solver can be used to advance the solution of (16) and (17) independently.

Depending on the method which is used to define these values on the ghost nodes, various methods of GSM are therefore developed. Three types of GSM-based algorithms are proposed and formulated below. It is worthwhile, however, to reiterate the lemma by Liu et al. [21]:

Lemma. The GSM Riemann problems (16) and (17) provide a solution which is identical to that of the Riemann problem (8) in their respective solid fields, provided that their ghost solid states are respectively defined as the exact interfacial states, $U^n_L = U_{IL}$ and $U^n_R = U_{IR}$.

2.3.2. On the Original Ghost Solid Method (OGSM)

This method follows the pioneering works by Fedkiw [18] on ghost fluid method. Here, the local real solid velocity and stress are simply copied to the corresponding ghost solid nodes. The Young modulus as well as the speed of sound is copied from the real solid to its corresponding ghost solid node, on the other side of the interface. By assuming that the interface lies between node $i$ and $i + 1$, and a band of 2–5 ghost nodes on each side of interface, this method can be formulated as:
\begin{align}
    U^n_L |_{x=x_j} &= U^n_R, \quad A^n_j = A_{i+1}, \quad i - 4 \leq j \leq i, \\
    U^n_R |_{x=x_j} &= U^n_R, \quad A^n_j = A_i, \quad i + 1 \leq j \leq i + 5.
\end{align}
Fig. 2 illustrates the process of defining the ghost nodes on the right-hand side of the interface. Remember, the minimum number of required ghost nodes, which in a particular application may be one or more than 5, depends on the computational cell of the single medium solver which is employed. For a first-order Godunov solver, only one ghost node is required on each side of the interface. The simplicity lies in that no Riemann problem needs to be solved to define the values of

\[ \text{Lemma.} \quad \text{The GSM Riemann problems (16) and (17) provide a solution which is identical to that of the Riemann problem (8) in their respective solid fields, provided that their ghost solid states are respectively defined as the exact interfacial states, } U^n_L = U_{IL}, \text{ and } U^n_R = U_{IR}. \]
Fig. 2. Schematic illustration of OGSM for defining ghost solid status for medium 1.

ghost nodes. Moreover, no system of linear equations need to be solved along with the solid solver in comparison to other methods such as the Immersed Interface Method [22] (the IIM requires solving a system of 12 equations for 12 unknowns in 1D, and 54 equation for 54 unknowns in 2D). However, as will be discussed below, under certain settings it can, and will, result in non-physical oscillations in the velocity as well as the stress waves.

In the following section on numerical experiments, it is shown the OGSM can lead to non-physical oscillations even for first-order methods. It is further demonstrated that these oscillations can become even more severe using the higher order schemes.

2.3.3. On the Modified Ghost Solid Method (MGSM)

Here, we assume the modulus of elasticity and the speed of sound are copied from real solid to its corresponding ghost solid nodes on the other side of the interface. We shall define the ghost node values such that the interfacial values predicted by the left and right single mediums will be identical to that of the multi-medium problem. In other words, we shall ensure the solutions to the problems (16) and (17) for the interfacial values are identical to those of (8).

Considering the left medium and the ghost nodes on the right side of the interface, similar to Section 2.2, characteristic analysis of (16) reveals:

\[
(p_I - p_L) + \frac{E_L}{c_L}(u_I - u_L) = 0, \quad (19)
\]

in the leftward wave region, and

\[
(p_I - p_R^*) - \frac{E_L}{c_L}(u_I - u_R^*) = 0, \quad (20)
\]

in the rightward region. \(u_I\) and \(p_I\) are the interfacial values obtained in Section 2.2. It can be seen that Eqs. (19) and (11) are identical. Hence, Eq. (19) is automatically satisfied. Moreover, it is clear that Eq. (20) will be satisfied if \(u_R^* = u_I\) and \(p_R^* = p_I\). Using a similar analysis for the right medium, and the ghost solid nodes on the left side of the interface, we can conclude that \(u_L^* = u_I\) and \(p_L^* = p_I\).

Now, assuming that the interface lies between the nodes \(i\) and \(i + 1\), the interfacial values of velocity and stress can be approximated by assuming that

\[
U_L = U^n_I, \quad U_R = U^n_{i+1} \quad (21)
\]

and similarly

\[
A_L = A_i, \quad A_R = A_{i+1} \quad (22)
\]

as in Section 2.2. The approximate interfacial values of \(u_I\) and \(p_I\) can be calculated using Eqs. (14) and (15) as

\[
u_I = \frac{c_i c_{i+1}(p^n_I - p^{i+1}_L) + E_i c_{i+1} u^n_I + E_{i+1} c_i u^n_{i+1}}{E_{i+1} c_i + E_{i+1} c_{i+1}}, \quad (23)
\]

\[
p_I = p^n_I - \frac{E_i}{c_i} \left(\frac{c_i c_{i+1}(p^n_I - p^{i+1}_L) + E_i c_{i+1} (u^n_{i+1} - u^n_I)}{E_{i+1} c_i + E_{i+1} c_{i+1}}\right). \quad (24)
\]

Therefore, the ghost nodes can be defined as:

\[
U^n_{L\mid x=x_j} = U^n_I, \quad A^n_{j} = A_{i+1}, \quad i - 4 \leq j \leq i.
\]

\[
U^n_{R\mid x=x_j} = U^n_I, \quad A^n_{j} = A_{i}, \quad i + 1 \leq j \leq i + 5. \quad (25)
\]

In Eqs. (23) to (24), a uniform profile of \(U\) is assumed from \(x_i\) to \(x_I\) and from \(x_I\) to \(x_{i+1}\). Hence, the interfacial values are calculated at time step \(n\), and they are used to update the solution from time step \(n\) to \(n + 1\). Given that CFL \(\leq 1\), and the
mesh is fine enough to capture the incident wave, the assumption leads to reasonably accurate results. Fig. 3 illustrates the method schematically. One may note that the same values of $u_I$ and $p_I$ are used for both the left and right ghost nodes. This means that only a single Riemann problem needs to be solved in order to define the ghost node values.

By numerical experiment, it will be shown that this method can greatly mitigate or eliminate the non-physical oscillations seen for the OGSM. The disadvantage of this method vis-a-vis the OGSM is that it involves solving for a Riemann problem at the interface.

