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ONE METHOD OF SOLVING LINEAR MULTIPARAMETER EIGENVALUE PROBLEM

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AUTHORS' CONTRIBUTIONS

This work was carried out in collaboration between all authors. Authors VVK and BMP designed the study, wrote the protocol and performed preliminary data analysis. Author OSY was conducted the field студії and prepared the initial draft. All authors read and approved the final manuscript.

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ABSTRACT

We consider a nonlinear multiparameter spectral problem in the real Euclidian space. In correspondence to this problem we put a variation problem on minimization of a specific functional. The equivalence of these two problems is proved. Beside that, based on a gradient procedure, we propose a numerical method for finding the eigenvalues and the eigenvectors of the spectral problem. Finally, we prove the convergence of this method and illustrate its application by several examples.

Keywords: Multiparameter eigenvalue problem; variation problem; iterative procedure.

Subject Classification AMS: 47J10, 47J30, 65F15, 65H17.

1. INTRODUCTION

Consider the following operator equation:

$$T(\lambda)x = f$$

with the function $T(\lambda): E^m \to X(H)$ and *X(H)* being a set of linear operators, that acts in the Hilbert space *H*. The function $T(\lambda)$ depends on several spectral parameters $\lambda_1, \lambda_2, \dots, \lambda_m$ in a linear or a nonlinear way.

The solvability analysis of such operator equations leads to the problem of finding such parameters λ_i, $i = 1, 2, \dots, m$, that there exists a non-trivial solution of the corresponding homogeneous equation $T(\lambda)x = 0$. Such problems arise in many areas of analysis and mathematical physics. In particular, a series of inverse problems concerning the synthesis of radiating systems is reduced to the nonlinear equations that have non-unique solution. During the study of quantitative and qualitative characteristics of such solutions it is necessary find appropriate solutions of homogeneous linear integral equations

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with non-linear (one or two) spectral parameters, ie, it is arise the problem of research and solving the generalized eigenvalue problems (see, for example [1-6]. Since the spectral parameters are the geometric and electromagnetic characteristics of radiating systems, the solution of this problem makes it possible to obtain the necessary information still at the design stage, choosing relatively of the size and of the electromagnetic characteristics of radiation systems, ie it is carry out computational experiment. Such problems are still insufficiently studied both from the theoretical point of view and from the point of view of the construction of numerical methods and algorithms to solve them.

In the literature it is widely explored the different formulations of such problems, the corresponding spectral theory as well as numerous methods and algorithms of solving them. See, for example [1-17].

In this paper we consider the multiparameter eigenvalue problem of the following structure

$$T(\lambda)x \equiv Ax - \sum_{i=1}^{m} \lambda_i B_i x = 0, \qquad (1)$$

where E^n is a real Euclidian space, and all the scalar parameters $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m) \in E^m$ are the spectral ones.

We also suggest a variation approach for solving this problem. The idea is to replace the given multi-parameter spectral problem by an equivalent variation problem of minimization of a specific functional. Note, that despite the similarity of the approach, our technique is different from one, presented in [9-11,13,15,18]. The point is that in order to find the minimum of the functional we propose to apply the gradient procedure to an expanded problem in the space $E^n \oplus E^m$. This allows us to simultaneously receive the eigenvalues and the eigenvectors of the initial problem. Another advantage of this method is the simplicity of calculation of one step of the gradient iteration. Finally, the practical application of the algorithm is illustrated on several examples.

2. EIGENVECTORS AND EIGENVELUES AS POINTS OF MINIMUM

Let E^n be the real Euclidian space with the scalar product $(\cdot,\cdot)_{E^n}$ and the norm $\|\cdot\|_{E^n}$, and $A,\ B_i : E^n \to E^n$, $i = 1, 2, \dots, m$ be a square $n \times n$ matrix.

