Pruning Sparse Signal Models Using Interference

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Abstract—Previous work on sparse approximations has shown
that in the pursuit of a signal model using greedy iterative
algorithms, the efficiency of the representation can be increased
by considering the interference between selected atoms. However,
in such interference-adaptive algorithms, atoms are still often
selected that necessitate correction by subsequently chosen atoms.
It is thus logical to remove these atoms from the representation
so that they do not diminish the efficiency of the pursued signal
model. In this paper, we propose to prune atoms from the
model based on the degree and type of interference, and test
its effectiveness in an interference-adaptive orthogonal matching
pursuit algorithm.

I. INTRODUCTION

Efficiently representing data and signals is of key
importance in many applications, such as analysis and compression.
Sparse approximation techniques [1]–[4] attempt to circum-
vent limitations associated with the use of linear and orthog-
onal transforms for representing signals that are not sparse
over a single basis. However, although one is free to define
the atoms in the dictionary and the process of decomposition,
the resulting nonorthogonal representation may contain many
atoms that exist more to compensate for poor atom selections
(or due to signal/model mismatches), than to represent features
of the signal [5]–[9].

In [8], [9], a measure based on interference was incorporated
into the atom selection criterion of orthogonal matching pur-
suit (OMP) [2], to generate more efficient signal models that
have a clearer connection between atoms and signal content.
Though poorly chosen atoms can eventually become irrelevant
as the model is constructed, they remain a part of the model,
and as a result affect future atom selections that can reduce
the model efficiency. In this paper, we exploit the interaction
between atoms to determine whether they facilitate data repre-
sentation, or work instead to “correct” for other atoms in the
signal model. Atoms that are determined to require significant
correction in the signal model by subsequently chosen atoms
are pruned from the representation during the pursuit. The
goal is to make a pursued signal model more efficient in that
it requires fewer atoms to represent the signal for a range of
levels of the residual energy (or distortion).

In Section II, we first review the signal model, OMP, and the
concept of interference. We also describe how interference can
be applied to the atom selection criterion of OMP. In Section
III, we present the proposed pruning algorithm that removes
specific atoms from the signal model during the decomposition
process. Section IV presents example computer simulations of
this algorithm, and shows that some poorly selected atoms are
successfully removed from the model. We conclude in Section
V with comments on the algorithm and proposed future work.

II. SIGNAL MODEL, OMP, AND INTERFERENCE

A. Signal Model and OMP

Sparse approximation decomposes a signal \(x \in \mathbb{C}^{N}\)
using elements from a dictionary \(D \triangleq \{d_i \in \mathbb{C}^{N} : \|d_i\|_2 = 1\}^N_{i=1}\),
represented in matrix form as \(D \triangleq [d_1|d_2|\cdots|d_N]_{K \times N}\), such
that \(x\) is described by the model

\[
x = Ds + r
\]  

(1)

where \(s\) is a vector of weights, and the residual \(r\) represents
that part of \(x\) not contained in \(Ds\) (i.e., it is the error between
the signal and the model).

Various algorithms have been proposed to find representa-
tive components for the signal model. Here we focus on greedy
iterative algorithms that yield the following \(n\)th-order model:

\[
x = H(n)a(n) + r(n).
\]  

(2)

In order to distinguish between iterative solutions to the de-
composition and the general description in (1), we denote the \(n\)
selected dictionary atoms as \(H_n \triangleq \{h_i \in D \}_{i=1}^{n}\), which form
the matrix \(H(n) \triangleq [h_0|h_1|\cdots|h_{n-1}]_{K \times n}\). The associated
weights are in the column vector \(a(n) \triangleq [a_0,a_1,\ldots,a_{n-1}]^T\),
and \(r(n)\) is the \(n\)th-order residual. The \(n\)th-order approxima-
tion of \(x\) is thus given by \(x(n) \triangleq H(n)a(n)\), and we denote the \(n\)th-order representation of \(x\) using the triple

\[
\mathcal{X}_n \triangleq \{H(n),a(n),r(n)\}.
\]  

(3)

OMP updates the \(n\)th-order representation using the following
iterative rule:

\[
\mathcal{X}_{n+1} = \begin{cases} 
H(n+1) = [H(n)|h_n], \\
a(n+1) = H^\dagger(n+1)x, \\
r(n+1) = x - H(n+1)a(n+1)
\end{cases}
\]  

(4)

where \(H^\dagger(n) \triangleq [H^H(n)H(n)]^{-1}H^H(n)\). The OMP atom
selection criterion is given by

\[
h_n = \arg \max_{d \in D^H} |d^Hr(n)|^2
\]  

(5)

where the superscript \(^H\) denotes complex conjugate transpose.
B. Interference