2.3.4. On the stability of the OGSM and MGSM

For the instability of the numerical solution to be caused by, or associated with, the use of the GSMs, and not by the solid solver of choice, the leading numerical error must be traced to the implemented GSMs. Here, we will follow the Lax–Richtmyer stability analysis [23,24].

As the effects of the GSMs are sensed closest to the interface at each time step, it is reasonable to assume that the maximum error, due to the GSMs, may occur at the interface. As discussed in Section 2.3.3, if the MGSM is employed, $U_I$ from the MGSM and the multi-medium solution will be identical. Hence, the error incurred due to the MGSM will be theoretically zero, if the exact solution of the Riemann problem is used. Consequently, the stability of the solution will unconditionally associated with the stability of the single medium solver employed; i.e. $\|E^n\|_\infty$ will not be determined by the use of MGSM. Similarly, we can conclude if an approximate Riemann solver is used, the stability of the solution will be determined by both the approximate Riemann solver and the single medium solver employed together with the MGSM.

However, if the OGSM is employed, the interfacial values calculated, from the left and right mediums, are not necessarily identical to those of the exact solution of the multi-medium problem. Using the ghost solid values, and the multi-medium Riemann problem values, the error in velocity, at the interface, for the left medium is:

$$|E_{uL}| = \left| \left( \frac{c_R c_L}{E_{LCR} + E_{RCL}} - \frac{c_L^2}{2E_L c_L} \right) (p_L - p_R) + \left( \frac{E_{LCR}}{E_{LCR} + E_{RCL}} - \frac{1}{2} \right) u_L + \left( \frac{E_{RCL}}{E_{LCR} + E_{RCL}} - \frac{1}{2} \right) u_R \right|. \quad (26)$$

As it can be seen, the above error has no upper bound in general. We can obtain similar relations for stress in the left medium. Moreover, we can obtain similar results for the right medium. This means that the error due to the use of the OGSM can be generally unbounded and lead to instabilities. In other words, $\|E^n\|_\infty$ may be determined by and associated with the error incurred by the OGSM. These large errors may become evident in the form of spurrious oscillations, or complete instabilities in the solution, even when a first-order solid solver is used. Higher order schemes may tend to switch to lower order schemes, specifically a first-order scheme, which is required by the Godunov theorem, to avoid spurrious oscillations [20] in the presence of large discontinuities. As a result, the use of a higher order scheme, may not be successful in rectifying the instabilities caused by the use of the OGSM.

If we want to minimize this error for the OGSM, we derive at the special case of the acoustic impedance matching where the error given by (26), and equivalents of it, will become identically zero and the stability becomes synonymous with the stability of the solid solver employed for the single medium. In this special case, the calculated interfacial values from the left and right medium become identical to those of the multi-medium problem. Moreover, it will be the only case where the interfacial values from the left and right mediums become identical. The special case of the acoustic impedance matching and its stable characteristics can be used as reference and guide to predict the stability of the (more general) OGSM. To quantify, a much simpler dimensionless parameter, $\vartheta$, (than that of equivalents of (26) made applicable for both velocity and stress) is proposed:

$$\vartheta = \max \left( \frac{|u_L - u_R|}{|u_L| + |u_R|}, \frac{|p_L - p_R|}{|p_L| + |p_R|} \right)$$

(27)

where the subscript $IL$ and $IR$ refer to the interfacial values obtained from the left and right single medium solutions, respectively. The $\vartheta$ value quantifies how close the numerical situation is to the acoustic impedance matching case. Our
extensive numerical tests indicate that the maximum permissible value of $\vartheta$, before non-physical oscillations are observed, is $\vartheta_{\text{crit}} \approx 0.1$. The above discussion not only explains the origin/reason for possible non-physical oscillations, but also provides a means forward to determine the applicability of the OGSM. For the latter, one can ascertain beforehand if $\vartheta$ is within the permissible range to ensure the non-physical oscillations are kept to an acceptable level as time progresses. If at any time step $\vartheta$ exceeds the value of 0.1, the OGSM as applied to across the interface can no longer be considered as a viable approach at that time step. One may then need an alternative GSM at that time step. Subsequently, if the value of $\vartheta$ drops below 0.1, the OGSM can be reinstated for use due to its simplicity.

It is worth noting that calculation of $\vartheta$ is not computationally expensive, specially, if a TVD solver is used; the first-order fluxes may be readily available to calculate $\vartheta$.

2.3.5. On the Double Riemann Ghost Solid Method (DRGSM)

This is a variant of the MGSM. A similar variant MGFM was recently developed by Liu et al. [21] to capture fluid–solid interaction due to very strong shock impacting the interface in the Eulerian–Lagrangian coupling. Here, instead of solving a single Riemann problem at the interface, a separate Riemann problem is defined and solved for each side of the interface. To obtain the ghost values on the ghost nodes, one has to solve the approximate Riemann problem by assuming that the interface lies on the ghost node just beside the actual interface. This interface can be referred to as a ghost interface. Next, using the real values of its neighboring nodes a Riemann problem is formulated. As soon as this Riemann problem is solved for the ghost node that lies just beside the interface, the values of $U = [u, p]^T$ on this node can be copied to its corresponding ghost nodes.

Similar to Section 2.3.3, it is assumed that the actual interface lies between the nodes $i$ and $i+1$. To obtain the values of $U^R_\text{IR}$, in (8), the following Riemann problem is defined:

$$\frac{\partial U}{\partial t} + A \frac{\partial U}{\partial x} = 0, \quad U(x, t = t^n) = \begin{cases} U^n_i, & x < x_{i+1}, \\ U^n_{i+2}, & x > x_{i+1}. \end{cases} \quad (28)$$

The physical properties of problem (33) are defined as

$$A(x) = \begin{cases} A_i, & x < x_{i+1}, \\ A_{i+2}, & x > x_{i+1}. \end{cases} \quad (29)$$

