International Journal of Wavelets, Multiresolution
and Information Processing
Vol. 6, No. 5 (2008) 665–674

THE REPLACEABILITY OF SAMPLING MATRIX FOR MULTIDIMENSIONAL PERFECT RECONSTRUCTION FILTER BANKS

BO YANG
Institute of Aerospace Science and Technology
Shanghai Jiaotong University, Shanghai, 200240, P. R. China
and
College of Information and Electrical Engineering
Shandong University of Science and Technology, 579 Qianwangang Road
Huangdao, Qingdao, 266510, P. R. China
yangbo.sd@gmail.com

ZHONGLIANG JING
Institute of Aerospace Science and Technology
Shanghai Jiaotong University, Shanghai, 200240, P. R. China
zljing@sjtu.edu.cn

Received 11 April 2007
Revised 26 March 2008

It is shown under a divisibility condition the sampling matrix for a filter bank can be replaced without loss of perfect reconstruction. This is the generalization of the common knowledge that removing up/downsampling will not alter perfect reconstruction. The result provides a simple way to implement redundant perfect reconstruction filter banks, which constitute tight frames of $l_2(Z^n)$ when iterated. As an example, a quincunx-sampled frame is presented, which is nearly shift-invariant as well as limited-redundancy.

Keywords: Multidimensional; filter banks; wavelets; frames; shift invariance.

AMS Subject Classification: 42C40, 52C05, 65F99, 68R99

1. Introduction

Filter banks and wavelets, which had developed independently, have converged to form a single theory more than a decade ago.[1,2] The Discrete Wavelet Transform (DWT) has proven to be highly successful for efficient image coding.[3,4] However, the nonredundant transform lacks Shift Invariance (SI), which may be more desirable for many other signal processing tasks.[5–11]

It is well known that removing up/downsampling for a filter bank will not lose the Perfect Reconstruction (PR), on which the Undecimated DWT (UDWT)[8–10] is based. As a result, the UDWT is exactly shift-invariant; however, the transform suffers from the high degree of redundancy, making it impractical for very large multidimensional data sets. Kingsbury[11] showed that the SI of the DWT could be dramatically improved by using a Dual-Tree Complex DWT (DTCWT), which is much less redundant than the UDWT.

In this paper, we show the replaceability of sampling matrix for multidimensional PR Filter Banks (PRFB's), which is the extension of the common knowledge that removing up/downsampling will not lose PR. One of the possible applications of the result would be in building tight frames of $l_2(Z^n)$. A useful example is given in Sec. 5. We derive a Quincunx-Sampled Frame Transform (QSFT) from the standard wavelet filters. Like the DTCWT, the transform has near SI and limited redundancy.

2. Multidimensional Perfect Reconstruction Filter Banks

The multidimensional PRFB's are briefly reviewed in this section; for more details, see Refs. 12 and 13. A general multidimensional filter bank is shown in Fig. 1. The system is critically sampled, i.e. $D = |\det(\boldsymbol{D})|$.

The vectors of the analysis and synthesis filter banks can be defined as:

$$\begin{aligned} \boldsymbol{h}(\boldsymbol{z}) &= [H_0(\boldsymbol{z}), \ldots, H_{D-1}(\boldsymbol{z})]^T, \\ \boldsymbol{g}(\boldsymbol{z}) &= [G_0(\boldsymbol{z}), \ldots, G_{D-1}(\boldsymbol{z})]^T \end{aligned} \tag{2.1}$$

with $\boldsymbol{z} = (z_1, \ldots, z_n)^T$. Modulation analysis for Fig. 1 gives the output of the system:

$$Y(\boldsymbol{z}) = \frac{1}{D}\boldsymbol{g}(\boldsymbol{z})^T \boldsymbol{H}_{AC}(\boldsymbol{z})\boldsymbol{x}_{AC}(\boldsymbol{z}) \tag{2.2}$$

where $\boldsymbol{x}_{AC}$ is the Aliasing-Component (AC) *vector* for the input, $\boldsymbol{H}_{AC}$ is the AC *matrix* for the analysis bank, that is,

