On sum of powers of the Laplacian eigenvalues of graphs

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Received 5 February 2008; accepted 13 June 2008

Submitted by M. Fiedler

Abstract

For a graph $G$ and a real $\alpha \neq 0$, we study the graph invariant $s_\alpha(G)$ – the sum of the $\alpha$th power of the non-zero Laplacian eigenvalues of $G$. The cases $\alpha = 2, \frac{1}{2}$ and $-1$ have appeared in different problems. Here we establish some properties for $s_\alpha$ with $\alpha \neq 0, 1$. We also discuss the cases $\alpha = 2, \frac{1}{2}$. © 2008 Elsevier Inc. All rights reserved.

AMS classification: 05C50; 05C90

Keywords: Laplacian eigenvalues; Laplacian energy; Energy; Subdivision graph

1. Introduction

Let $G$ be a simple finite undirected graph with vertex set $V(G)$. Let $A(G)$ be the $(0, 1)$ adjacency matrix of $G$ and $D(G)$ the diagonal matrix of vertex degrees. Then $L(G) = D(G) - A(G)$ is called the Laplacian matrix of $G$. It is symmetric, positive semidefinite and singular. The Laplacian eigenvalues of $G$ are the eigenvalues of $L(G)$. Let $\mu_1, \mu_2, \ldots, \mu_n$ be the Laplacian eigenvalues of $G$ arranged in a non-increasing manner, where $n = |V(G)|$. When more than one graph is under discussion, we write $\mu_i(G)$ instead of $\mu_i$. It is known that $\mu_n = 0$ and the multiplicity of 0 is equal to the number of connected components of $G$.

Let $\alpha$ be a non-zero real number. Let $G$ be a graph with $n$ vertices. Let $s_\alpha(G)$ be the sum of the $\alpha$th power of the non-zero Laplacian eigenvalues of $G$, i.e.,

$\sum_{i=1}^{n-\text{mult}(0)} \mu_i^\alpha$
where $h$ is the number of non-zero Laplacian eigenvalues of $G$. The case $\alpha = 1$ is trivial as $s_1(G) = 2m$, where $m$ is the number of edges. Some properties for $s_2$ were established in [9], where Laziji called it the Laplacian energy of the graph. Recall that the energy of a graph is equal to the sum of the absolute values of its ordinary eigenvalues [3] and that an energy–like quantity was proposed and studied in [7] based on the Laplacian eigenvalues. Recently, some properties of $s_1$ were given in [11]. We also note that for a connected graph $G$ with $n$ vertices, $ns_{n-1}(G)$ is equal to its Kirhoff index or quasi-Wiener index, which found applications in electric circuit, probabilistic theory and chemistry [6,15].

In this paper, we establish some properties for $s_\alpha$, where $\alpha$ is a real number with $\alpha/\neq 0, 1$. We also discuss further properties for $s_2$ and $s_{1/2}$.

2. Preliminaries

Let $K_n$ and $P_n$ be respectively the complete graph and the path on $n$ vertices. Let $K_{a,b}$ be the complete bipartite graph with two partite sets having $a$ and $b$ vertices, respectively.

We need some properties of the Laplacian eigenvalues. For more details, see [12,13].

Let $\overline{G}$ be the complement of the graph $G$ with $n$ vertices. The Laplacian eigenvalues of $\overline{G}$ are $n - \mu_{n-1}(G), n - \mu_{n-2}(G), \ldots, n - \mu_1(G), 0$.

**Lemma 1** [13]. Let $G$ be a non-complete graph with $n$ vertices. If $G^*$ is obtained from $G$ by adding an edge, then

$$\mu_1(G^*) \geq \mu_1(G) \geq \mu_2(G^*) \geq \mu_2(G) \geq \cdots \geq \mu_{n-1}(G^*) \geq \mu_{n-1}(G) \geq \mu_n(G^*) = \mu_n(G) = 0.$$

**Lemma 2** [2]. Let $G$ be a graph with at least one edge and maximum vertex degree $\Delta$. Then

$$\mu_1 \geq 1 + \Delta$$

with equality for connected graph if and only if $\Delta = n - 1$.

**Lemma 3** [12]. Let $G$ be a connected graph with diameter $d$. Then $G$ has at least $d + 1$ distinct Laplacian eigenvalues.

**Lemma 4.** Let $G$ be a graph with $n$ vertices. Then $\mu_1 = \cdots = \mu_{n-1}$ if and only if $G \cong K_n$ or $G \cong K_n$.

