On recurrent solutions in high-dimensional non-dissipative Lorenz models: the role of nonlinear feedback loop

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10th Chaotic Modeling and Simulation International Conference
(CHAOS2017) Barcelona, Spain
June 1, 2017

Presentations at the Chaos 2017

9.00-10.30	**Chair: Ángela Jiménez-Casas**	**Models and Modeling I**
	Inertial Manifold of the Nonlinear Dynamical System Governing a Thermosyphon Model	Ángela Jiménez-Casas
	Modeling of summarization process according to basic concepts of text invariants	Elena Y. Aivas (Buriak), Olga V. Lazarenko, Dmitrii I. Panchenko
	On quasi-periodic solutions associated with the extended nonlinear feedback loop in the five-dimensional non-dissipative Lorenz model	Sara Faghih-Naini, Bo-Wen Shen
	New Method for Constructing Solutions of Nonlinear Partial Differential Equations	Iosif Andrushkevich
	Operative Scheme of the Short-Range Complex Weather Forecasting and Its Applications to Prediction in Medicine and in Electric Power Industry	Philipp L. Bykov, Vladimir A. Gordin

14.00-15.30	**Chair: Beatrice Venturi, Co-Chair: Bo-Wen Shen**	**Models and Modeling II**
	On recurrent solutions in high-dimensional non-dissipative Lorenz models	Bo-Wen Shen, Sara Faghih-Naini
	CHAOTIC SOLUTIONS AND GLOBAL INDETERMINACY IN THE ROMER ENDOGENOUS GROWTH MODEL	Beatrice Venturi, Giovanni Bella, Paolo Mattana
	ANALYSIS OF A DISCRETE MODEL OF PREY-PREDATOR SYSTEM WITH PREY REFUGE	Prasenjit Das
	Chaos and a quantitative modeling of the kinetics of phase transitions on the final measure areas	Ivan G. Grabar, Olga I. Grabar, Yuri O. Kubrak, Mykola M. Marchuk
	Kleptoparasitism and complexity in a multi-trophic web	Massimo Materassi

Lorenz Models

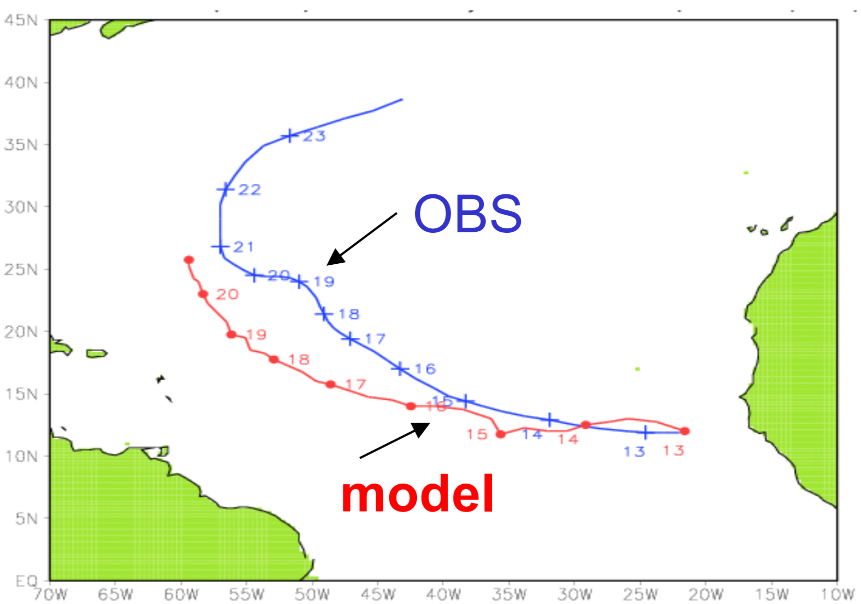
model	r_c	heating terms	solutions	references
3DLM	24.74	rX	steady or chaotic	Lorenz (1963)
3D-NLM	n/a	rX	periodic	Shen (2014b)
5DLM	42.9	rX	steady or chaotic	Shen (2014a,2015a,b)
5D-NLM	n/a	rX	quasi-periodic	Faghih-Naini and Shen (2017)
6DLM	41.1	rX, rX_1	steady or chaotic	Shen (2015a,b)
7DLM	116.9	rX	steady or chaotic	Shen (2016a,b)
7D-NLM	n/a	rX	quasi-periodic	Shen and Faghih-Naini (2017)
8DLM	103.4	rX, rX_1	steady or chaotic	Shen(2016b)
9DLM	102.9	rX, rX_1, rX_2	steady or chaotic	Shen(2016b)
9DLMr	679.8	rX	steady or chaotic	In preparation

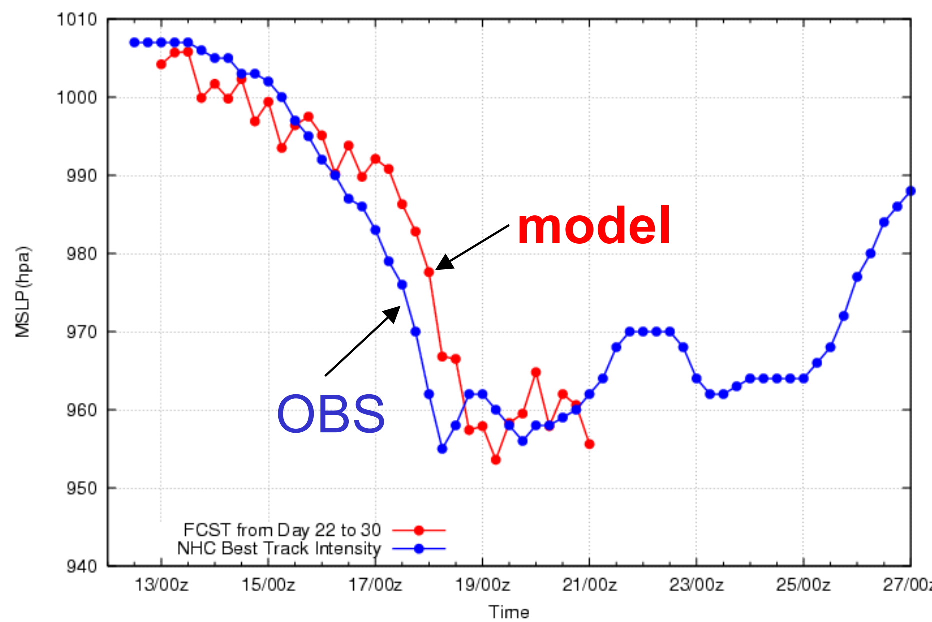
rc: a critical value of Raleigh parameter for the onset of chaos