$$\boldsymbol{x}_{AC}(\boldsymbol{z}) = \left\{X[\boldsymbol{e}(2\pi\boldsymbol{D}^{-T}\boldsymbol{k}_0) \circ \boldsymbol{z}], \ldots, X[\boldsymbol{e}(2\pi\boldsymbol{D}^{-T}\boldsymbol{k}_{D-1}) \circ \boldsymbol{z}]\right\}^T, \tag{2.3}$$

$$\boldsymbol{H}_{AC}(\boldsymbol{z}) = \left\{\boldsymbol{h}[\boldsymbol{e}(2\pi\boldsymbol{D}^{-T}\boldsymbol{k}_0) \circ \boldsymbol{z}], \ldots, \boldsymbol{h}[\boldsymbol{e}(2\pi\boldsymbol{D}^{-T}\boldsymbol{k}_{D-1}) \circ \boldsymbol{z}]\right\}. \tag{2.4}$$

In (2.3) and (2.4), the *Schur* product of two complex vectors is defined as:

$$\begin{aligned} \boldsymbol{e}(\boldsymbol{\omega}) \circ \boldsymbol{z} &= \left(e^{-j\omega_1}, \ldots, e^{-j\omega_n}\right)^T \circ (z_1, \ldots, z_n)^T \\ &= \left(e^{-j\omega_1} z_1, \ldots, e^{-j\omega_n} z_n\right)^T \end{aligned} \tag{2.5}$$

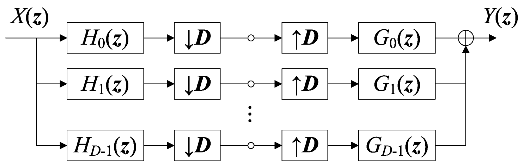


Fig. 1. A general multidimensional D-channel filter bank.

with $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n)^T$; the vectors $\boldsymbol{k}_i$ are a set of samples belonging to $\mathrm{FP}(\boldsymbol{D}^T)$ (the *Fundamental Parallelepiped* (FP) of the transposed matrix)[13] and shall always be chosen such that $\boldsymbol{k}_0 = \boldsymbol{0}$; $2\pi\boldsymbol{D}^{-T}\boldsymbol{k}_i$ represent physically *aliasing offsets* in frequency.

Equation (2.2) yields the following result on PR[12,13]:

Theorem 2.1. *PR Condition*: *Perfect reconstruction is achieved iff*

$$\boldsymbol{g}(\boldsymbol{z})^T \boldsymbol{H}_{AC}(\boldsymbol{z}) = D(\underbrace{1\ 0\ \cdots\ 0}_{D}). \tag{2.6}$$

The AC matrix is *paraunitary* for an orthogonal filter bank, that is,

$$\boldsymbol{H}_{AC}(\boldsymbol{z})\boldsymbol{H}^*_{AC}(\boldsymbol{z}) = \boldsymbol{H}^*_{AC}(\boldsymbol{z})\boldsymbol{H}_{AC}(\boldsymbol{z}) = D\boldsymbol{I}, \tag{2.7}$$

$$G_i(\boldsymbol{z}) = H_i^*(\boldsymbol{z}), \quad i = 0, \ldots, D-1. \tag{2.8}$$

3. The Replaceability of Sampling Matrix

We now use $\boldsymbol{D'}$ to replace the sampling matrix $\boldsymbol{D}$, while the original analysis/synthesis filters are still used. Analyzing the replaced system as in Sec. 2, we obtain

$$Y(\boldsymbol{z}) = \frac{1}{D'}\boldsymbol{g}(\boldsymbol{z})^T \boldsymbol{H}'_{AC}(\boldsymbol{z})\boldsymbol{x}'_{AC}(\boldsymbol{z}), \tag{3.1}$$

$$\boldsymbol{x'}_{AC}(\boldsymbol{z}) = \left\{X[\boldsymbol{e}(2\pi\boldsymbol{D'}^{-T}\boldsymbol{k'}_0) \circ \boldsymbol{z}], \ldots, X[\boldsymbol{e}(2\pi\boldsymbol{D'}^{-T}\boldsymbol{k'}_{D'-1}) \circ \boldsymbol{z}]\right\}^T, \tag{3.2}$$