Fig. 4 illustrates the position of the ghost interface as well as the closest ghost node to the actual interface. One can then solve for the values of $u$ and $p$ on the node $i+1$. By comparing Eqs. (28) and (29) with the problem (8), one can solve for the solution at the ghost interface:

$$u_{IR} = \frac{c_i c_{i+2} (p^n_i - p^n_{i+2}) + E_i c_{i+2} u^n_i + E_{i+2} c_i u^n_{i+2}}{E_i c_{i+2} + E_{i+2} c_i} \quad (30)$$

and

$$p_{IR} = p^n_i - \frac{E_i}{c_i} \left( c_i c_{i+2} (p^n_i - p^n_{i+2}) + E_i c_{i+2} (u^n_{i+2} - u^n_i) \right) \quad (31)$$

where the index $l_R$ refers to the values on the ghost interface applicable on the node on the right side of the interface. In this setting, this index refers to the ghost interface at the location $x = x_{i+1}$. Using these values, one can then properly define values for the rest of the ghost nodes on the right side:
Fig. 5. Schematic illustration of the definition of the ghost properties in DRGSM method, for (a) the ghost nodes on the right side of the interface, and (b) the ghost nodes on the left-hand side of the interface.

\[ U^*_{R|j} = \begin{bmatrix} u_{IR} \\ p_{IR} \end{bmatrix}, \quad A_j = A_{i+1}, \quad i + 1 \leq j \leq i + 5. \quad (32) \]

Similarly, one can then define the following Riemann problem for the node just on the left side of the interface. Assuming that the ghost interface lies on the node \( i \), we have

\[ \frac{\partial U}{\partial t} + A \frac{\partial U}{\partial x} = 0, \quad U(x, t = t^n) = \begin{cases} U^n_{i-1} & x < x_i, \\ U^n_{i+1} & x > x_i, \end{cases} \quad (33) \]

and the material properties as

\[ A(x) = \begin{cases} A_{i-1} & x < x_i, \\ A_{i+1} & x > x_i. \end{cases} \quad (34) \]

Therefore, by comparing (33) and (34) with (8) one can get the solution for the ghost interface at \( x = x_i \):

\[ u_{IL} = \frac{c_{i-1}c_{i+1}(p^n_{i-1} - p^n_{i+1}) + E_{i-1}c_{i+1}u^n_{i-1} + E_{i+1}c_{i-1}u^n_{i+1}}{E_{i-1}c_{i+1} + E_{i+1}c_{i-1}} \quad (35) \]

and

\[ p_{IL} = p^n_{i-1} - \frac{E_{i-1}c_{i+1}(c_{i-1}c_{i+1}(p^n_{i-1} - p^n_{i+1}) + E_{i-1}c_{i+1}(u^n_{i-1} - u^n_{i+1}))}{E_{i-1}c_{i+1} + E_{i+1}c_{i-1}} \quad (36) \]

where the subscript \( IL \) refers to the ghost interface on the left-hand side of the real interface. Next, these values are copied to the ghost nodes on the left-hand side of the interface:

\[ U^*_{L|j} = \begin{bmatrix} u_{IL} \\ p_{IL} \end{bmatrix}, \quad A_j = A_{i+1}, \quad i - 4 \leq j \leq i. \quad (37) \]

Fig. 5 shows schematically how the ghost properties are copied to the ghost nodes on the right and left side of the interface.

Numerical experiments below will show that DRGSM is able to eliminate the non-physical oscillations which occur for the GSM. However, comparing to the MGSM and GSM, this method is more complicated and involved. Numerical experiments will also show that the improvements are not so significant when compared to MGSM, despite the greater effort required.

3. 2D implementation of the GSMs

By using the techniques developed in Section 2, in the normal direction of the interface, one can readily extend to multi-dimensional GSM-based algorithms. One should note, however, that there are additional boundary conditions, more specifically the slip and the no-slip boundary conditions along the interface not applicable for the 1D problem.
3.1. Governing equation

The governing equation for an isotropic, linearly elastic solid, in a Cartesian frame of reference, can be formulated as the following, in the conservative form. 

\[
\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} + \frac{\partial G(U)}{\partial y} = 0.
\]  
(38)

Here, 

\[U = \begin{bmatrix}
\rho u_x \\
\rho u_y \\
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{xy}
\end{bmatrix}, \quad F(U) = \begin{bmatrix}
\sigma_{xx} \\
\sigma_{xy} \\
\rho \alpha^2 u_x \\
(\alpha^2 - 2\beta^2) \rho u_x \\
\rho \beta^2 u_y
\end{bmatrix}, \quad G(U) = \begin{bmatrix}
\sigma_{xy} \\
\sigma_{yy} \\
(\alpha^2 - 2\beta^2) \rho u_y \\
\rho \alpha^2 u_y \\
\rho \beta^2 u_x
\end{bmatrix}
\]  
(39)

where \(\rho\) is the density, \(u_x\) and \(u_y\) are velocity at of each point in the \(x\) and \(y\) direction, respectively; \(\sigma_{xx}\) and \(\sigma_{yy}\) are the normal components of the stress tensor in the \(x\) and \(y\) direction; \(\sigma_{xy}\) is the tangential component of the stress and \(\alpha\) and \(\beta\) are the longitudinal and transverse wave speeds, respectively. These are 

\[
\alpha = \sqrt{\frac{2\mu + \lambda}{\rho}}, \quad \beta = \sqrt{\frac{\mu}{\rho}}
\]  
(40)

where \(\mu\) and \(\lambda\) are the Lamé constants.

Eq. (38) can be written in the normal–tangential frame of reference (\(\eta–\xi\)) as 

\[
\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial \eta} + \frac{\partial G(U)}{\partial \xi} = 0.
\]  
(41)

Here, the variables \(\eta\) and \(\xi\) are used to denote normal and tangential coordinates (see Fig. 6). Under this framework, 

\[U = \begin{bmatrix}
\rho u \\
\rho v \\
\sigma_{\eta\eta} \\
\sigma_{\xi\xi} \\
\sigma_{\eta\xi}
\end{bmatrix}, \quad F(U) = \begin{bmatrix}
\sigma_{\eta\eta} \\
\sigma_{\xi\xi} \\
\rho \alpha^2 u \\
(\alpha^2 - 2\beta^2) \rho u \\
\rho \beta^2 v
\end{bmatrix}, \quad G(U) = \begin{bmatrix}
\sigma_{\eta\xi} \\
\sigma_{\xi\xi} \\
(\alpha^2 - 2\beta^2) \rho v \\
\rho \alpha^2 v \\
\rho \beta^2 u
\end{bmatrix}
\]  
(42)

where \(u\) is the velocity in the normal direction (\(\eta\)), \(v\) is the velocity in the tangential direction (\(\xi\)), and \(\sigma_{\eta\eta}, \sigma_{\xi\xi}, \) and \(\sigma_{\eta\xi}\) are the stress components in the normal–tangential coordinate reference frame.