**Proof.** Suppose that $\mu_1 = \cdots = \mu_{n-1}$. If $G$ is connected, then by Lemma 3, $G \cong K_n$. If $G$ is not connected, then $\mu_{n-1} = 0$ and so all Laplacian eigenvalues are equal to zero, which obviously implies that $G \cong K_n$. Conversely, it is easily seen that $\mu_1 = \cdots = \mu_{n-1}$ if $G \cong K_n$ or $G \cong K_n$. □

**Lemma 5.** Let $G$ be a connected graph with $n \geq 2$ vertices. Then $\mu_2 = \cdots = \mu_{n-1}$ and $\mu_1 = 1 + \Delta$ if and only if $G \cong K_n$ or $G \cong K_{1,n-1}$. 
Proof. Suppose that $\mu_2 = \cdots = \mu_{n-1}$ and $\mu_1 = 1 + \Delta$. By Lemma 2, $\Delta = n - 1$. Then $\overline{G}$ has an isolated vertex, say $v$, and the Laplacian eigenvalues of $\overline{G} - v$ are $n - \mu_{n-1}, \ldots, n - \mu_2, 0$. By Lemma 4, $\overline{G} - v \cong K_{n-1}$ or $\overline{G} - v \cong K_{n-1}$. Thus $G \cong K_n$ or $G \cong K_{1,n-1}$.

Conversely, it is easy to see that $\mu_2 = \cdots = \mu_{n-1}$ and $\mu_1 = 1 + \Delta$ if $G \cong K_n$ or $G \cong K_{1,n-1}$.

□

For a graph $G$, let $Z(G) = \sum_{u \in V(G)} d_u^2$, where $d_u$ stands for the degree of vertex $u$ in $G$.

**Lemma 6** [8]. Let $G$ be a connected bipartite graph with $n$ vertices. Then $\mu_1 \geq 2 \sqrt{\frac{Z(G)}{n}}$ with equality if and only if $G$ is a regular bipartite graph.

The subdivision graph $S(G)$ of a graph $G$ is obtained by inserting a new vertex (of degree 2) on each edge of $G$. The ordinary spectrum of a graph $G$ is the spectrum of its adjacency matrix.

**Lemma 7** [16]. Let $G$ be a bipartite graph with $n$ vertices and $m$ edges. If the non-zero Laplacian eigenvalues of $G$ are $\mu_i, i = 1, \ldots, h$, then the ordinary spectrum of $S(G)$ consists of the numbers $\pm \sqrt{\mu_i}, i = 1, \ldots, h$, and of $n + m - 2h$ zeros.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the ordinary eigenvalues of the graph $G$, where $n = |V(G)|$. Then the energy of $G$ is defined as [3,4]

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$  

**Lemma 8** [14,17]. Let $G$ be a graph with $n$ vertices, $m \geq 1$ edges and $q$ quadrangles. Then

$$E(G) \geq \sqrt{\frac{(2m)^3}{2Z(G) - 2m + 8q}}$$  

with equality if and only if $G$ is the vertex-disjoint union of $K_{a_1,b_1}, \ldots, K_{a_r,b_r}$ with $a_1b_1 = \cdots = a_rb_r$ and $r \geq 1$, and isolated vertices.

3. Results

It is obvious that for any graph $G$ with $n$ vertices, $s_\alpha(G) \geq 0$ for $\alpha \neq 0$ with equality if and only if $G \cong K_n$.

**Theorem 1.** (i) For any non-complete graph $G$, if $G^*$ is obtained from $G$ by adding an edge, then $s_\alpha(G) < s_\alpha(G^*)$ for $\alpha > 0$ and $s_\alpha(G) > s_\alpha(G^*)$ for $\alpha < 0$.

(ii) For any graph $G$ with $n$ vertices

$$s_\alpha(G) \leq (n - 1)n^\alpha \text{ if } \alpha > 0,$$
$$s_\alpha(G) \geq (n - 1)n^\alpha \text{ if } \alpha < 0$$  

with either equality if and only if $G$ is the complete graph $K_n$. 

Proof. Note that \( \sum_{i=1}^{n-1} \mu_i(G^*) = \sum_{i=1}^{n-1} \mu_i(G) = 2 \). By Lemma 1, the result in (i) follows.

Note that \( \mu_1(K_n) = \cdots = \mu_{n-1}(K_n) = n \) and \( \mu_n(K_n) = 0 \). From (i), we have (ii). \( \square \)

Theorem 2. Let \( \alpha \) be a real number with \( \alpha \neq 0, 1 \), and let \( G \) be a connected graph with \( n \geq 3 \) vertices, \( t \) spanning trees and maximum vertex degree \( \Delta \). Then

\[
s_\alpha(G) \geq (1 + \Delta)^\alpha + (n - 2) \left( \frac{tn}{1 + \Delta} \right)^{\frac{\alpha}{n-2}} \tag{1}
\]

with equality if and only if \( G \cong K_n \) or \( G \cong K_{1,n-1} \).