$$\boldsymbol{H'}_{AC}(\boldsymbol{z}) = \left\{\boldsymbol{h}[\boldsymbol{e}(2\pi\boldsymbol{D'}^{-T}\boldsymbol{k'}_0) \circ \boldsymbol{z}], \ldots, \boldsymbol{h}[\boldsymbol{e}(2\pi\boldsymbol{D'}^{-T}\boldsymbol{k'}_{D'-1}) \circ \boldsymbol{z}]\right\}, \tag{3.3}$$

with $D' = |\det(\boldsymbol{D'})|$ and $\boldsymbol{k'}_i \in \mathrm{FP}(\boldsymbol{D'}^T)$.

From (2.4) and (3.3), we observe that $\boldsymbol{H}_{AC}$ and $\boldsymbol{H'}_{AC}$ have the same first-column, which is the alias-free version of the analysis bank. If all aliased versions contained by $\boldsymbol{H'}_{AC}$ (from the second to the last column) are also contained by $\boldsymbol{H}_{AC}$, from (2.6) one can derive:

$$\boldsymbol{g}(\boldsymbol{z})^T \boldsymbol{H'}_{AC}(\boldsymbol{z}) = D(\underbrace{1\ 0\ \cdots\ 0}_{D'}). \tag{3.4}$$

Substituting into (3.1) and equalizing the proportion, PR is achieved for the replaced system.

Next, we have to find when the set of aliasing offsets generated in (3.3) is a subset of that generated in (2.4). We define the set associated with $\boldsymbol{D}$ as $\mathrm{AO}(\boldsymbol{D})$, that is,

$$\begin{aligned}\mathrm{AO}(\boldsymbol{D}) &= \{2\pi\boldsymbol{D}^{-T}\boldsymbol{k} \,|\, \boldsymbol{k} \in \mathrm{FP}(\boldsymbol{D}^T)\} \\ &= \{2\pi\boldsymbol{D}^{-T}\boldsymbol{k} \in [0, 2\pi)^n \,|\, \boldsymbol{k} \in Z^n\}.\end{aligned} \tag{3.5}$$

Proposition 3.1. *Assuming nonsingular integer matrices $\boldsymbol{D}$ and $\boldsymbol{D'}$, there is* $\mathrm{AO}(\boldsymbol{D'}) \subseteq \mathrm{AO}(\boldsymbol{D})$ *iff $\boldsymbol{D'}^{-1}\boldsymbol{D}$ is an integer matrix, i.e. $\boldsymbol{D}$ is left-divisible by $\boldsymbol{D'}$.*

Proof. Prove sufficiency first. Letting $\boldsymbol{M} = \boldsymbol{D'}^{-1}\boldsymbol{D}$, using (3.5), one can write

$$\mathrm{AO}(\boldsymbol{D'}) = \left\{2\pi\boldsymbol{D}^{-T}(\boldsymbol{M}^T\boldsymbol{k}) \in [0, 2\pi)^n \ \mid \boldsymbol{k} \in Z^n\right\}. \tag{3.6}$$

Since $\boldsymbol{M}$ is an integer matrix, one has

$$\left\{\boldsymbol{k} \mid \boldsymbol{M}^T\boldsymbol{k} \in Z^n\right\} \supseteq Z^n. \tag{3.7}$$

Consequently, one can write

$$\begin{aligned}&\left\{2\pi\boldsymbol{D}^{-T}(\boldsymbol{M}^T\boldsymbol{k}) \in [0, 2\pi)^n \ \mid \boldsymbol{k} \in Z^n\right\}\\ &\quad\subseteq \left\{2\pi\boldsymbol{D}^{-T}(\boldsymbol{M}^T\boldsymbol{k}) \in [0, 2\pi)^n \ \mid (\boldsymbol{M}^T\boldsymbol{k}) \in Z^n\right\}\end{aligned} \tag{3.8}$$

that is, $\mathrm{AO}(\boldsymbol{D'}) \subseteq \mathrm{AO}(\boldsymbol{D})$.