Outline

1. Introduction (butterfly effect)
   - The Three-Dimensional Lorenz Model (3DLM, Lorenz, 1963)
     -- the nonlinear feedback loop (NFL)
   - The 5DLM (Shen 2014)
     -- an extension of nonlinear feedback loop
     -- negative nonlinear feedback (stabilization)
   - The 6DLM (Shen 2015b)
     -- impact of additional heating term (destabilization)
2. Results: the role of the extended nonlinear feedback loop
   - The 7DLM and 9DLM (Shen 2016a)
     -- hierarchical scale dependence
   - The 7D nondissipative LM (7D-NLM)
     -- quasi-periodic solutions with three incommensurate frequencies
3. Summary

Promising 30-day Simulations of Hurricane Helene





1. Are the simulations of tropical cyclone (TC) genesis consistent with Chaos theory?

2. Why can the high-resolution global model have skills?

Shen, B.-W., W.-K. Tao, and M.-L. Wu, 2010b: African Easterly Waves and African Easterly Jet in 30-day High-resolution Global Simulations. A Case Study during the 2006 NAMMA period. Geophys. Res. Lett., L18803, doi:10.1029/2010GL044355.

(Helene: 12-24 September, 2006)

Chaos and Butterfly Effect

The following folklore has been used as an analogy of the butterfly effect (Gleick, 1987):

"For want of a nail, the shoe was lost.
For want of a shoe, the horse was lost.
For want of a horse, the rider was lost.
For want of a rider, the battle was lost.
For want of a battle, the kingdom was lost.
And all for the want of a horseshoe nail."

However, Lorenz (2008) made the following comments:

1. *Let me say right now that I do not feel that this verse is describing true chaos, but better illustrates the simpler phenomenon of instability.*

2. *The implication is that subsequent small events will not reverse the outcome.*

- Gleick, J., 1987: Chaos: Making a New Science. New York. Penguin. 360pp.
- Lorenz, E., 2008: The butterfly effect. Premio Felice Pietro Chisesi e Caterina Tomassoni award lecture, University of Rome, Rome, April 2008.

“Responses”

Lorenz’s comments support the view that the verse neither describes negative (nonlinear) feedback nor indicates recurrence, the latter of which is required for the appearance of a butterfly pattern.

Our studies have been performed to understand their individual and/or collective impact of nonlinearity, heating and dissipation on the following characteristics of a chaotic system defined by Devaney (1989): (1) sensitivity to initial conditions; (2) topological transitivity; and (3) dense periodic points.

We have illustrated the following roles by the nonlinear feedback loop and its interactions with dissipative and heating terms.

1. Negative nonlinear feedback using 5D, 7D, and 9DLMs
2. Positive nonlinear feedback using 6D and 8DLMs
3. Recurrence (in periodic and quasi-periodic solutions) using non-dissipative 3D, 5D and 7D LMs (with Sara)

Objectives

- Determine the physical process that is responsible for the recurrence of the solution.
    - the nonlinear feedback loop (NFL)
- Propose a systematic approach for selecting new (spatial) modes to improve the stability of high-dimensional Lorenz models.
    - an analysis on the extension of the nonlinear feedback loop
- Examine the impact of the extended nonlinear feedback loop on the appearance of quasi-periodic solutions with incommensurate frequencies.
- Understand whether a steady solution, a chaotic solution, a limit cycle solution or a high-dimensional torus can better describe weather.

Equations for Rayleigh-Benard Convection

By assuming 2D (x,z), incompressible and Boussinesq flow, the following equations that describe the so-called Rayleigh-Benard Convection were used in Lorenz (1963):

This does not appear explicitly in the Lorenz model.

$$\frac{\partial}{\partial t}\nabla^2\psi = -J(\psi, \nabla^2\psi) + \nu\nabla^4\psi + g\alpha\frac{\partial\theta}{\partial x},$$

Non-linear terms

$$\frac{\partial\theta}{\partial t} = -J(\psi, \theta) + \frac{\Delta T}{H}\frac{\partial\psi}{\partial x} + \kappa\nabla^2\theta,$$

Boundary Forcing, which is represented by 'r'.

Here ψ is the streamfunction that gives $u = -\psi_z$ and $w = \psi_x$. θ is the temperature perturbation. The constants, g, α, ν, and κ denote the acceleration of gravity, the coefficient of thermal expansion, the kinematic viscosity, and the thermal conductivity, respectively.

- Navier-Stokes equation with constant viscosity
- Heat transfer equation with constant thermal conductivity

3-Dimenaional Lorenz Model (3DLM)

1) **r** – Rayleigh number: (Ra/Rc)

a dimensionless measure of temperature difference between top and bottom surfaces of liquid; proportional to effective force on fluid;

2) **σ** – Prandtl number: (v/κ)

the ratio of the kinetic viscosity (κ, momentum diffusivity) to thermal diffusivity (v);

3) **b** – Physical proportion: $(4/(1+a^2))$, b=8/3;

4) **a** – a=l/m, the ratio of the vertical height h of the fluid layer to the horizontal size of the convection rolls. b =8/3;

- l=aπ/H and m=π/H.

$$\frac{dX}{d\tau} = -\sigma X + \sigma Y, \qquad M_1$$

$$\frac{dY}{d\tau} = -XZ + rX - Y, \qquad M_2$$

$$\frac{dZ}{d\tau} = XY - bZ. \qquad M_3$$

-XZ is associated with the J(M1, M3), indicating the impact of the M3 mode. With no –XZ, the above system is reduced to become a system with linear terms only, leading to an unstable solution as r>1.