Proof. By the matrix-tree theorem (see [13]), \( \prod_{i=1}^{n-1} \mu_i = tn \). By the arithmetic–geometric mean inequality

\[ s_\alpha(G) = \mu_1^\alpha + \sum_{i=2}^{n-1} \mu_i^\alpha \geq \mu_1^\alpha + (n - 2) \left( \prod_{i=2}^{n-1} \mu_i^\alpha \right)^{\frac{1}{n-2}} = \mu_1^\alpha + (n - 2) \left( \frac{tn}{\mu_1} \right)^{\frac{\alpha}{n-2}} \]

with equality if and only if \( \mu_2 = \cdots = \mu_{n-1} \). Let \( f(x) = x^\alpha + (n - 2) \left( \frac{tn}{x} \right)^{\frac{\alpha}{n-2}} \). By solving \( f'(x) = \alpha(x^{\alpha-1} - (tn)^{\frac{\alpha}{n-2}}x^{-\frac{\alpha}{n-2}} - 1) \geq 0 \), it may be easily seen that \( f(x) \) is increasing for \( x \geq (tn)^{\frac{1}{n-2}} \) whether \( \alpha > 0 \) or \( \alpha < 0 \). Obviously, \( 2m \leq n\Delta \leq (n - 1)(1 + \Delta) \). By Lemma 2

\[ \mu_1 \geq 1 + \Delta \geq \frac{2m}{n - 1} = \frac{\sum_{i=1}^{n-1} \mu_i}{n - 1} \geq \left( \prod_{i=1}^{n-1} \mu_i \right)^{\frac{1}{n-1}} = (tn)^{\frac{1}{n-1}} \]

and then \( s_\alpha(G) \geq f(1 + \Delta) = (1 + \Delta)^\alpha + (n - 2) \left( \frac{tn}{1 + \Delta} \right)^{\frac{\alpha}{n-2}} \). Hence (1) follows, and equality holds in (1) if and only if \( \mu_2 = \cdots = \mu_{n-1} \) and \( \mu_1 = 1 + \Delta \), which, by Lemma 5, is equivalent to \( G \cong K_n \) or \( G \cong K_{1,n-1} \). \( \square \)

Theorem 3. Let \( G \) be a connected graph with \( n \geq 3 \) vertices, \( m \) edges and maximum vertex degree \( \Delta \):

(i) If \( \alpha < 0 \) or \( \alpha > 1 \), then

\[
s_\alpha(G) \geq (1 + \Delta)^\alpha + \frac{(2m - 1 - \Delta)^\alpha}{(n - 2)^{\alpha-1}} \tag{2}
\]

with equality if and only if \( G \cong K_n \) or \( G \cong K_{1,n-1} \).

(ii) If \( 0 < \alpha < 1 \), then

\[
s_\alpha(G) \leq (1 + \Delta)^\alpha + \frac{(2m - 1 - \Delta)^\alpha}{(n - 2)^{\alpha-1}} \tag{3}
\]

with equality if and only if \( G \cong K_n \) or \( G \cong K_{1,n-1} \).
Proof. Observe that for $\alpha \neq 0, 1$ and $x > 0$, $x^\alpha$ is a strictly convex function if and only if $\alpha < 0$ or $\alpha > 1$.

Suppose that $\alpha < 0$ or $\alpha > 1$. Then

$$\left(\sum_{i=2}^{n-1} \frac{1}{n-2} \mu_i\right)^\alpha \leq \sum_{i=2}^{n-1} \frac{1}{n-2} \mu_i^\alpha,$$

i.e.,

$$\sum_{i=2}^{n-1} \mu_i^\alpha \geq \frac{1}{(n-2)^{\alpha-1}} \left(\sum_{i=2}^{n-1} \mu_i\right)^\alpha$$

with equality if and only if $\mu_2 = \cdots = \mu_{n-1}$. It follows that

$$s^\alpha(G) = \mu_1^\alpha + \frac{1}{(n-2)^{\alpha-1}} \left(\sum_{i=2}^{n-1} \mu_i\right)^\alpha$$

$$= \mu_1^\alpha + \frac{(2m-\mu_1)^\alpha}{(n-2)^{\alpha-1}}.$$

Let $g(x) = x^\alpha + \frac{(2m-x)^\alpha}{(n-2)^{\alpha-1}}$. By solving $g'(x) = \alpha x^{\alpha-1} - \frac{(2m-x)^{\alpha-1}}{(n-2)^{\alpha-1}} \geq 0$, it is easily seen that $g(x)$ is increasing for $x \geq \frac{2m}{n-1}$. Note that $(n-1)(1 + \Delta) \geq 2m$. By Lemma 2, $\mu_1 \geq 1 + \Delta \geq \frac{2m}{n-1}$ and then