When $\mathcal{D}$ is an overcomplete dictionary, such that it can represent any signal in the signal space, and if its elements are linearly dependent, then it is likely that the atoms of the signal model in (2) will interact in nontrivial ways. In particular, the energy of the superposition of these atoms will differ from the sum of their individual energies. In the extreme case, some atoms may destructively interfere such that the energy of their superposition is zero [7]. We measure this interaction using interference: the interference associated with the $n$th atom $\mathbf{h}_m \in \mathcal{H}_n$ of the $n$th-order representation $\mathcal{X}_m = \{\mathbf{H}(n), \mathbf{a}(n), \mathbf{r}(n)\}$ is defined for $m = 0, 1, \ldots, n - 1$ as follows [8]:

$$
\Delta(m) = \frac{1}{2} \left\{ ||\hat{\mathbf{x}}(n)||^2 - ||\hat{\mathbf{x}}(n) - a_m \mathbf{h}_m||^2 \right\} = \text{Real}\left\{ a_m^* \mathbf{h}_m^H (\mathbf{a}(n) - a_m \mathbf{h}_m) \right\}
$$

(6)

where the superscript $^*$ denotes complex conjugation. When $\Delta(m) < 0$, $\mathbf{h}_m$ destructively interferes with the rest of the model, and when $\Delta(m) > 0$, it constructively interferes. An atom is orthogonal to the model when $\Delta(m) = 0$.

C. Interference-Adaptive OMP

Interference provides information about the relationship of an atom to the current signal model, and can be incorporated into the atom selection rule in (5) according to [8], [9]

$$
\mathbf{h}_m = \arg \max_{\mathbf{d} \in \mathcal{D}} ||\mathbf{r}^H(n)\mathbf{d} + \lambda(n)\text{Real}\{\mathbf{r}^H(n)\mathbf{d}\mathbf{d}^H\hat{\mathbf{x}}(n)\} \quad (7)
$$

where $-\infty < \lambda(n) < \infty$ weights the influence of interference in the atom selection. When the pursuit update rule is (4), this change in the atom selection criterion can produce more efficient signal models [8], [9].

Figure 1 shows examples of representations created by (4) using (7) and a multiscale dictionary of Gabor and Dirac atoms. Arrows point to the first atom found in each decomposition, which have the same frequency, scale, and translation for $n = 0$ for both signals. The interference weighting in each case is a constant $\lambda(n) = \lambda \geq 0$. Observe for Attack that for small $\lambda$, many atoms appear prior to the signal onset; these atoms function entirely to correct for the first atom selected.

As $\lambda$ is increased in (7), i.e., as more emphasis is placed on the influence of constructive interference within the model in (2), the model created for Attack becomes a better reflection of the signal itself, and uses fewer atoms to correct for poor atom selections. For Bimodal, the first atom selected requires correction by several subsequent atoms such that it becomes a trough. However, although the amplitude of the first atom in Bimodal decreases as $\lambda$ is increased, it might be better if this atom is not a part of the signal representation if it requires such extensive correction.

The influence of $\lambda$ on the residual energy and the convergence of each signal model can be seen in Fig. 2. For a signal-to-residual energy ratio (SRR) of 60 dB, the order of the signal model for Attack is reduced from 65 to 45 atoms. This increase in efficiency occurs because the selection criterion in (7) uses information about how the signal has been modeled so far (via interference), and does not rely only on the structures in the current residual in (5) [9].

III. PRUNING USING INTERFERENCE

At some point in a pursuit (or even periodically), it makes sense to examine the current elements of the signal model in (2) and determine which parts are functioning well together and which are not. The examples described above indicate that it might be beneficial to prune a signal model during a pursuit such as OMP. Atoms selected based on little or no information about how a signal is related to other atoms in the dictionary could yield a less efficient signal model, such as the atoms in Fig. 1(a) preceding the signal onset. Even though the contributions of such problematic atoms appear to diminish as the pursuit proceeds, they still remain a part of the signal model. We expect to increase the model’s efficiency by pruning such atoms during the decomposition.