Physical Processes in Lorenz Models

- Dissipative terms (e.g., -bZ)
- Heating terms (e.g., rX)
- Nonlinear terms. (e.g., -XZ and XY)

(A) Dissipative Lorenz models

- small r, steady state solutions
- moderate r, chaotic solutions
- large r, limit cycles

(B) non-Dissipative Lorenz models

- "Periodic" solutions or homoclinic orbits
- Quasi-periodic solutions in high-dimensional Lorenz models

Dissipative Lorenz models

- large dissipations, steady state solutions
- moderate dissipations, chaotic solutions
- small dissipations, limit cycles

Nonlinear Feedback Loop

- -XZ and XY form a nonlinear feedback loop (NFL).
- Mathematically, the NFL leads to complex eigenvalues in locally linear systems.
- Physically, the NFL acts as a nonlinear restoring force to produce nonlinear oscillatory solutions (e.g., 3D-NLM).

Six Modes and their Derivatives

These three modes were used in the original Lorenz model

mode	$\frac{\partial}{\partial x}$	$\frac{\partial^*}{\partial x}$	$\frac{\partial}{\partial z}$
$M_1 = \sqrt{2} sin(lx) sin(mz)$	$\sqrt{2} l cos(lx) sin(mz)$	lM_2	$\sqrt{2} m sin(lx) cos(mz)$
$M_2 = \sqrt{2} cos(lx) sin(mz)$	$-\sqrt{2} l sin(lx) sin(mz)$	$-lM_1$	$\sqrt{2} m cos(lx) cos(mz)$
$M_3 = sin(2mz)$			$2m cos(2mz)$
$M_4 = \sqrt{2} sin(lx) sin(3mz)$	$\sqrt{2} l cos(lx) sin(3mz)$	lM_5	$3\sqrt{2} m sin(lx) cos(3mz)$
$M_5 = \sqrt{2} cos(lx) sin(3mz)$	$-\sqrt{2} l sin(lx) sin(3mz)$	$-lM_4$	$3\sqrt{2} m cos(lx) cos(3mz)$
$M_6 = sin(4mz)$			$4m cos(4mz)$

Three additional modes with two additional vertical wavenumbers are included in this study.

The Nonlinear Feedback Loop in the 3DLM

In the original 3D Lorenz model, the Jacobian has only two terms:

$$J(\psi,\theta) = C_1 C_2 \Big(XY\, J(M_1, M_2) - XZ\, J(M_1, M_3) \Big).$$

$J(M_1, M_3) \approx -mlM_2 \rightarrow \dfrac{dY}{d\tau} = -XZ + rX - Y,$ M_2

$J(M_1, M_2) = mlM_3 \rightarrow \dfrac{dZ}{d\tau} = XY - bZ.$ M_3

A loop appears as $M_2 \rightarrow M_3 \rightarrow (-M_2)$

- In the original 3D Lorenz model, a feedback loop forms with $J(M_1,M_2)$ and $J(M_1,M_3)$.
- In the 5DLM, the inclusion of M_5 extends the feedback loop.
- Note that the feedback loop also indicates that any error growth of small-scale perturbations will be remained within the system. Namely, <u>they can not *upscale further to alter large-scale (or environmental) flow.*</u>

Five-Dimensional Lorenz Model (5DLM)

$\tau = \kappa(1+a^2)(\pi / H)^2 t$ (dimensionless time)

1) **r** – Rayleigh number: (Ra/Rc)

   a dimensionless measure of temperature difference between top and bottom surfaces of liquid; proportional to effective force on fluid

2) **σ** – Prandtl number: (ν/κ)

   the ratio of the kinetic viscosity (κ, momentum diffusivity) to thermal diffusivity (ν)

3) **b** – Physical proportion: $(4/(1+a^2))$

4) **d** – $(9+a^2) /(1+a^2)$

5) **a** – the ratio of the vertical height h of the fluid layer to the horizontal size of the convection rolls. It turns out that for b =8/3, the convection begins for the smallest value of the Rayleigh number, that is, for the smallest value of the temperature difference.