To prove necessity, we first prove a fact that for any $\boldsymbol{k} \in Z^n$ and nonsingular integer matrix $\boldsymbol{D}$, there exist $\boldsymbol{l} \in Z^n$ and $\boldsymbol{a} \in \mathrm{AO}(\boldsymbol{D})$ such that

$$2\pi\boldsymbol{D}^{-T}\boldsymbol{k} = 2\pi\boldsymbol{l} + \boldsymbol{a}. \tag{3.9}$$

It is clear that there exists $\boldsymbol{\xi} \in [0, 2\pi)^n$ such that

$$2\pi\boldsymbol{D}^{-T}\boldsymbol{k} = 2\pi\boldsymbol{l} + \boldsymbol{\xi}. \tag{3.10}$$

Left multiplying both sides of (3.10) by $\boldsymbol{D}^T$, one obtains

$$(2\pi)^{-1}\boldsymbol{D}^T\boldsymbol{\xi} = \boldsymbol{D}^T(\boldsymbol{D}^{-T}\boldsymbol{k} - \boldsymbol{l}) \in Z^n. \tag{3.11}$$

From (3.5) and (3.11), one can derive $\xi \in \mathrm{AO}(\boldsymbol{D})$, thus, the fact is true. Next rewriting (3.9) with $\boldsymbol{D'}$, one obtains

$$2\pi\boldsymbol{D'}^{-T}\boldsymbol{k} = 2\pi\boldsymbol{l} + \boldsymbol{a'}, \quad \boldsymbol{a'} \in \mathrm{AO}(\boldsymbol{D'}). \tag{3.12}$$

For $\mathrm{AO}(\boldsymbol{D'}) \subseteq \mathrm{AO}(\boldsymbol{D})$, one has $\boldsymbol{a'} \in \mathrm{AO}(\boldsymbol{D})$, and thus $(2\pi)^{-1}\boldsymbol{D}^T\boldsymbol{a'} \in Z^n$. Left multiplying (3.12) by $(2\pi)^{-1}\boldsymbol{D}^T$, one can write

$$\forall\, \boldsymbol{k} \in Z^n, \quad \boldsymbol{D}^T\boldsymbol{D'}^{-T}\boldsymbol{k} = [\boldsymbol{D}^T\boldsymbol{l} + (2\pi)^{-1}\boldsymbol{D}^T\boldsymbol{a'}] \in Z^n. \tag{3.13}$$

Therefore, $\boldsymbol{D'}^{-1}\boldsymbol{D}$ must be an integer matrix. □

Now we are ready to state the following theorem.

Theorem 3.1 (Replaceability Theorem (RT)). *The sampling matrix for a filter bank can be replaced without loss of PR if the original sampling matrix is left-divisible by the new one.*

Let us consider two special cases, both of which have been already used in the fields of filter banks and wavelets.

Lemma 3.1. *Removing sampling* (*i.e.* $\boldsymbol{D'} = \boldsymbol{I}$) *will not alter PR, on which undecimated systems are based.*

Lemma 3.2. *Adding a unimodular*[14] *sampler* (*i.e.* $\boldsymbol{D'} = \boldsymbol{DV}$ *with unimodular* $\boldsymbol{V}$) *will not alter PR, which is used to remove the frequency scrambling for directional filter banks.*[15]

In addition, there is a link between the replaced system and the *cycle-spinning* method.[16] Let $\mathrm{LAT}(\boldsymbol{D})$ denote a *sampling lattice* generated by $\boldsymbol{D}$. The divisibility essentially means $\mathrm{LAT}(\boldsymbol{D}) \subseteq \mathrm{LAT}(\boldsymbol{D'})$. From lattice theory, we know that $\mathrm{LAT}(\boldsymbol{D'})$ in this case can be generated by the union of D/D' sublattices $\mathrm{LAT}(\boldsymbol{D})$, related to the conception of *cosets*.[17] In this sense, the replacement of sample matrix (for the $1 < D' < D$ case) is equivalent to an *incomplete* cycle-spinning; the iterative approach, however, is avoided for the same quality.