The multiparameter eigenvalue problem consists in finding a set of spectral parameters $\lambda = \{\lambda_1, \dots, \lambda_m\} \in E^m$ such, that the exists a non-trivial solution $x \neq 0$ of the equation (1). The set of spectral parameters $\lambda = \{\lambda_1, \dots, \lambda_m\}$ we will name the generalized eigenvalue or eigenvalue set, and the corresponding vector x we will name the generalized eigenvector of the problem (1). The manifold of all such sets $\lambda = \{\lambda_1, \dots, \lambda_m\}$ of *m*-dimensional vector space E^m is called the "eigenvalue surface", and for *m*=2 it is called "eigenvalue curve". For *m* = 1 we obtain the classical eigenvalue problem of the form

$$Ax = \lambda B_1 x$$

Along with the problem (1) we consider the problem of finding such set of parameters $\lambda = \{\lambda_1, \dots, \lambda_m\}$ and such vectors x on which functional

$$F(u) = \frac{1}{2}\| T(\lambda)x \|^2, \quad \forall u = \{x, \lambda\} \in H = E^n \oplus E^m,\ x \neq 0 \qquad (2)$$

reaches its minimum value, i.e.

$$F(u) \to \min_u, \quad u \in U \subset H, (u \neq 0 \qquad (3)$$

where U is a set which contains the points $u^* = \{x^*, \lambda^*\}$ that satisfy the equation (1), H is the Hilbert space in which the scalar product and norm are defined as follows:

$$(u,v)_H = (u_1,u_2)_{E^n} + (v_1,v_2)_{E^m},\ \|u\|_H = \sqrt{\|u_1\|_{E^n}^2 + \|v_1\|_{E^m}^2},$$
$$u = \{u_1, v_1\},\ v = \{u_2, v_2\},\ u_1, u_2 \in E^n,\ v_1, v_2 \in E^m.$$

Finally, let us denote the set of points of minimum of functional $F(u)$ on U as

$$U_* = \{u : u \in U, \quad F(u) = 0\}.$$

Now we prove that the problems (1) and (3) are equivalent/

Theorem 1. *Every eigenvector* x^* *that corresponds to its eigenvalue set* $\lambda^* = (\lambda_1^*, \ldots, \lambda_m^*)$ *of problem* (1) *is the stationary point* $u^* = \{x^*, \lambda^*\}$ *of functional* (2) *and, conversely, every stationary point* $u^* = \{x^*, \lambda^*\}$ *of functional* (2) *corresponds to eigenvector* x^* *and its eigenvalue set* $\lambda^* = (\lambda_1^*, \ldots, \lambda_m^*)$ *of problem* (1).

Proof. Let us analyze the increment of the functional (2) $F(u + \Delta u) - F(u) = = F(x + h, \lambda + q) - F(x, \lambda)$ for arbitrary $u,\ u + \Delta u \in U$, where $\Delta u = \{h, q\} \in U$.

After simple transformations we obtain that

$$F(u + \Delta u) - F(u) =$$

$$= F(x+h, \lambda+q) - F(x,\lambda) = (T(\lambda)x, T(\lambda)h)_{E^n} - (T(\lambda)x, \sum_{i=1}^{m} B_i x q_i)_{E^n} +$$

$$+\frac{1}{2}\left\{(T(\lambda)h, T(\lambda)h)_{E^n} - 2(T(\lambda)h, \sum_{i=1}^{m} B_i x q_i)_{E^n} - 2(T(\lambda)x, \sum_{i=1}^{m} B_i h q_i)_{E^n} + \right.$$

$$\left. + 2(\sum_{i=1}^{m} B_i x q_i, \sum_{i=1}^{m} B_i x q_i)_{E^n}\right\} + o(\|\Delta u\|_H^2).$$

Thus, the first differential of $F(u)$ can be described by the formula below:

$$d\{F(x,\lambda);(h,q)\} = (T(\lambda)x, T(\lambda)h)_{E^n} - \sum_{i=1}^{m} (T(\lambda)x, B_i x)_{E^n} q_i =$$

$$= (T^*(\lambda)T(\lambda)x, h)_{E^n} + (f(\lambda,x), q)_{E^m} = (u_g, \Delta u)_H,$$

Where

$$f(\lambda,x) = (f_1(\lambda,x), f_2(\lambda,x), \ldots, f_m(\lambda,x)), \quad f_i(\lambda,x) = -(T(\lambda)x, B_i x)_{E^n},\ i = 1,2,\ldots, m,$$

From here for the gradient of functional (2) we obtain the expression

$$grad\, F(u) = \nabla F(u) = u_g = \left\{(T^*(\lambda)T(\lambda)x, e_1), \ldots (T^*(\lambda)T(\lambda)x, e_n),\right.$$

$$\left. f_1(\lambda,x), f_2(\lambda,x), \ldots, f_m(\lambda,x)\right\}, \tag{4}$$

where $e_i \in E^n$ is such vector that its *i*-th element is equal to 1, and the rest is equal to zero.