3.2. No-slip and perfect-slip conditions at the interface

Different boundary conditions can arise at the interface. We shall only discuss the no-slip and perfect-slip boundary conditions.
3.2.1. No-slip condition at the interface

If the two solids cannot slide at the interface and in the absence of any gap at the interface, then a no-slip boundary condition is appropriate.

The no-gap formation at the interface implies the continuity of the normal velocity $u$ at the interface

$$u_{IL} = u_{IR} = u_I. \quad (43)$$

Moreover, it means that the normal component of the traction can be non-zero, and equal for both solids. Consequently, the boundary force balance implies

$$\sigma_{\eta\eta I_L} = \sigma_{\eta\eta I_R} = \sigma_{\eta\eta I}. \quad (44)$$

The no-sliding between the two solids suggests that the relative tangential velocity is zero at the interface. Hence, the tangential velocity $v$ will be continuous across the interface

$$v_{IL} = v_{IR} = v_I. \quad (45)$$

It also implies that the tangential component of the traction can be non-zero and equal for both solids. Subsequently, boundary force balance leads to

$$\sigma_{\xi\eta I_L} = \sigma_{\xi\eta I_R} = \sigma_{\xi\eta I}. \quad (46)$$

3.2.2. Perfect-slip condition at the interface

For this interfacial boundary condition, no gap is allowed to be formed at the interface. However, the solids can slide against each other.

Similar to the previous section, the requirement of an absence of gap at the interface leads to conditions identical to (43) and (44) for the normal velocity ($u$) and the stress component ($\sigma_{\eta\eta}$). However, allowing the solids to slide, without any friction at the interface, will render conditions (45) and (46) inapplicable.

3.3. Coupled and uncoupled variables

Consider a variable $\chi$. The subscripts $I_L$ and $I_R$ are used to indicate if an interfacial value is calculated on the left or right side of the interface, respectively. If due to the boundary conditions at $x$, there exists a relation $\kappa$ such that

$$\kappa(\chi_{IL}, \chi_{IR}) = 0, \quad (47)$$

then $\chi$ is considered to be a coupled variable across the interface at that point. Otherwise, it is uncoupled.

3.4. On the 2D OGSM

Here, the extension of the OGSM method given in Section 2.3.2 is presented. Similar to its 1D counterpart, it can be easily applied in practice. Following Fedkiw [8,1], for the coupled variables one has to copy the values of the real nodes to the ghost nodes in the same region just like for the one-dimensional setting. Variables which are not coupled across the interface, such as material properties, are generally discontinuous and need to be extrapolated across the interface into the ghost nodes.

3.4.1. The OGSM for the no-slip condition at the interface

Conditions (43), (44), (45), and (46) must be satisfied at the interface for the values of $u$, $\sigma_{\eta\eta}$, $v$, and $\sigma_{\xi\eta}$. Moreover, the Cauchy equation of motion must hold which constraints the admissible values of $\sigma_{\xi\xi}$. As such, the variable $U$ can be considered as a coupled variable. To define the ghost values $U_{IL}^*$ and $U_{IR}^*$, one has to simply copy the values of $U^n$ at each time step from the closest real node, on the same side of the interface, to the corresponding ghost node. In Fig. 7, consider the ghost node $A$ on the right-hand side of the interface. In order to define the ghost values at this node, one has to first find the closest real node, $B$. As soon as this node is found, one can use a simple copy to define the ghost values as

$$U_{IR}^*|_A = U_B. \quad (48)$$

To circumvent a lengthy search process, it is possible to define the location of the ghost nodes such that they coincide with real solid nodes. In this way, only a simple copy is necessary for defining the ghost values.

3.4.2. The OGSM for the perfect-slip condition at the interface

According to Section 3.2.2, for this case, only the conditions (43) and (44) need to be satisfied at the interface. As such, the only coupled variables are $u$ and $\sigma_{\eta\eta}$. Hence, at the time step $t = t^n$ only the values of $u^n$ and $\sigma_{\eta\eta}^n$ need to be copied from the closest real node to the ghost node. The values of $v^n$, $\sigma_{\xi\xi}^n$, and $\sigma_{\xi\eta}^n$, as well as the material properties need to be extrapolated from the real nodes, across the interface, into the ghost nodes.
3.5. On the 2D MGSM

This is the extension of MGSM in 1D which was developed in Section 2.3.3. One has to construct and solve an appropriate Riemann problem to determine the values of the coupled variables at the interface to be copied to the ghost nodes. In addition, special attention needs to be paid to $\sigma_{\xi \xi}$, the normal component of the stress tensor, which is in the tangential direction of the interface. Similar to OGSM, the uncoupled variables are extrapolated across the interface into the ghost nodes.

The interfacial points are Lagrangian points and their locations are known at each time step. The locus of the interfacial points forms a curve. If the coordinates of the interfacial points are $(X_I, Y_I)$ and $S$ is the parametrization variable used to parametrize the curve, the unit normal to the interface $(N)$ is given by

$$N = \left( \frac{\partial Y_I}{\partial S} \mathbf{i} - \frac{\partial X_I}{\partial S} \mathbf{j} \right) \sqrt{\frac{\partial X_I^2}{\partial S} + \frac{\partial Y_I^2}{\partial S}}$$

(49)

where $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors in the $x$–$y$ coordinate system.

Considering node $A$ (see Fig. 8), which is just bordering the interface, we have the $U_L$ which is equal to $U_A$. Next, we search for another node, $B$, that is bordering the right side of the interface such that the angle between the line, $AB$, and the interface normal is minimal. The value of $U_B$ at this point is considered as the value $U_R$ on the interface normal. Usually, interpolation may be necessary to find a proper value for $U_R$. Wang et al. [15] have provided a correction algorithm for the values on the real side, specially for critical problems such as shock impedance matching for compressible flow. However, our critical tests in shock impedance matching indicate that such corrections are not quite essential for the elastic solid–solid interactions. Otherwise, one may adopt [15] which has been shown to be reasonably robust.