$$\frac{dX}{d\tau} = -\sigma X + \sigma Y,$$

$$\frac{dY}{d\tau} = -XZ + rX - Y,$$

$$\frac{dZ}{d\tau} = XY - XY_1 - bZ,$$

$$\frac{dY_1}{d\tau} = XZ - 2XZ_1 - dY_1,$$

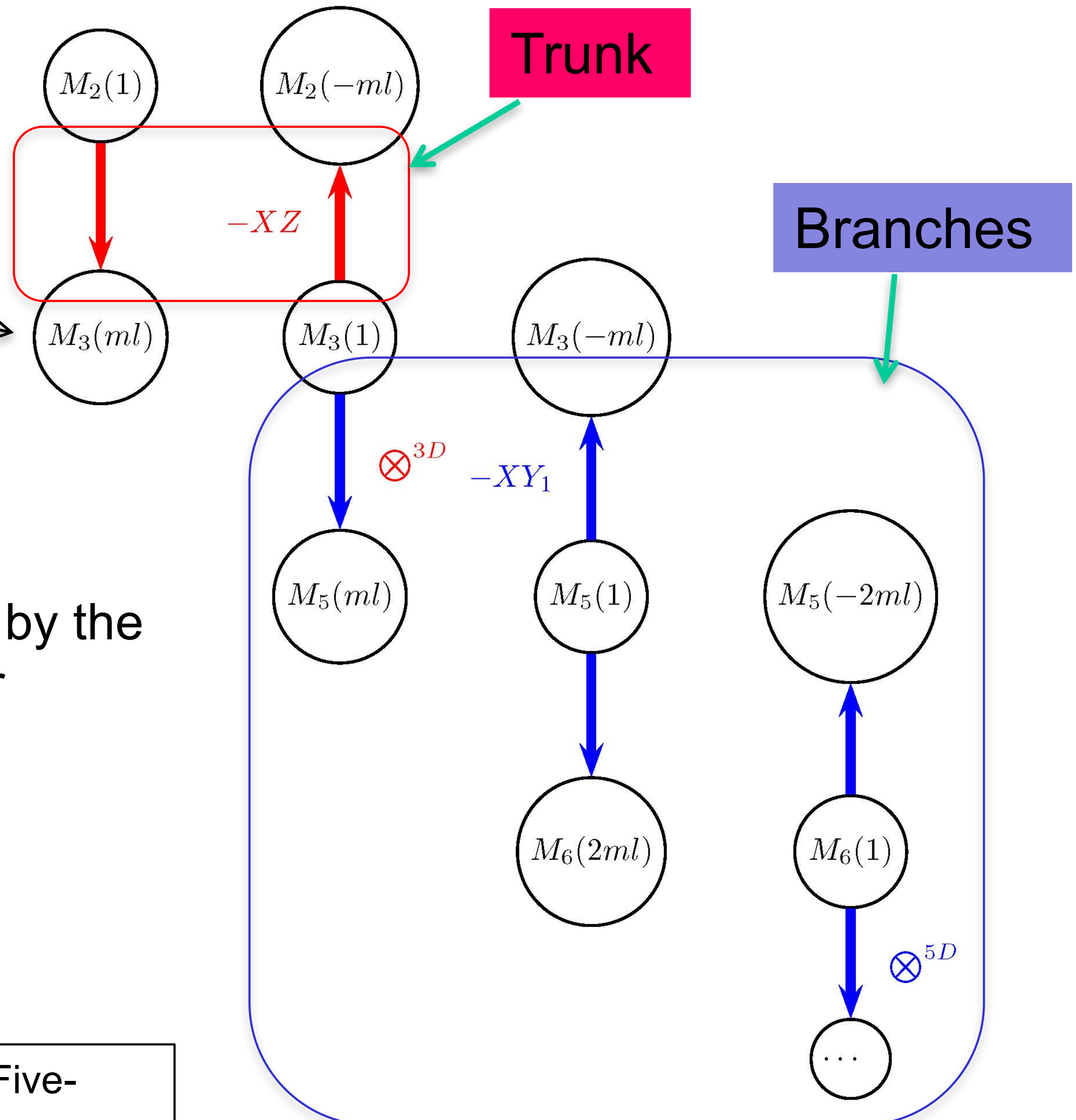
$$\frac{dZ_1}{d\tau} = 2XY_1 - 4bZ_1.$$

Major negative feedback term

An Extension of Nonlinear Feedback Loop

Scale Interactions via $J(M_1,M_j)$

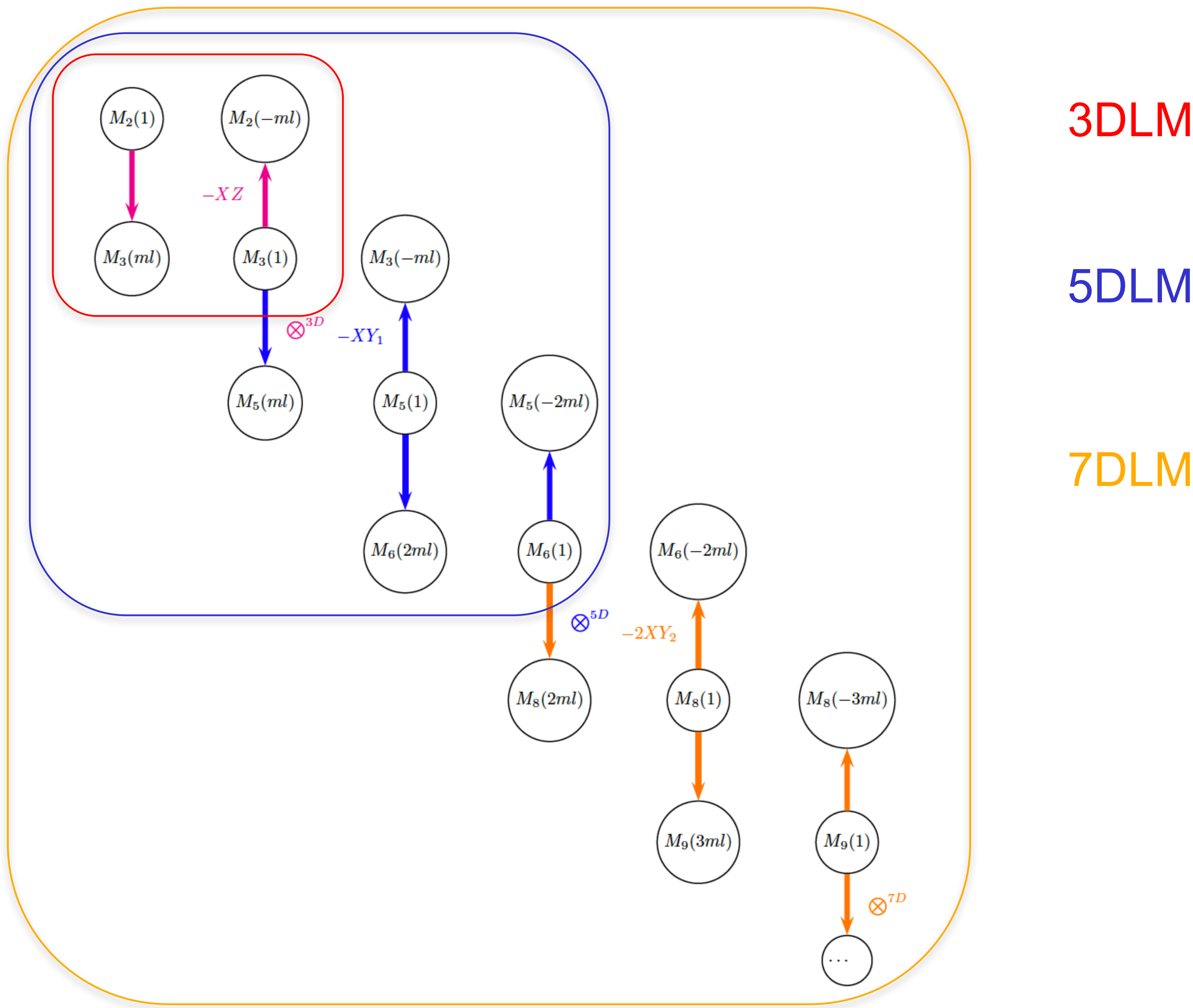
Jacobian	Outcome	Coef
$J(M_1, M_2)$	mlM_3	XY
$J(M_1, M_3)$	$ml(M_5 - M_2)$	XZ