One strategy for rejecting atoms from a signal model is given by backward-optimized OMP in [10], which removes the atom that minimally increases the model distortion. Such a measure, however, does not adequately embody the notion of how well each atom represents the signal together with the rest of the atoms in the model. Thus, we propose using a measure of interference to make such a decision: atoms...
that need “too much” correction by subsequent atoms through destructive interference should be removed from the signal model. Toward this end, we propose the following pursuit algorithm with pruning:

0) Initialization ($i = 0$): $\mathcal{H}_0 = \emptyset$, $a(0) = [ ]$, $r(0) = x$.
1) Pursue the representation for $L$ steps yielding $x_{i+L} = \{H(i + L), a(i + L), r(i + L)\}$.
2) Inspect and prune atoms on the basis of negative interference with threshold $\alpha \leq 0$:
   a) Let $\Gamma = \{l : \Delta(l) < \alpha(l + L)^{L-1}\}$ contain the indexes of those atoms with a sufficient level of negative interference.
   b) If $\Gamma = \emptyset$, then return to step 1).
      Else:
      i) $k = i + L - |\Gamma|$. 
      ii) Remove elements from the representation basis such that $\mathcal{H}_k = \mathcal{H}_{i+L}\setminus \Gamma$, where $\setminus \Gamma$ denotes that elements associated with $\Gamma$ have been removed.
      iii) Recompute weights $a(k)$.
      iv) Recompute residual $r(k) = x - H(k)a(k)$.
      v) (Optional) Remove atoms associated with indexes $\Gamma$ from the dictionary.
3) $i \leftarrow k$ and return to step 1) until some stopping criterion is satisfied (e.g., based on the SRR)

Observe that $L$ and $\alpha$ are new parameters that must be judiciously chosen for improved performance.

IV. COMPUTER SIMULATIONS

We have evaluated the performance of the proposed pursuit using interference-adaptive OMP with atom pruning for the test signals shown in Fig. 1. Each pursuit used the same dictionary of Gabor and Dirac atoms. Figures 3(a) and (c) show the SRR decay as a function of the pursuit iteration for different values of the interference weighting $\lambda$ and $\alpha = 0$. Once the model reaches SRR = 60 dB, or 500 iterations have been performed, the decomposition algorithm is stopped. In most cases, the residual energy temporarily increases every $L$ iterations because the model order is reduced with pruning. There is one case in Bimodal (for $\lambda = 4$) where no atoms are removed from the model because none of them have negative interference.

At the end of each pursuit, we reordered the signal model based on decreasing magnitude of the weights $a(n)$. We then computed the SRR as a function of the order, yielding the results shown in Figs. 3(b) and (d). Observe that with an appropriate choice of $\lambda$, there can be an increase in the efficiency of the models over those created by basic OMP. However, we find that the resulting model orders can be a bit higher than those in Fig. 2 for the same level of SRR. For example, for the model of Attack created using $\lambda = 15$, SRR = 60 dB is achieved with about 47 atoms in Fig. 2(a); with the proposed model pruning, this order increases to 51 atoms. Also, the model of Bimodal created using $\lambda = 1$ with pruning is slightly less efficient than basic OMP without pruning.

Comparing the time-domain distribution of atoms in the models of Attack in Fig. 4(a) with those in Fig. 1(a), observe that the first atom selected by OMP is removed from the models when $\lambda = 3$ and 15, which we expect since it destructively interferes with most of the atoms that precede the onset. For Bimodal and $\lambda = 1$, the first atom — which links together the two modes — is not pruned from the signal model because it eventually is a useful contribution to the model (i.e., with constructive interference) with the addition of smaller-scale atoms. Finally, the wivigrams of these representations, which are the superpositions of each atom’s Wigner-Ville distribution [1], [7], are shown in Fig. 5. For Attack, pruning the first atom selected by OMP with $\lambda = 3$ reduces the amount of energy in the time-region preceding the signal onset.

Without proper choice of the interference weighting $\lambda$, the number of iterations $L$ for pruning, and the degree of interference parameter $\alpha$, a pursuit may not converge to a good solution. This can be seen in the periodic behavior of the SRR in Figs. 3(a) and (c). Even when atoms are pruned from the dictionary such that they can never be selected, the pursuit still performs poorly. As demonstrated in the case of Bimodal, the pursuit can select initial atoms that destructively interfere with all subsequent atoms, which then results in the
We have addressed the issue of pursuit algorithms for sparse signal approximations making poor atom selections that are corrected by subsequent atoms. Our prior work has shown that incorporating an interference measure between atoms can lead to representations that are more efficient because they reduce the need for correction. In this paper, we apply these concepts to the pruning of initial atom selections. Specifically, the modified pursuit algorithm periodically reviews the atoms in a model and prunes those associated with excessive destructive interference. Such atoms are usually the result of either a mismatch of the model to the signal, or due to poor selections directed more by algorithm greed rather than efficient signal representation. Future work will explore other measures with which to prune a signal model, and making the number of iterations $L$ between pruning adapt to the performance of the pursuit.

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**REFERENCES**

Fig. 4. Time-distribution of atoms in signal models created by OMP with interference adaptation for various $\lambda$ and periodic atom pruning.

Fig. 5. Wvigrams of models created by OMP with interference adaptation for several $\lambda$ and periodic atom pruning.