Note that all results derived above hold true in one dimension, where the matrices reduce to scalars.

4. Connection to Tight Frames

It is well known that an orthogonal filter bank constructs a discrete orthonormal *basis* when iterated.[18] Similarly, the replaced version of an orthogonal bank constructs a tight *frame*[19] of $l_2(Z^n)$. In this case, the equivalent synthesis filters for i iteration steps are written as

$$\begin{aligned} G_0'^{(i)}(\boldsymbol{z}) &= \prod_{m=0}^{i-1} G_0(\boldsymbol{z}^{\boldsymbol{D'}^m}), \\ G_j'^{(i)}(\boldsymbol{z}) &= G_j(\boldsymbol{z}^{\boldsymbol{D'}^{i-1}}) G_0'^{i-1}(\boldsymbol{z}). \end{aligned} \tag{4.1}$$

The discrete frame functions are represented as:

$$\begin{aligned} \varphi'_{i,\boldsymbol{l}} &= \alpha \cdot g_0'^{(i)}(\boldsymbol{n} - \boldsymbol{D'}^i \boldsymbol{l}), \\ \psi'_{i,j,\boldsymbol{l}} &= \alpha \cdot g_j'^{(i)}(\boldsymbol{n} - \boldsymbol{D'}^i \boldsymbol{l}), \quad \boldsymbol{l} \in Z^n, \end{aligned} \tag{4.2}$$

where $g_j'^{(i)}(\boldsymbol{n})$ is the impulse responses of $G_j'^{(i)}(\boldsymbol{z})$ and $\alpha = (D'/D)^{i/2}$ is chosen to offset the power increase resulting from the change of sampling.

An overcomplete set is then obtained:

$$\Gamma = \left\{ \psi'_{i,j,\boldsymbol{l}},\ \varphi'_{I,\boldsymbol{l}} \mid i = 1, 2, \ldots, I; j = 1, 2, \ldots, D-1; \boldsymbol{l} \in Z^n \right\}. \tag{4.3}$$

If the original filter bank is orthogonal, from the PR of the replaced system we derive that for $\forall\, x(\boldsymbol{n}) \in l_2(Z^n)$, one has

$$x = \sum_i \sum_j \sum_{\boldsymbol{l}} \langle \psi'_{i,j,\boldsymbol{l}}, x \rangle\, \psi'_{i,j,\boldsymbol{l}} + \sum_{\boldsymbol{l}} \langle \varphi'_{I,\boldsymbol{l}}, x \rangle\, \varphi'_{I,\boldsymbol{l}}. \tag{4.4}$$

From (4.4), one can easily derive that

$$\|x\|_2^2 = \sum_i \sum_j \sum_{\boldsymbol{l}} \left| \langle \psi'_{i,j,\boldsymbol{l}}, x \rangle \right|^2 + \sum_{\boldsymbol{l}} \left| \langle \varphi'_{I,\boldsymbol{l}}, x \rangle \right|^2. \tag{4.5}$$

As a result, we obtain the following lemma:

Lemma 4.1. *Assuming an orthogonal filter bank, the set* Γ *constitutes a tight frame of* $l_2(Z^n)$.

Such a frame analysis can also be extended to the biorthogonal case. In this case, the equivalent analysis filters also construct a group of frame functions

$$\begin{aligned} \tilde{\varphi}'_{i,\boldsymbol{l}} &= \alpha \cdot h_0'^{(i)}(\boldsymbol{D'}^i\boldsymbol{l} - \boldsymbol{n}), \\ \tilde{\psi}'_{i,j,\boldsymbol{l}} &= \alpha \cdot h_j'^{(i)}(\boldsymbol{D'}^i\boldsymbol{l} - \boldsymbol{n}), \end{aligned} \tag{4.6}$$

which corresponds to another set

$$\tilde{\Gamma} = \{\tilde{\psi}'_{i,j,\boldsymbol{l}},\ \tilde{\varphi}'_{I,\boldsymbol{l}} \mid i = 1, 2, \ldots, I; j = 1, 2, \ldots, D-1; \boldsymbol{l} \in Z^n\}. \tag{4.7}$$