Next, we shall define the following Riemann problem in the perpendicular direction to the interface,

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial \eta} = 0,$$

$$U(\eta, t = t^n) = \begin{cases} U^m_L & \eta < 0, \\ U^m_R & \eta > 0. \end{cases}$$

(50)
3.5.1. On the no-slip condition at the interface and MGSM

By integrating the characteristics equations of the Riemann problem (50), and applying the conditions (43) and (44), we obtain

\[ u_I = \frac{1}{\rho_L \alpha_L + \rho_R \alpha_R} (\sigma_{\eta\eta} - \sigma_{\eta\eta_R} + \rho_L \alpha_L u_L + \rho_R \alpha_R u_R), \] (51)

\[ \sigma_{\eta\eta} = \frac{1}{\rho_L \alpha_L + \rho_R \alpha_R} \left[ \rho_L \alpha_L \sigma_{\eta\eta_R} + \rho_R \alpha_R \sigma_{\eta\eta_R} + \rho_L \alpha_L \rho_R \alpha_R (u_L - u_R) \right], \] (52)

\[ v_I = \frac{1}{\rho_L \beta_L + \rho_R \beta_R} (\sigma_{\xi\eta} - \sigma_{\xi\eta_R} + \rho_L \beta_L v_L + \rho_R \beta_R v_R), \] (53)

and

\[ \sigma_{\xi\eta} = \frac{1}{\rho_L \beta_L + \rho_R \beta_R} \left[ \rho_L \beta_L \sigma_{\xi\eta_R} + \rho_R \beta_R \sigma_{\xi\eta_R} + \rho_L \beta_L \rho_R \alpha_R (v_L - v_R) \right]. \] (54)

Integrating the zero characteristic of Eq. (50) on the left side results in

\[ q_I = q_L + \frac{\gamma_L}{\alpha_L^2} (p_I - p_L). \] (55)

However, if it is integrated on the right-hand side, one can get

\[ q_R = q_R + \frac{\gamma_R}{\alpha_R^2} (p_I - p_R). \] (56)

Using the obtained values of \( u_I, v_I, p_I, q_I, q_{I_R} \) and \( \tau_I \), one can then construct \( U_{I_L} \) and \( U_{I_R} \) as

\[
U_{I_L} = \begin{bmatrix}
u_I \\ v_I \\ p_I \\ q_{I_L} \\ \tau_I
\end{bmatrix}
\quad \text{and} \quad
U_{I_R} = \begin{bmatrix}
u_I \\ v_I \\ p_I \\ q_{I_R} \\ \tau_I
\end{bmatrix}.
\] (57)

Values of \( U_{I_L} \) can be copied to the ghost nodes on the right-hand side of the interface which are closest to the normal line that exits the point where \( U_{I_L} \) was computed at (like the ghost node C in Fig. 8). In a similar manner, \( U_{I_R} \) values will be copied to the proper ghost nodes on the left side of the interface.

3.5.2. On the slip condition at the interface and MGSM

Here, only the conditions (43) and (44) must be satisfied for this type of interface condition. Hence, only the values of \( u \) and \( p \) are coupled.

Similar to Section 3.5.1, \( u_I \) and \( \sigma_{\eta\eta} \) are identical to Eqs. (51) and (52), respectively. Moreover, the zero characteristic of Eq. (50) are integrated on the left and right side of the interface to obtain results identical to Eqs. (55) and (56).

The values of \( u_I \) and \( \sigma_{\eta\eta} \) are then copied to the ghost nodes on the right-hand side and left-hand side of the interface which are closest to the normal line that exits at the point where they are calculated. In a similar manner, \( \sigma_{\xi\eta} \) and \( \sigma_{\xi\xi} \) values will be copied to the proper ghost nodes on the right side and the left side of the interface, respectively.

The values of uncoupled variables, \( v, \sigma_{\xi\eta}, \) and material properties are extrapolated across the interface, accordingly.

4. Numerical experiments

In the following numerical experiments, all computations are carried out in non-dimensional form and the solid mediums on each side of the interface are considered to be homogeneous, isotropic, and linearly elastic solids.

4.1. Test Example 1: On possible non-physical oscillations on the use of OGS and the critical \( \vartheta \) value

This experiment is designed to show the non-physical oscillations which may rise due to the use of the OGS. Moreover, as a test example, it indicates broadly how the critical permissible \( \vartheta \) value of 0.1 has been determined. The domain of the solution is \([0, 10]\) and the interface is located at \( x_I = 5 \). The initial velocity \( u(x, 0) = 1 \) for \( x \in [0, 5] \) and zero otherwise, and the initial normal stress \( p(x, 0) \) is zero for all \( x \). The boundary conditions are \( u(0, t) = 1 \) and \( u(10, t) = 0 \).

Two sets of material properties are assumed:

1. \( \rho_L = 1 \) and \( E_L = 1 \) on the left-hand side of the interface, and \( \rho_R = 1.4 \) and \( E_R = 1.4 \) on the right side;
2. \( \rho_L = 1 \) and \( E_L = 1 \) on the left-hand side of the interface, and \( \rho_R = 5 \) and \( E_R = 5 \) on the right side.
Fig. 9. Test Example 1: Comparison of the velocity and stress profiles between the exact solution, OGSM, MGSM, DRGSM and CLAWPACK ($\rho_L = 1$, $E_L = 1$, $\rho_R = 1.4$, $E_R = 1.4$, and $t_f = 0.3$).

The first-order Godunov method is used as the solid solver for each solid medium with the spatial discretization of $\Delta x = 0.01$. The maximum CFL number [25] is considered to be ($c_{\text{max}} \Delta t / \Delta x = 0.98$).

Fig. 9 shows the velocity and stress profile for the first set of materials at $t_f = 0.3$, calculated using the OGSM, MGSM, and DRGSM. The results are compared against the analytical as well as the CLAWPACK [26] solution. The calculated value of $\vartheta$ (introduced in Section 2.3.4) for this problem when the OGSM is employed reaches a maximum value of 0.09 and remains below 0.1 (the proposed critical value $\vartheta_{\text{crit}}$) for all time steps. The results for the OGSM concur well with the analysis and there is no observed oscillation in the velocity and stress predictions.