The degree of nonlinearity is measured by the degree of the extension of the nonlinear feedback loop.

Shen, Bo-Wen, 2014a: Nonlinear Feedback in a Five-dimensional Lorenz Model. (JAS)

Extended Nonlinear Feedback Loop in the 7DLM



Nondissipative Lorenz Models

5D-NLM

$$\frac{dX}{d\tau} = \cancel{-\sigma X} + \sigma Y,$$

$$\frac{dY}{d\tau} = -XZ + rX - \cancel{Y},$$

$$\frac{dZ}{d\tau} = XY - XY_1 - \cancel{bZ},$$

$$\frac{dY_1}{d\tau} = XZ - 2XZ_1 - \cancel{d_0 Y_1},$$

$$\frac{dZ_1}{d\tau} = 2XY_1 - \cancel{4bZ_1}.$$

7D-NLM

$$\frac{dX}{d\tau} = \cancel{-\sigma X} + \sigma Y,$$

$$\frac{dY}{d\tau} = -XZ + rX - \cancel{Y},$$

$$\frac{dZ}{d\tau} = XY - XY_1 - \cancel{bZ},$$

$$\frac{dY_1}{d\tau} = XZ - 2XZ_1 - \cancel{d_0 Y_1},$$

$$\frac{dZ_1}{d\tau} = 2XY_1 - 2XY_2 - \cancel{4bZ_1},$$

$$\frac{dY_2}{d\tau} = 2XZ_1 - 3XZ_2 - \cancel{d_0 Y_2},$$

$$\frac{dZ_2}{d\tau} = 3XY_2 - \cancel{9bZ_2}.$$

Locally Linear Systems

$$A^{5D} = \begin{pmatrix} 0 & \sigma & 0 & 0 & 0 \\ 0 & 0 & -X_c & 0 & 0 \\ 0 & X_c & 0 & \boxed{-X_c} & 0 \\ 0 & 0 & \boxed{X_c} & 0 & -2X_c \\ 0 & 0 & 0 & 2X_c & 0 \end{pmatrix}$$

$$A^{7D} = \begin{pmatrix} 0 & \sigma & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -X_c & 0 & 0 & 0 & 0 \\ 0 & X_c & 0 & \boxed{-X_c} & 0 & 0 & 0 \\ 0 & 0 & \boxed{X_c} & 0 & -2X_c & 0 & 0 \\ 0 & 0 & 0 & 2X_c & 0 & \boxed{-2X_c} & 0 \\ 0 & 0 & 0 & 0 & \boxed{-2X_c} & 0 & -3X_c \\ 0 & 0 & 0 & 0 & 0 & 3X_c & 0 \end{pmatrix}$$

Uncoupled Locally Linear Systems

$$A^{5D} = \begin{pmatrix} 0 & \sigma & 0 & 0 & 0 \\ 0 & 0 & -X_c & 0 & 0 \\ 0 & X_c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2X_c \\ 0 & 0 & 0 & 2X_c & 0 \end{pmatrix}$$

Two frequencies, X_c and $2X_c$

$$A^{7D} = \begin{pmatrix} 0 & \sigma & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -X_c & 0 & 0 & 0 & 0 \\ 0 & X_c & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2X_c & 0 & 0 \\ 0 & 0 & 0 & 2X_c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3X_c \\ 0 & 0 & 0 & 0 & 0 & 3X_c & 0 \end{pmatrix}$$

Three frequencies, X_c , $2X_c$, and $3X_c$

Locally Linear Systems

$$A^{5D} = \begin{pmatrix} 0 & \sigma & 0 & 0 & 0 \\ 0 & 0 & -X_c & 0 & 0 \\ 0 & X_c & 0 & \boxed{-X_c} & 0 \\ 0 & 0 & \boxed{X_c} & 0 & -2X_c \\ 0 & 0 & 0 & 2X_c & 0 \end{pmatrix}$$

Two incommensurate frequencies

$$A^{7D} = \begin{pmatrix} 0 & \sigma & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -X_c & 0 & 0 & 0 & 0 \\ 0 & X_c & 0 & \boxed{-X_c} & 0 & 0 & 0 \\ 0 & 0 & \boxed{X_c} & 0 & -2X_c & 0 & 0 \\ 0 & 0 & 0 & 2X_c & 0 & \boxed{-2X_c} & 0 \\ 0 & 0 & 0 & 0 & \boxed{-2X_c} & 0 & -3X_c \\ 0 & 0 & 0 & 0 & 0 & 3X_c & 0 \end{pmatrix}$$