$$s^\alpha(G) \geq g(1 + \Delta) = (1 + \Delta)^\alpha + \frac{(2m - 1 - \Delta)^\alpha}{(n-2)^{\alpha-1}}$$

with equality if and only if $\mu_2 = \cdots = \mu_{n-1}$ and $\mu_1 = 1 + \Delta$. By Lemma 5, equality holds in (2) if and only if $G \cong K_n$ or $G \cong K_{1,n-1}$.

Now suppose that $0 < \alpha < 1$. Then

$$\left(\sum_{i=2}^{n-1} \frac{1}{n-2} \mu_i\right)^\alpha \geq \sum_{i=2}^{n-1} \frac{1}{n-2} \mu_i^\alpha$$

with equality if and only if $\mu_2 = \cdots = \mu_{n-1}$, and $g(x)$ is decreasing for $x \geq \frac{2m}{n-1}$. By similar arguments as above, the second part of the theorem follows. □

Now we consider bipartite graphs.

Theorem 4. Let $\alpha$ be a real number with $\alpha \neq 0, 1$, and let $G$ be a connected bipartite graph with $n \geq 3$ vertices, $t$ spanning trees. Then

$$s^\alpha \geq \left(2\sqrt{\frac{Z(G)}{n}}\right)^\alpha + (n-2) \left(\frac{\ln n}{2\sqrt{\frac{Z(G)}{n}}}\right)^\frac{\alpha}{n-2}$$

with equality if and only if $G \cong K_{\frac{n}{2}, \frac{n}{2}}$. 

(4)
Proof. By Lemma 6, we have $\mu_1 \geq 2\sqrt{Z(G)/n} \geq \frac{4m}{n} \geq \frac{2m}{n-1} \geq (tn)^{1/n}$. Thus, by similar arguments as in the proof of Theorem 2, we have $s_\alpha \geq f\left(2\sqrt{Z(G)/n}\right)$, from which (4) follows, and equality holds in (4) if and only if $\mu_2 = \cdots = \mu_{n-1}$ and $\lambda_1 = 2\sqrt{Z(G)/n}$.

Suppose that equality holds in (4). Then $G$ is a regular bipartite graph with at most three distinct Laplacian eigenvalues. Thus, by Lemma 3, $G$ is a regular bipartite graph with at most diameter 2, i.e., $G \cong K_{\frac{n}{2}, \frac{n}{2}}$.

Conversely, it is easily seen that $\mu_2 = \cdots = \mu_{n-1}, \lambda_1 = 2\sqrt{Z(G)/n}$, and then (4) is an equality if $G \cong K_{\frac{n}{2}, \frac{n}{2}}$. □

Theorem 5. Let $G$ be a connected bipartite graph with $n \geq 3$ vertices, $m$ edges:

(i) If $\alpha < 0$ or $\alpha > 1$, then

$$s_\alpha(G) \geq \left(2\sqrt{Z(G)/n}\right)^\alpha + \frac{\left(2m - 2\sqrt{Z(G)/n}\right)^\alpha}{(n-2)^{\alpha-1}}$$

with equality if and only if $G \cong K_{\frac{n}{2}, \frac{n}{2}}$.

(ii) If $0 < \alpha < 1$, then

$$s_\alpha(G) \leq \left(2\sqrt{Z(G)/n}\right)^\alpha + \frac{\left(2m - 2\sqrt{Z(G)/n}\right)^\alpha}{(n-2)^{\alpha-1}}$$

with equality if and only if $G \cong K_{\frac{n}{2}, \frac{n}{2}}$.

Proof. By Lemma 6, we have $\mu_1 \geq 2\sqrt{Z(G)/n} \geq \frac{4m}{n} \geq \frac{2m}{n-1}$. Thus, by similar arguments as in the proof of Theorem 3, we have $s_\alpha \geq f\left(2\sqrt{Z(G)/n}\right)$ for $\alpha < 0$ or $\alpha > 1$, and then (5) follows.

Similarly, $s_\alpha \leq g\left(2\sqrt{Z(G)/n}\right)$ for $0 < \alpha < 1$, and then (6) follows.