Fig. 10 shows the solution obtained for the second set of data at $t_f = 0.3$. For almost all the time steps, the calculated value of $\vartheta$ when the OGSM is employed varies between 1.0 and 0.1. Thus, it exceeds the proposed critical value of 0.1 employed to avoid any non-physical oscillation. In Fig. 10, the employment of OGSM has lead to severe non-physical oscillation in the stress and velocity distributions. However, these oscillations are completely removed when either MGSM or DRGSM is employed. It is interesting to note that the MGSM and DRGSM, in Figs. 9 and 10, enable a better concurrence with the analytical solution compared to the solution from CLAWPACK [26].

This test indicates the applicability of the $\vartheta$-criterion as well as the maximum permissible value of $\vartheta$. It shows how the combination of the material properties of the interacting solids can lead to non-physical oscillations when the OGSM is employed. We have carried out numerous other tests for various material properties and shock conditions and found that the proposed critical value of $\vartheta \approx 0.1$ serves as a good guide to determine if the application of the OGSM will likely lead to non-physical oscillations.

4.2. Test Example 2: On the effect of the incident wave

This experiment is designed to show the non-physical oscillations which may rise due to the effect of an incident wave on the interface when one applies the OGSM. Moreover, it indicates the robustness of the $\vartheta$-criterion and the applicability of the $\vartheta_{\text{crit}} \approx 0.1$.

The domain of the solution is $[0, 10]$ and the interface is located at $x_I = 5$. The material properties of the mediums are $\rho_L = 1$ and $E_L = 1$ on the left-hand side of the interface, and $\rho_R = 5$ and $E_R = 5$ on the right side. The spatial discretization is $\Delta x = 0.5$ and the maximum CFL number is 0.99. The initial condition for this problem is $p(x, 0) = u(x, 0) = 0$. The boundary conditions are $p(10, t) = 0$ and

$$p(0, t) = \begin{cases} 
0 & 0 \leq t \leq t_r, \\
1 & \text{otherwise,}
\end{cases}$$

where two cases of the reference time $t_r = 0.1$ and $t_r = 0.2$ are considered. The solution is obtained for the final time $t_f = 8$. Fig. 11 shows the stress and velocity profiles with the reference time of $t_r = 0.2$ in Eq. (58). The $\vartheta$ parameter,
Fig. 10. Test Example 1: Comparison of the velocity and stress profiles between the exact solution, OGSM, MGSM, DRGSM, and CLAWPACK ($\rho_L = 1$, $E_L = 1$, $\rho_R = 5$, $E_R = 5$, and $t_f = 0.3$).

Fig. 11. Test Example 2: Comparison of the velocity stress profiles between the exact solution, OGSM, MGSM, DRGSM, and CLAWPACK ($\rho_L = 1$, $E_L = 1$, $\rho_R = 5$, $E_R = 5$, the final time $t_f = 8$, and reference time of $t_R = 0.2$).

Fig. 12 shows the stress and velocity profiles with the reference time of $t_R = 0.1$ in Eq. (58). The $\vartheta$ parameter, when the OGSM is employed, is always below $\vartheta_{\text{crit}} \approx 0.1$ for all the time steps except for three time steps that reaches a maximum of 0.5 at $t = 5$ when the wave impacts on the interface. After these three time steps, $\vartheta$ quickly drops below $\vartheta_{\text{crit}} \approx 0.1$. Although there are no apparent non-physical oscillations in the OGSM solutions of velocity and stress profile, the above-mentioned three time steps have led to a perceptible numerical error in the velocity and stress profile when compared against the exact solution. The MGSM and DRGSM solutions remain stable, and have a good agreement with the analytical solution as well as CLAWPACK.
Then, it spikes to 0.5 and then increases to 0.55 and slowly descends below $\vartheta_{\text{crit}} \approx 0.1$ after 11 time steps. This figure indicates severe oscillations in the stress and velocity profiles when the OGSM is employed. The MGSM and DRGSM solutions remain stable. Moreover, they concur well with analytical solution and CLAWPACK. These test examples show the importance of the impacting wave, on the interface, on the non-physical oscillations that may rise due to the employment of the OGSM. Moreover, it indicates the applicability and robustness of the $\vartheta$-criterion. It is worth mentioning that we have carried out many more tests for various material properties and various types of waves impacting the interface and found that the proposed critical value of $\vartheta \approx 0.1$ serves as a good guide to determine if the OGSM leads to oscillations.

4.3. Test Example 3: On the effect of solver

This numerical experiment is designed to compare the effect of the solver on the results obtained by OGSM and MGSM. The domain of the solution is $[0, 10]$ and the interface is located at $x_I = 5$. The initial velocity and stress $u(x, 0)$ is 1 for $x \in [0, 5]$ and zero otherwise. The boundary conditions are $p(0, t) = 1$ and $p(10, t) = 0$. The material properties on the left-hand side of the interface are $\rho_L = 1$, and $E_L = 1$, and on the right side are $\rho_R = 5$ and $E_R = 10$. The spatial discretization is $\Delta x = 0.01$. The solution is obtained for $t_f = 0.3$. Firstly, the first-order Godunov is used as the solver. Next, the second-order MUSCL solver [27,20] is used. Then, the obtained results are compared for this two solvers.

It is noticed that the maximum calculated value of $\vartheta$ for this problem is close to 1.0 and far exceeds the $\vartheta_{\text{crit}} \approx 0.1$. It can be seen, in Fig. 13, that using OGSM leads to non-physical oscillations in the stress and velocity profiles. Furthermore, it is noticed that using the higher order method in fact intensifies the non-physical oscillations associated with the OGSM method.

It is clear that the order of the numerical solver does not affect the inherent characteristics of the OGSM. Fig. 14 shows that even as the order of accuracy of the solver is increased, the MGSM continues to remain stable with a solution which concurs well with the analysis.

Finally, when no GSM is used (i.e. the boundary conditions are applied directly) with the specified CFL number, both the first-order Godunov and the MUSCL solver will become completely unstable. The order of maximum numerical error for velocity and normal stress will be $10^{-11}$ to $10^{-12}$.