It is not difficult to show that $\{\tilde{\Gamma}, \Gamma\}$ constructs a pair of *dual frames*,[19] that is, for $\forall\, x(\boldsymbol{n}) \in l_2(Z^n)$, there is a more general expansion form

$$x = \sum_i \sum_j \sum_{\boldsymbol{l}} \left\langle \tilde{\psi}'_{i,j,\boldsymbol{l}}, x \right\rangle \psi'_{i,j,\boldsymbol{l}} + \sum_{\boldsymbol{l}} \left\langle \tilde{\varphi}'_{I,\boldsymbol{l}}, x \right\rangle \varphi'_{I,\boldsymbol{l}}. \tag{4.8}$$

Different with the standard wavelet expansion, the representations in (4.4) and (4.8) are redundant by a factor r, where

$$r = (D-1) \sum_{i=1}^{I} D'^{-i} + D'^{-I}. \tag{4.9}$$

If $D > D' > 1$, the redundancy is *bounded* by $1 < r \le (D-1)/(D'-1)$.

5. A Useful Example: The QSFT

The replaceability of sampling matrix provides a simple and direct way to implement redundant PRFB's and frame transforms. We consider in this section a quincunx-sampled example, i.e. the QSFT, which is derived from the standard DWT using this method.

The rectangular sampling for the standard DWT is shown in Fig. 2(a), and one can write

$$\boldsymbol{D}_R = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad \mathrm{AO}(\boldsymbol{D}_R) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \pi \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \pi \end{pmatrix}, \begin{pmatrix} \pi \\ \pi \end{pmatrix} \right\}. \tag{5.1}$$

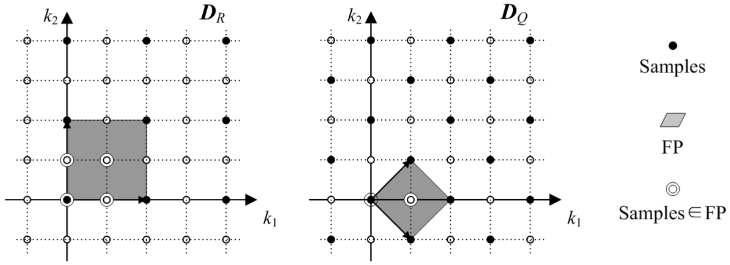
The matrix $\boldsymbol{D}_R$ is clearly divisible by any integer matrix with determinant ± 2, such as $\boldsymbol{D}_Q$, where

$$\boldsymbol{D}_Q = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \mathrm{AO}(\boldsymbol{D}_Q) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \pi \\ \pi \end{pmatrix} \right\} \tag{5.2}$$

corresponds to the quincunx sampling,[13,17] see Fig. 2(b).

It is clear that $\mathrm{AO}(\boldsymbol{D}_Q) \subseteq \mathrm{AO}(\boldsymbol{D}_R)$, consistent with RT. The matrix $\boldsymbol{D}_Q$ provides unique advantages over other possible matrices, besides satisfying the divisibility condition:

- All eigenvalues of $\boldsymbol{D}_Q$ have magnitude greater than one, making sure an associated frame is dilated in all dimensions;



(a) Rectangular sampling (b) Quincunx sampling

Fig. 2. Rectangular and quincunx sampling lattices generated by $\boldsymbol{D}_R$ and $\boldsymbol{D}_Q$, respectively.

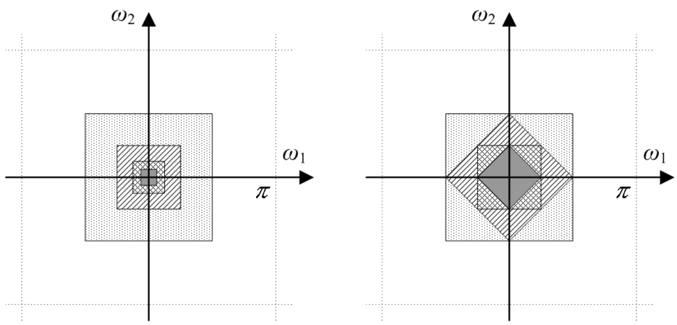
- The second power of $\boldsymbol{D}_Q$ is a diagonal matrix (i.e. $\boldsymbol{D}_Q^2 = \boldsymbol{D}_R$), implying that a rectangular sampling is achieved after every other iteration step.