Three incommensurate frequencies

Lorenz Models

model	r_c	heating terms	solutions	references
3DLM	24.74	rX	steady or chaotic	Lorenz (1963)
3D-NLM	n/a	rX	periodic	Shen (2014b)
5DLM	42.9	rX	steady or chaotic	Shen (2014a,2015a,b)
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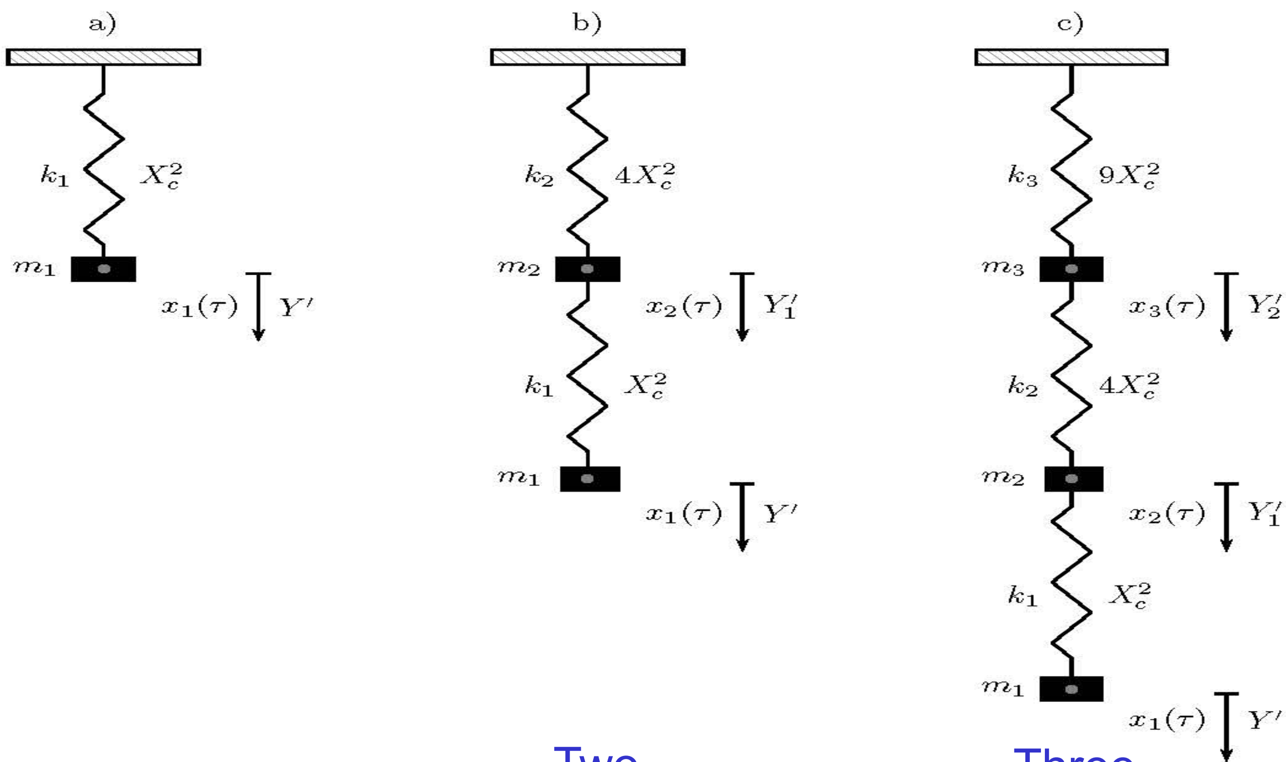
9DLMr	679.8	rX	steady or chaotic	In preparation