4.4. Under the special case of acoustic impedance matching conditions

The acoustic impedance or the characteristic acoustic impedance is a material property defined as

$$Z = \rho c.$$  \hspace{1cm} (59)

If the acoustic impedance of two different materials is the same, the incident wave at the interface of the materials in contact should just pass through, without any reflection at the interface [3].
Moreover, for elastic solid–solid interactions under the acoustic impedance matching conditions, if the \( \vartheta \) value is calculated analytically, it is found that \( \vartheta = 0 \) regardless of the shock wave conditions hitting the interface. Therefore, it is interesting to test the validity of the OGSM as applied to the two numerical problems with matched acoustic impedance shown below in Test Example 3. Likewise, it would be interesting to see how the MGSM and DRGSM perform.

The material properties of the two mediums are \( \rho_L = 1 \) and \( E_L = 1 \) for the left solid and \( \rho_R = 2 \) and \( E_R = 0.5 \) for the right medium. Hence, both the left and right mediums have a unit acoustic impedance. The domain of the solution is \( x \in [0, 10] \) and the interface is at \( x_I = 5 \). GSMs are applied and a second-order MUSCL solver is used to solve for the elastic solid governing equation.
Fig. 15. Test Example 4: Velocity and stress profile obtained using a MUSCL solver together with OGSM, MGSM, DRGSM, and a second-order CLAWPACK solver (at \( t_f = 1.2 \)).

Fig. 16. Test Example 5: Velocity and stress profile obtained using a MUSCL solver together with OGSM, MGSM, and DRGSM (at \( t_f = 7 \)).
4.4.1. Test Example 4: A unit pulse hitting the interface

The initial conditions for this case are:

\[ u(x, 0) = p(x, 0) = \begin{cases} 
1 & 4 \leq x \leq 5, \\
0 & \text{otherwise}, 
\end{cases} \]

and the boundary conditions case are \( p(0, t) = p(10, t) = 0 \). The solution is obtained for \( t_f = 1.2 \). The grid size is \( \Delta x = 0.01 \).

The calculated value of \( \vartheta \) remains identically zero for all the time steps, which is less than \( \vartheta_{\text{crit}} \approx 0.1 \). From Fig. 15, it is clear that all the wave energy passes through unimpeded and that no wave is reflected at the interface. All the proposed
GSMs remain stable and are actually successful, in properly capturing the solid–solid interaction, in this acoustic impedance matching test.

4.4.2. Test Example 5: A sinusoidal wave hitting the interface

The initial conditions for this case are \( p(x, 0) = u(x, 0) = 0 \), and the boundary conditions for this case are \( p(0, t) = \sin(4\pi t) \) and \( p(10, t) = 0 \). The grid size is \( \Delta x_L = 0.02 \) and \( \Delta x_R = 0.01 \), for the left and right medium, respectively. The solution is obtained for \( t_f = 7 \).

In this experiment, the calculated value of \( \theta \) remains identically zero for all the time steps, which is less than \( \theta_{\text{crit}} \approx 0.1 \). Fig. 16 clearly indicates that no non-physical oscillation is observed for any of the proposed GSMs. Furthermore, it is noted that all the wave energy passes through and no wave is reflected at the interface. The proposed GSMs are successful in properly capturing the solid–solid interaction.

4.5. Test Example 6: On a general wave propagation

The material properties of the interacting mediums are \( \rho_L = 1 \) and \( E_L = 1 \) for the left solid and \( \rho_R = 4 \) and \( E_R = 2 \) for the right medium. The initial conditions for this case are \( p(x, 0) = u(x, 0) = 0 \). The boundary conditions for this case are

Fig. 18. Test Example 7: Comparison of the stress component results obtained using OGSM and MGSM for no-slip condition \( (t_f = 1) \).

Fig. 19. Test Example 7: Comparison of the velocity component results obtained using OGSM and MGSM for no-slip condition ($t_f = 1$).

$$p(0, t) = \begin{cases} 0.2 & t \leq 1, \\ 0.8t - 0.6 & 1 < t \leq 2, \\ 1 & 2 < t, \end{cases}$$

$$p(10, t) = 0.$$

GSMs are applied and second-order MUSCL solver is used to solve for the elastic solid governing equation. The grid size is $\Delta x = 0.04$, the CFL number is 0.98, and the solution is obtained for $t_f = 8$.

The calculated value of $\theta$ for this experiment is either less than 0.1 or close to zero for almost all time steps, except only for only five time steps at $t = 5$ that reaches a maximum of 0.31 which is greater than $\theta_{\text{crit}} \approx 0.1$. Since the shocks which hit the interface are not very strong and there is supposedly adequate numerical viscosity to damp out these non-physical oscillations, $\theta$ quickly falls below $\theta_{\text{crit}} \approx 0.1$ for the rest of the temporal calculations. Fig. 17 at $t_f = 8$ shows that although the oscillations are no longer apparent, the numerical inaccuracies introduced when $\theta$ exceeds 0.1 at the mentioned time steps, have led to slightly less accurate solution than the MGSM (where no oscillation is found throughout) in comparison to the analytical solution. In this figure, where the DRGSM is employed, it can provide more accurate results for the reflected waves similar to MGSM compared to the OGSM. Still, the DRGSM can result in a slightly yet perceptible time lead as the wave passes through the interface (see Fig. 17). Moreover, in the vicinity of large discontinuities in the solution, it is observed that all the GSMs concur slightly better with the exact solution compared to CLAWPACK.

4.6. Test Example 7: 2D Experiment-1

In this section, a numerical experiment is devised to compare OGSM and MGSM for a 2D problem. The setup of the experiment is such that it is identical to a 1D problem, in the normal direction of the interface. The results are calculated in the $x$–$y$ coordinates. However, they are converted to the $\xi$–$\eta$ directions and then re-plotted for ease of comparison to the 1D solution, also to ascertain the viability of the 2D solution.

The primary interest is in studying the robustness of GSMs for capturing interface interactions. The solution domain is chosen as $\Omega = \{(x, y) \mid x \in [0, 10] \text{ and } y \in [-5, 6]\}$, however, the results are only plotted for the region $y \in [0, 1]$ to eliminate the effects of top and bottom boundary conditions on the solution. The interface is defined by the line $y = 5.5 - x$. The boundary conditions at the left and right boundaries are

$$u_x(0, y) = u_y(0, y) = u_x(10, y) = u_y(10, y) = 0.$$
Fig. 20. Test Example 7: Comparison of the stress component results obtained using OGSM and MGSM for perfect-slip condition ($t_f = 1$).

and the initial conditions are

$$u(x, y, 0) = \sigma_{\eta\eta}(x, y, 0) = \begin{cases} 1 & (5.5 - \sqrt{2}/2 - x) \leq y \leq (5.5 - x), \\ 0 & \text{otherwise}, \end{cases}$$

$$\forall (x, y) \in \Omega, \quad v(x, y, 0) = \sigma_{\xi\xi}(x, y, 0) = \sigma_{\xi\eta}(x, y, 0) = 0.$$ 

The material properties of the left medium are $\rho_L = 1$, $\alpha_L = 1$, and $\beta_L = 0.3$; and those of the right medium are $\rho_R = 5$, $\alpha_L = 1$, and $\beta_L = 0.3$.