Let $T(\lambda)x = 0,\ \ x \neq 0$. Then from (4) immediately follows that $\nabla F(u) = 0$. Let $\nabla F(u) = 0$. Then from (4) we also obtaine that

$$T^*(\lambda)T(\lambda)x = 0 \Rightarrow$$
$$\Rightarrow (T^*(\lambda)T(\lambda)x, x)_{E^n} = 0 \Rightarrow (T(\lambda)x, T(\lambda)x)_{E^n} = 0 \Rightarrow T(\lambda)x = 0,$$

that proves the theorem statement.

Remark. Since $F(u) \geq 0$, $F(u^*) = 0$ for $u, u^* \in U$, then each stationary point u^* of functional $F(u)$ is the point of its local (and global) minimum.

Therefore, we showed that the problem (1) and the problem of finding stationary points of the functional $F(u)$ are equivalent.

3. ALGORITHM AND ITS CONVERGENCE

This result allows us to construct a gradient procedure for numerical solution of the problem (3), and hence the problem (1) in the form

$$u_{k+1} = u_k - \gamma(u_k)\nabla F(u_k)\,, \quad k = 0, 1, 2, \dots . \tag{5}$$

The relation (5) describes the whole class of methods which differs only by the choice of the step $\gamma_k = \gamma(x_k)$. In this paper we propose to calculate the value γ_k at each iteration by using the formula:

$$\gamma_k = \frac{F(u_k)}{\| \nabla F(u_k) \|_H^2} . \tag{6}$$

Further, to simplify writing, the index H in the notation of scalar product and norm we will omit.

Finally, the iterative process can be written in the following form:

$$u_{k+1} = u_k - \frac{F(u_k)}{\left\|\nabla F(u_k)\right\|_H^2}\nabla F(u_k), \quad k = 0,1,2,... \tag{7}$$

It would be interesting to mention that such choice of the coefficient γ_k is not a random one. It can be shown that the expression in the numerator of (6) descends to zero faster than one in the denominator of (6). It gives us the practical proof of the convergence of the method. Beside that, the proposed iterative process (7) converges to the approximate solution of (3) faster than the similar process, presented in [11,13,14], where the coefficient γ_k is calculated as $\gamma_k = F(u_k) / \| \nabla F(u_0) \|_H^2$.

As for the theoretical aspect, the local convergence of the method (7) is confirmed by the following theorem.

Theorem 2. *Let the matrix T(λ) of the eigenvalue problem (1) be such that the gradient of the functional (2) satisfies the Lipschitz condition*

$$\left\|\nabla F(u) - \nabla F(z)\right\| \leq L\left\|u - z\right\|, \quad \forall u, z \in U,\ L > 0\,, \tag{8}$$

where U is some closed convex set that includes a point of minimum u^* *of the functional (2). If for some initial approximation* $u_0 = (x_0, \lambda^{(0)}) \in U$ *the following evaluation is fulfilled*

$$0 < \gamma_0 \equiv \gamma(u_0) \le 1/2L, \tag{9}$$

then the iterative process (7) converges to the point of minimum of the functional (2) $u^* = \{x^*, \lambda^*\}$ and, thus, *the process converges to the pair of the eigenvector* x* and the set of eigenvalues $\lambda^* = (\lambda_1^*, \dots, \lambda_m^*)$ of the eigenvalue problem (1), i.e.

$$\lim_{k\to\infty} \rho(u_k, U_*) = \lim_{k\to\infty} \rho(u_k, u^*) = 0, \ \lim_{k\to\infty} F(u_k) = F(u^*) = 0, \tag{10}$$

and the estimation

$$F(u_k) \le 2^{-2k} F(u_0), \quad k = 1, 2, \dots, \tag{11}$$

is true.