rc: a critical value of Raleigh parameter for the onset of chaos

Linearized Lorenz Models vs. Coupled Systems

3D-NLM 5D-NLM 7D-NLM

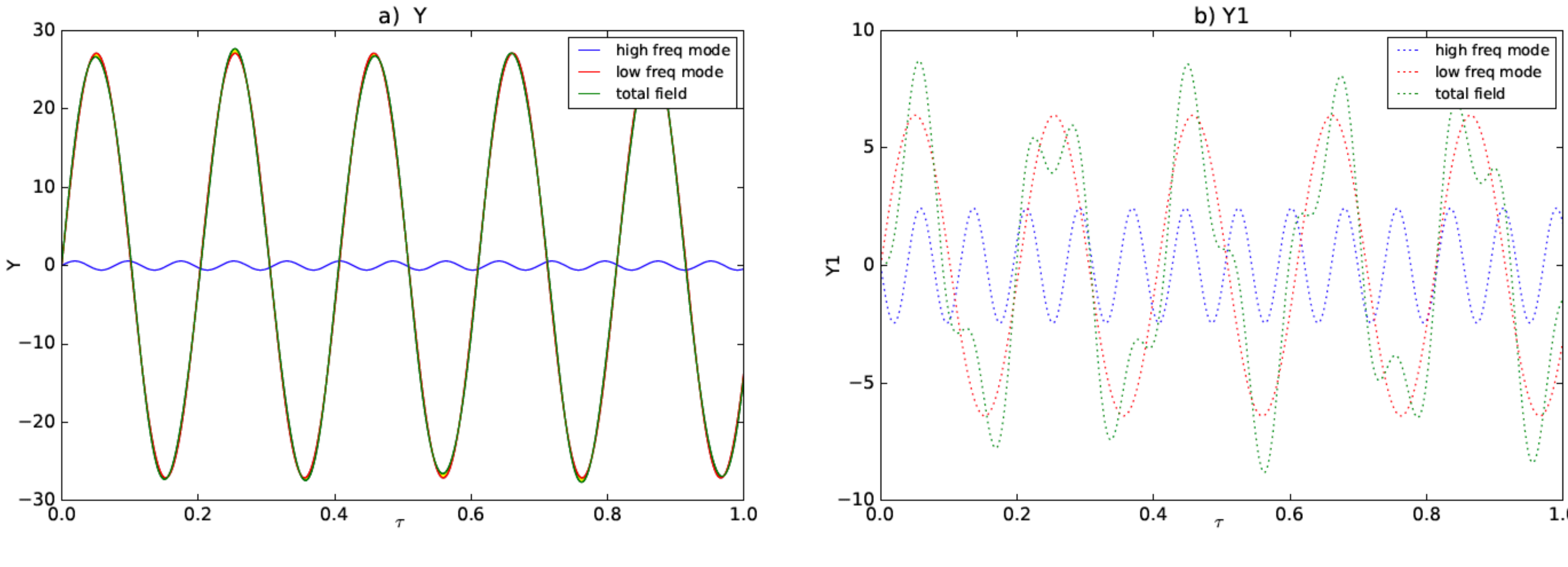


Two incommensurate frequencies

Three incommensurate frequencies

Systems with one mass and one spring (a), two masses and two springs (b) and three masses and three springs (c). Three masses are identical, i.e. $m_1=m_2=m_3$. Three spring constants k_1, k_2 and k_3 are selected as X_c^2, $4X_c^2$, and $9X_c^2$, respectively. It is shown that the governing equations for the above systems in panels (a)-(c) are identical to those for the locally linear 3D-NLM, 5D-NLM, and 7D-NLM, respectively. This comparison illustrates how the nonlinear feedback loop and its extension enabled by a proper selection of high wavenumber modes can produce recurrent (i.e., periodic or quasi-periodic) solutions.

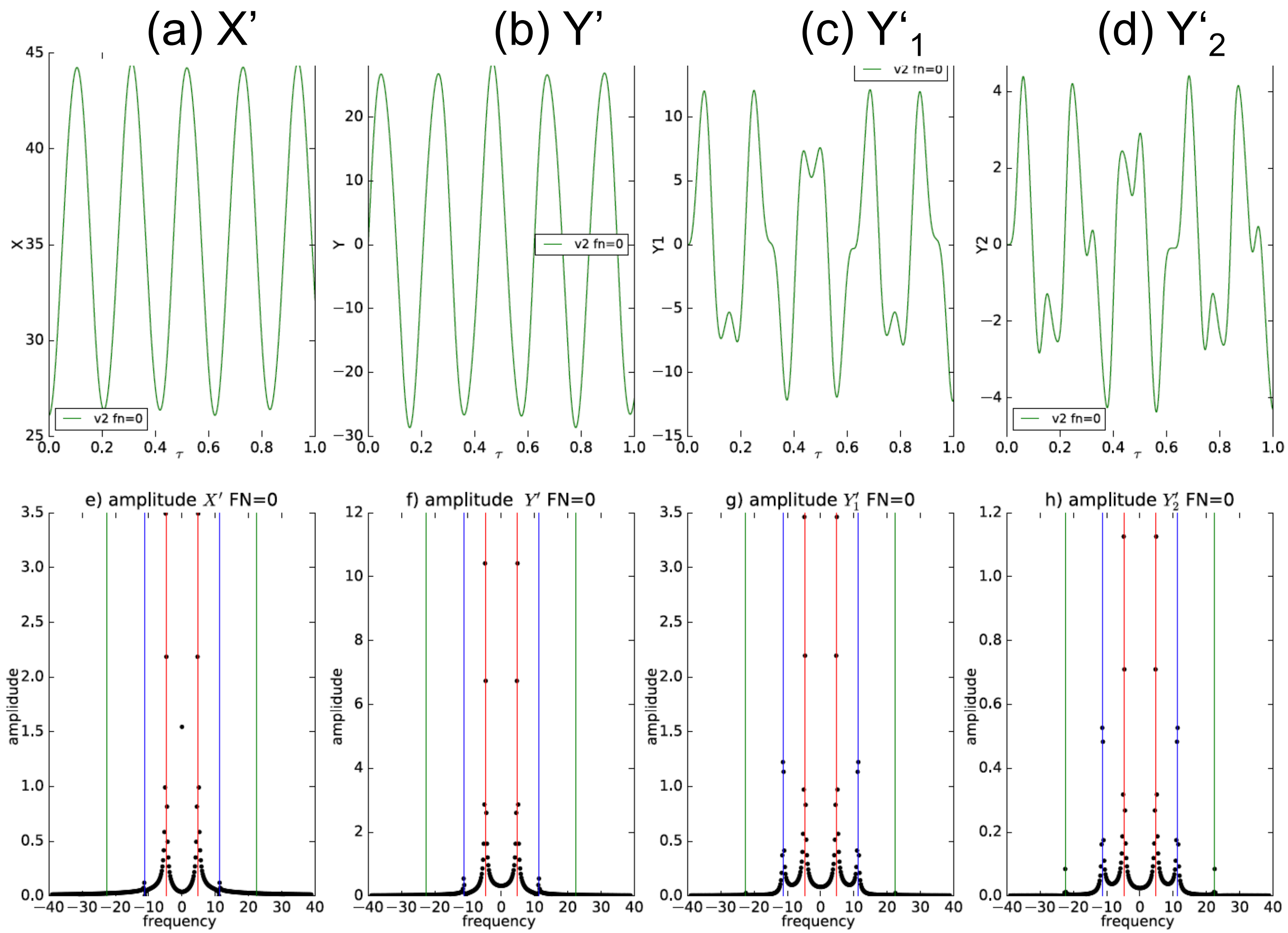
Two Spatial Modes in the 5D-NLM



Our the linear analytical solutions suggest that:

- The primary spatial mode (Y’), which is dominated by the low frequency component, appears ”periodic” (panel a).
- The secondary spatial mode (Y'_1), which has a ratio of 0.38 between the amplitude of the high-frequency component and that of the low-frequency component, displays “quasi-periodicity”, appearing “more irregular” as compared to the primary spatial mode (Y’) (panel b).

Three Spatial Modes in the 7D-NLM



spectral analysis

Three prominent frequencies for Y'_2.