A second-order MUSCL solver together with a grid size of $\Delta x = \Delta y = 0.05$ and CFL number of 0.58 is employed for solving the elastic solid equation in each medium. The solution is obtained at $t_f = 1$. Both the OGSM and MGSM are tested for no-slip and perfect-slip conditions at the interface.

The calculated value of $\vartheta$ for this test is close to 1 for almost all the time steps which is greater than $\vartheta_{\text{crit}} \approx 0.1$. As such, this will lead to non-physical oscillations for the OGSM.

In Figs. 18 and 19, the velocity and stress profiles obtained using OGSM and MGSM for the no-slip condition at the interface, respectively, are shown. It is clear that quantities of $u$, $\sigma_{\eta\eta}$ and $\sigma_{\xi\xi}$ computed values suffer from non-physical
Fig. 21. Test Example 7: Comparison of the velocity component results obtained using OGSM and MGSM for slip condition ($t_f = 1$).

oscillations when the OGSM is used which lends support to the proposed $\vartheta$-criterion. It can be seen that the MGSM has successfully removed these mentioned oscillations.

Next, Figs. 20 and 21 show the velocity and stress profiles obtained using OGSM and MGSM for the slip condition at the interface. It can be observed that the computed $u$, $\sigma_{yy}$ and $\sigma_{xz}$ suffer from non-physical oscillations when the OGSM is employed. It is noticed, however, the MGSM does not suffer from any of these oscillations.

It is apparent that both methods, for both the no-slip and the slip interface conditions, can predict the zero value of $v$ and $\sigma_{yz}$ reasonably well. However, it should be noted that due to the construct of the experiment, all of these values are consistently zero in the entire domain. Hence, minimum error is incurred while still using the OGSM (zero values were copied to the ghost nodes for both MGSM and OGSM). This situation will differ in a general problem.

Finally, Fig. 22 compares the 2D results against the analytic equivalent 1D solution, along the normal direction to the interface. It can be seen when the MGSM is employed, the numerical results agree with the analytical solution to a greater extent. However, the numerical results when the OGSM is employed suffer from severe non-physical oscillations close to the interface. These non-physical oscillations are rectified when the MGSM is employed.

4.7. Test Example 8: 2D Experiment-2

In this section, a numerical experiment is designed to study the robustness of the GSMs as well as their stability when they are applied to more complex geometry and stress wave interactions.

The solution domain is $\Omega = \{(x, y) \mid x \in [0, 10] \text{ and } y \in [0, 10]\}$ which comprises two solids

$$\Omega_1 = \{(x, y) \mid x < 5 \text{ and } y < (x + 1) \text{ and } y > (9 - x)\}$$

and

$$\Omega_2 = \Omega - \Omega_1.$$ 

The material properties are $\rho_1 = 5$, $\alpha_1 = 1$, $\beta_1 = 0.3$, $\rho_2 = 1$, $\alpha_2 = 1$, and $\beta_2 = 0.3$. The initial conditions $u_x = \sigma_{xx} = 1$ when $x \in [3, 4]$ and zero otherwise, and $u_y = \sigma_{yy} = \sigma_{xy} = 0$ everywhere in the domain. On the left and right boundaries, free surface conditions are imposed, and on the top and bottom boundaries, symmetry conditions are assumed. No-slip condition is assumed at the interface of the two solids. A second-order MUSCL solver is used together with a grid size of $\Delta x = \Delta y = 0.01$ and a CFL number of 0.65. The solution is obtained for $t = 1.5$. 

123
Fig. 22. Test Example 7: Normal velocity and normal stress along the line \( y = x - 2.5 \) with respect to the normal coordinate, \( \eta \). Results are obtained for \( t_f = 1 \).

The calculated value of \( \vartheta \) for this test is close to 1 for almost all the time steps which is greater than \( \vartheta_{\text{crit}} \approx 0.1 \). As such, this will lead to non-physical oscillations for the OGSM.

The problem setting provides \( y = 5 \) as the line of symmetry. Full calculations are carried out with no assumption of symmetry in the methods. Fig. 23 shows that the MGSM gives a very stable and smooth solution for \( \sigma_{xy} \) while a good symmetry is preserved about \( y = 5 \). However, it is observed the OGSM is completely unstable. This further attests to the applicability and robustness of the MGSM in comparison to the OGSM when it is applied to more complex geometries and wave interactions.

5. Conclusion and summary

In this work, three GSM-based algorithms were developed for the wave interaction at the solid–solid medium, with each medium governed by an isotropic, linearly elastic solid material. The OGSM method does not involve any Riemann solver to define the ghost values and hence is very easy to apply. However, it should be noted that the method can lead to severe non-physical oscillations in predicting the stress and velocity values. A non-dimensional criterion was introduced to ascertain the occurrence of these oscillations and its maximum permissible value was also determined via numerous numerical experiments carried out. These non-physical oscillations may become even more severe when the OGSM is combined with a higher order solver. One may in essence increase the numerical viscosity of the software to get rid of these oscillations. However, it should be noted this remedy will always lead to smearing of the wave fronts. The MGSM and the DRGSM developed in this work, were shown to successfully and completely remove the non-physical oscillations seen when employing the OGSM. As such, no addition of numerical viscosity is necessary when the MGSM or DRGSM are employed. This also means that a larger possible time step can be used and the wave front will be still fairly sharp. It was also shown that MGSM can robustly be used in acoustic impedance matching experiments. It was noted, DRGSM does not seem to provide significant improvements over MGSM.
Fig. 23. Test Example 8: Contour plots of $\sigma_{xy}$ when the MGSM and the OGSM are employed. Solution is obtained for $t = 1.5$.

References