In addition, the matrix can be easily extended to multi-dimensions (for more details, see Ref. 20).

The replacement of $\boldsymbol{D}_R$ by $\boldsymbol{D}_Q$ makes the QSFT provide some desirable properties, as follows:

Higher Frequency-Resolution: The QSFT has a higher sampling not only in time but also in frequency. This can be well understood by examining the frequency supports of equivalent lowpass filters. As shown in Fig. 3, the basic spectrum for the QSFT rotates by $45°$ and reduces by a factor $\sqrt{2}$ after every iteration step, more gradually than it does in the DWT case. As a result, the transform has intermediate scales, similar to the critically sampled quincunx DWT.[13,21]

Near SI: Having more samples within the equivalent scale makes the QSFT less shift-sensitive than the DWT. Following Kingsbury's illustration,[11] the improved SI can be indicated by reconstructing a disc image from only its wavelet or scaling function coefficients at a single scale. Figure 4 shows the reconstruction components



(a) DWT (b) QSFT

Fig. 3. Frequency supports of equivalent lowpass filters for DWT vs. QSFT.

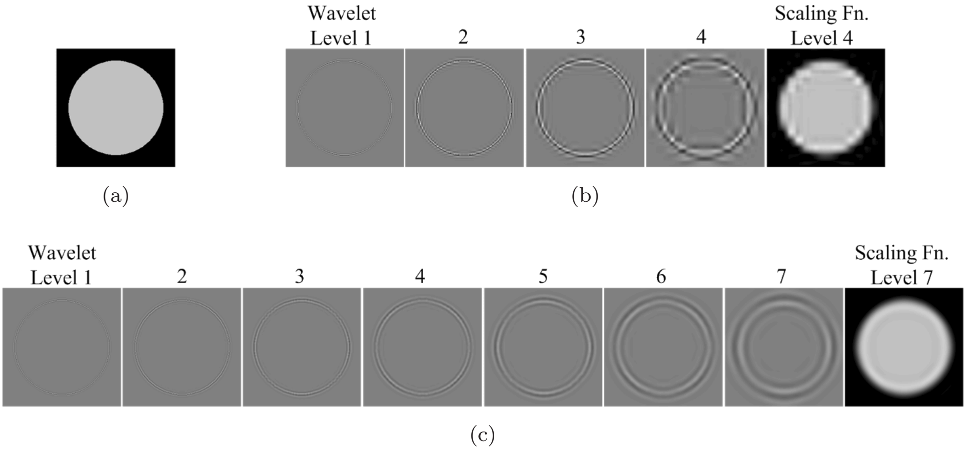


Fig. 4. SI for QSFT vs. DWT. (a) Input image; (b) Reconstruction components for DWT; (c) Reconstruction components for QSFT.

Table 1. Redundancy and complexity for various wavelets in two-dimensions.

Wavelet Methods	Redundancy Ratio	Complexity
DWT	1	$O(N)$
UDWT	$3I + 1$	$O(N \log N)$
DTCWT	4	$O(N)$
QSFT	$3 - 2^{1-I}$	$O(N)$

for the QSFT and the DWT under the same wavelets, DB2 (the Daubechies wavelet of order 2). Near-perfect circular arcs are generated by the QSFT; contrast these with the severely distorted arcs for the DWT.

Low Redundancy and Complexity: Let N refer to the size of the image and I refer to the number of the decomposition levels. In Table 1, we compare the redundancy and computational complexity of various transforms. The QSFT is at most 3-times redundant, better than the DTCWT and the UDWT. Note that the redundancy for the QSFT and the DTCWT is limited, whereas for the UDWT it increases infinitely with the number of the decomposition levels. In the complexity, the QSFT is of order $O(N)$, same as the DWT.