The tertiary mode (Y'_2) with the smallest spatial scale displays three prominent incommensurate frequencies, appearing the most "irregular" solution (right panels).

Summary

Challenging questions that have been (partially) addressed are:

- how can we build models (e.g., increasing resolutions) to improve the extended-range (15-30 days) numerical weather predictions?
- Is weather chaotic?

In additional to real-world modeling studies, we have derived high-dimensional Lorenz models to illustrate the following processes that are associated with the nonlinear feedback loop and its interactions with dissipative and heating terms:

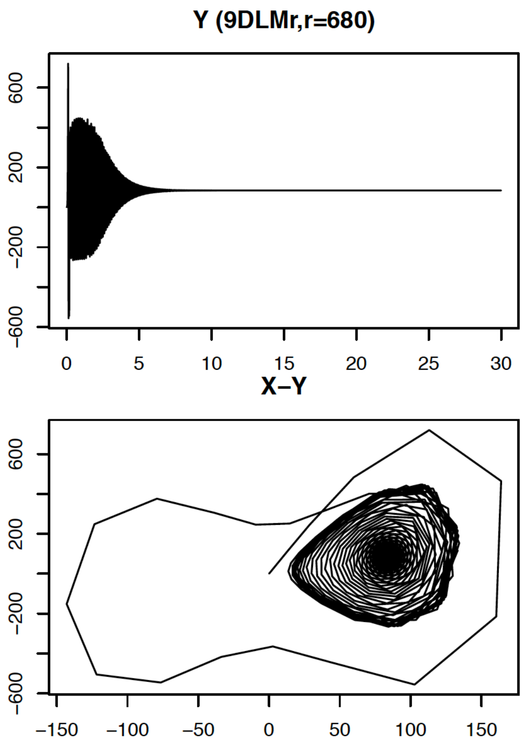
1. Negative nonlinear feedback using the 5D, 7D, and 9DLMs;
2. Positive nonlinear feedback using the 6D and 8DLMs;
3. Recurrence (in periodic and quasi-periodic solutions) using the non-dissipative 3D, 5D and 7D LMs.

Summary: the Role of Nonlinear Feedback Loop

- Using the dissipative LMs or non-dissipative LMs (NLMs), we have discussed the association of recurrent solutions (e.g., quasi-periodic solutions and limit cycles) with the nonlinear feedback loop (NFL).
- Mathematically, the NFL leads to complex eigenvalues in locally linear systems.
- Physically, it acts as a nonlinear restoring force to produce nonlinear oscillatory solutions (e.g., in the 3D-NLM).
- In the 5D-NLM, the occurrence of two incommensurate frequencies is associated with the extended NFL that provides two-way interactions between the primary (X,Y,Z) and secondary Fourier modes (Y_1, Z_1).
- Using the 7D-NLM (9D-NLM) that was derived based on the further extension of the nonlinear feedback loop, we obtain quasi-periodic solutions with three (four) incommensurate frequencies.
- A mathematical analogy between the linearized NLMs and the systems with various springs is derived, indicating the association of recurrent solutions with the NFL.

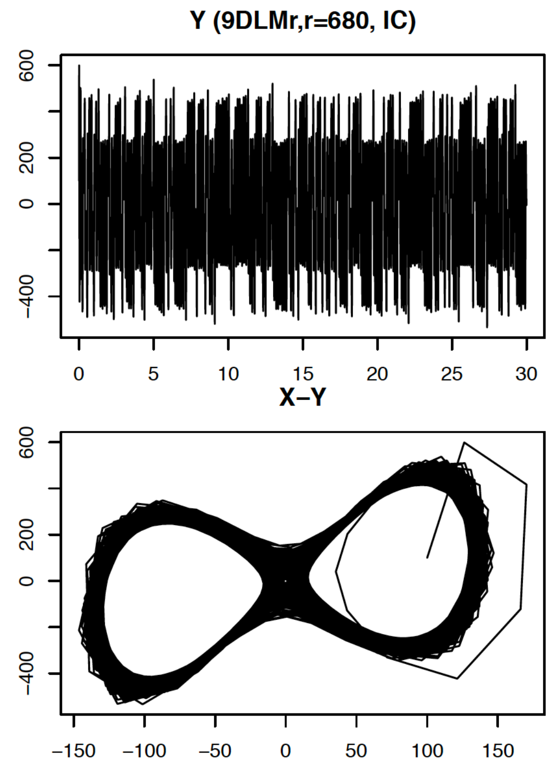
Future Work

- The 9DLM is derived based on the extension of the nonlinear feedback loop in the 7DLM.
- A linear stability analysis suggests that non-trivial critical points are stable.
- An analysis suggests that the ensemble Lyapunov exponent (eLE) becomes positive when r is equal or greater than 679.8.
- Is the solution with r=680 chaotic? A strange attractor or "Hamiltonian"-type chaos?



(x=0, y=1, z=0, y1=0, z1=0, y2=0, z2=0, y3=0, z3=0)

a strange attractor,
a limit cycle or
a high-dimensional
torus?



(x=100, y=100, z=100, y1=100, z1=100, y2=100, z2=100, y3=100, z3=100)

Acknowledgements

- SDSU: Ricardo Carretero, Joey Lin and Sam Shen
- UTA: Zdzislaw E. Musielak
- CU: Roger Pielke Sr.
- The University of Arizona: Xubin Zeng
- Federal University of Rio de Janeiro: Julio Buchnann
- UCAR: Richard Anthes
- University of Hamburg: V. Lucarini
- NC A&T State U.: Yuh-Lang Lin
- NASA/JPL: Frank Li
- NASA/GSFC: Chung-Lin Shie
- UAH: Yu-Ling Wu
- NASA/ARC: Samson Cheung and David Kao
- Royal Meteorological Institute of Belgium: S. Vannitsem

- Funding Sources: **SDSU** College of Sciences, NASA ESTO (Earth Science Technology Office) **AIST (Advanced Information System Technology) Program**