Either equality in (5) or (6) holds if and only if $\mu_2 = \cdots = \mu_{n-1}$ and $\lambda_1 = 2\sqrt{Z(G)/n}$, which, by the arguments in the proof of Theorem 4, is equivalent to $G \cong K_{\frac{n}{2}, \frac{n}{2}}$. □

Now we consider the special case $\alpha = 2$. Note that $s_2(G)$ is equal to the trace of $L^2$ where $L = L(G)$, from which it may be shown that [9]

$$s_2(G) = \sum_{u \in V(G)} (d_u^2 + d_u) = Z(G) + 2m,$$

where $m$ is the number of edges of $G$. Thus, if both the number of vertices and the number of edges are given, then the study of $s_2(G)$ is equivalent to that of $Z(G)$.

Let $G$ be a graph with $n$ vertices and $m$ edges. As restatements of the results in [5,10] on $Z(G)$, respectively, we have
\[ s_2(G) \geq 2m \left( \left\lceil \frac{2m}{n} \right\rceil + \left\lfloor \frac{2m}{n} \right\rfloor + 1 \right) - \left\lfloor \frac{2m}{n} \right\rfloor \left\lceil \frac{2m}{n} \right\rceil n \]

with equality if and only if any degree of \( G \) is either \( \left\lfloor \frac{2m}{n} \right\rfloor \) or \( \left\lceil \frac{2m}{n} \right\rceil \), and

\[ s_2(G) \leq m \left( \frac{2m}{n-1} + n \right) \]

with equality if and only if \( G \) is \( K_{1,n-1} \) or \( K_n \).

Let \( G \) be a connected graph with \( n \geq 2 \) vertices. It was proved in [9] that

\[ s_2(G) \geq 6n - 8 \]

with equality if and only if \( G \) is the path \( P_n \). An alternate argument is as follows: By Theorem 1, if \( s_2(G) \geq s_2(T) \) with equality if and only if \( G = T \), where \( T \) is a spanning tree of \( G \). Note that \( T \) has at least two vertices of degree one. By the Cauchy–Schwarz inequality, we have

\[ s_2(G) \geq s_2(T) = Z(T) + 2(n - 1) \]

\[ \geq 2 + \frac{(2(n - 1) - 2)^2}{n - 2} + 2(n - 1) = 6n - 8 \]

and then \( s_2(G) \geq 6n - 8 \) with equality if and only if \( G \) is a tree that has exactly two vertices of degree one and all other vertices have equal degrees, i.e., \( G \) is the path \( P_n \).

Finally, we turn to the special case \( \alpha = \frac{1}{2} \).

**Theorem 6.** Let \( G \) be a bipartite graph with \( n \) vertices and \( m \geq 1 \) edges. Then

\[ s_{\frac{1}{2}}(G) \geq \frac{2\sqrt{2m}}{\sqrt{n + 2}} \]

with equality if and only if \( G \cong K_2 \).

**Proof.** For \( u, v \in V(G) \), \( u \sim v \) means that \( u \) and \( v \) are adjacent in \( G \). It follows that

\[ Z(G) = \sum_{u \sim v} (d_u + d_v) \leq \sum_{u \sim v} n = mn \]

with equality if and only if \( d_u + d_v = n \) for any edge \( uv \) of \( G \), i.e., \( G \) is a complete bipartite graph. Note that \( S(G) \) possesses \( 2m \) edges, it is quadrangle-free and \( Z(S(G)) = Z(G) + 4m \). By Lemma 8

\[ E(S(G)) \geq \sqrt{\frac{(2 \cdot 2m)^3}{2Z(S(G)) - 2 \cdot 2m}} \]

\[ = \sqrt{\frac{(4m)^3}{2Z(G) + 4m} - 4m} \]

\[ \geq \sqrt{\frac{(4m)^3}{2(mn + 4m) - 4m}} \]

\[ = \frac{4\sqrt{2m}}{\sqrt{n + 2}}. \]
By Lemma 7, we have $s_{\frac{1}{2}}(G) = \frac{1}{2} E(S(G))$ and thus (7) follows. From the proof above, equality in (7) if and only if $G$ and $S(G)$ are both complete bipartite graphs, i.e., $G \cong K_2$. □

The lower bound in Theorem 6 is asymptotically best possible. For example, let $G$ be the complete bipartite graph $K_{k,k}$ and then $s_{\frac{1}{2}}(G) = \sqrt{2k} + (2k - 2)\sqrt{k}$ and the corresponding lower bound is equal to $c = \frac{2k^2}{\sqrt{k+1}}$. Obviously, $\lim_{k \to \infty} \frac{s_{\frac{1}{2}}(G)}{c} = \lim_{k \to \infty} \frac{(2k - 2 + \sqrt{2})\sqrt{k+1}}{2k\sqrt{k}} = 1$.

References