6. Conclusion

We have shown the replaceability of the sampling matrix for general filter banks, which is the generalization of the common knowledge that removing up/downsampling will not lose PR. The replaceability provides a simple way to build redundant filter banks and tight frames of $l_2(Z^n)$. As an example, we implemented the QSFT using the standard wavelet filters. The transform has a higher sampling in both time and frequency; consequently, it provides higher frequency-resolution, near SI, besides low redundancy and complexity.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (60775022) and the Scientific Research Startup Fund of Shandong University of Science and Technology (011123131).

References

1. M. Vetterli and C. Herley, Wavelets and filter banks: Theory and design, *IEEE Trans. Signal Process.* **40**(9) (1992) 2207–2232.
2. S. Mallat, A theory for multiresolution signal decomposition: The wavelet representation, *IEEE Trans. Pattern. Anal. Mach. Intell.* **11**(7) (1989) 674–693.
3. M. Antonini, M. Barlaud, P. Mathieu and I. Daubechies, Image coding using wavelet transform, *IEEE Trans. Image Process.* **1**(2) (1992) 205–220.
4. M. Vetterli and J. Kovacevic, *Wavelets and Subband Coding* (Prentice-Hall Englewood, NJ, 1995).
5. G. T. Strang, Wavelets and dilation equations: A brief introduction, *SIAM Rev.* **31**(4) (1989) 614–627.
6. S. Mallat, Zero-crossing of a wavelet transform, *IEEE Trans. Inform. Theory* **37**(4) (1991) 1019–1033.
7. E. P. Simoncelli, W. T. Freeman, E. H. Adelson and D. J. Heeger, Shiftable multiscale transforms, *IEEE Trans. Inform. Theory* **38**(2) (1992) 587–607.
8. M. J. Shensa, The discrete wavelet transform: Wedding the à trous and Mallat algorithm, *IEEE Trans. Signal Process.* **40**(10) (1992) 2464–2482.
9. M. Unser, Texture classification and segmentation using wavelet frames, *IEEE Trans. Image Process.* **4**(11) (1995) 1549–1560.
10. O. Rockinger, Image sequence fusion using a shift invariant wavelet transform, in *Proc. IEEE ICIP*, ed. I. Torwick, Vol. 3 (IEEE, 1997), pp. 288–291.
11. N. G. Kingsbury, Complex wavelets for shift invariant analysis and filtering of signals, *Appl. Comput. Harmon. Anal.* **10**(3) (2001) 234–253.
12. E. Viscito and J. P. Allebach, The analysis and design of multidimensional FIR perfect reconstruction filter banks for arbitrary sampling lattices, *IEEE Trans. Circuits Syst.* **38**(1) (1991) 29–42.
13. J. Kovacevic and M. Vetterli, Nonseparable multidimensional perfect reconstruction filter banks and wavelet bases for R_n, *IEEE Trans. Inform. Theory* **38**(2) (1992) 533–555.
14. M. Newman, *Integral Matrices* (Academic Press, New York, 1972).
15. S. Park, M. J. T. Smith and R. M. Mersereau, Improved structures of maximally decimated directional filter banks for spatial image analysis, *IEEE Trans. Image Process.* **13**(11) (2004) 1424–1431.
16. R. R. Coifman and D. L. Donoho, Translation-invariant denoising, in *Lecture Notes in Statistics*: *Wavelets and Statistics*, eds. A. Antoine and G. Oppenheim (Springer-Verlag, 1995), pp. 125–150.
17. J. Cassels, *An Introduction to the Geometry of Numbers* (Springer-Verlag, Berlin, 1971).
18. O. Rioul, A discrete-time multiresolution theory, *IEEE Trans. Signal Process.* **41**(8) (1993) 2591–2606.

19. I. Daubechies, *Ten Lectures on Wavelets* (SIAM, Philadelphia, 1992).
20. J. Kovacevic and M. Vetterli, FCO sampling of digital video using perfect reconstruction filter banks, *IEEE Trans. Image Process.* **2**(1) (1993) 118–122.
21. M. Feilner, D. Van De Ville and M. Unser, An orthogonal family of quincunx wavelets with continuously adjustable order, *IEEE Trans. Image Process.* **14**(4) (2005) 499–510.