Proof. Under the condition of Theorem the gradient of the functional satisfies the Lipschitz condition (8) and hence the inequality holds

$$\| F(u) - F(z) - (\nabla F(z), u - z) \| \le L\|u - z\|^2 / 2. \tag{12}$$

Now, if for some $k \ge 0$ it appears that $\nabla F(x_k) = 0$, then from (7) formally we obtain that

$$u_k = u_{k+1} = \dots$$

and the statement of the theorem holds. Therefore, we assume that $\nabla F(u_k) \ne 0$ for $k = 0, 1, \dots$., and the proof will be provided by the method of mathematical induction.

For $k = 0$ we consider

$$F(u_0) - F(u_1) = F(u_0) - F(u_0 - \gamma_0 \nabla F(u_0)). \tag{13}$$

By taking into account the inequality (12) for $z = u_0$, $u = u_1 = u_0 - \gamma_0 \nabla F(u_0)$, we obtain that

$$F(u_0) - F(u_1) \ge \gamma_0 (1 - L\gamma_0 / 2) \|\nabla F(u_0)\|^2.$$

Thus, given the condition (9), we obtain

$$F(u_0) - F(u_1) \ge \frac{3}{4}\gamma_0 \| \nabla F(u_0) \|^2 = \frac{3}{4} \frac{F(u_0)}{\| \nabla F(u_0) \|^2} \| \nabla F(u_0) \|^2$$

whence

$$\frac{1}{4} F(u_0) \ge F(u_1) \tag{14}$$

i.e.

$$F(u_1) < F(u_0). \tag{15}$$

Now let's show that $\gamma_1 < \gamma_0$. For this reason we will divide both parts of the inequality (14) onto the value $\| \nabla F(u_1) \|^2 \geq 0$. We obtain

$$\frac{F(u_1)}{\| \nabla F(u_1) \|^2} \leq \frac{1}{4} \frac{F(u_0)}{\| \nabla F(u_1) \|^2} = \frac{1}{4} \frac{F(u_0)}{\| \nabla F(u_1) \|^2} \cdot \frac{\| \nabla F(u_0) \|^2}{\| \nabla F(u_0) \|^2},$$

i.e.

$$\gamma_1 \leq \frac{1}{4} \gamma_0 \cdot \frac{\| \nabla F(u_0) \|^2}{\| \nabla F(u_1) \|^2}. \tag{16}$$

Further from the Lipschitz condition (8) we have

$$\nabla F(u_0) - \nabla F(u_1) \leq L(u_0 - u_1).$$

In view of (7), we obtain

$$\nabla F(u_0) - \nabla F(u_1) \leq L\gamma_0 \nabla F(u_0).$$

Now, considering (9), we can write that

$$\nabla F(u_0) - \nabla F(u_1) \leq \frac{1}{2} \nabla F(u_0)$$

or

$$\frac{1}{2} \nabla F(u_0) \leq \nabla F(u_1),$$

i.e.

$$\frac{1}{2} \frac{\nabla F(u_0)}{\nabla F(u_1)} \leq 1 \Rightarrow \frac{\nabla F(u_0)}{\nabla F(u_1)} \leq 2.$$

Finally, considering the last inequality, from (16) we obtain that

$$\gamma_1 < \gamma_0.$$

Now let (15) hold for $k = m$, i.e. $F(u_m) < F(u_{m-1})$ and $\gamma_m < \gamma_{m-1}$. We show that (15) holds and for $k = m + 1$, that is

$$F(u_{m+1}) < F(u_m), \quad \gamma_{m+1} < \gamma_m. \tag{17}$$

Like that for an arbitrary $k = m$ the relation (13) can be written as

$$F(u_m) - F(u_{m+1}) = F(u_m) - F(u_m - \gamma_m \nabla F(u_m)),$$

whence similarly as above, we find that

$$F(u_m) - F(u_{m+1}) \geq \gamma_m (1 - L\gamma_m / 2) \left\| \nabla F(u_m) \right\|^2.$$

Since

$$0 < \gamma_m < \gamma_{m-1} < ... < \gamma_0 \le 1/2L, \tag{18}$$

then

$$F(u_m) - F(u_{m+1}) \ge \frac{3}{4}\gamma_m \left\|\nabla F(u_m)\right\|^2 > 0. \tag{19}$$

It follows that

$$\frac{1}{4}F(u_m) \ge F(u_{m+1}), \tag{20}$$

i.e.

$$F(u_m) \ge F(u_{m+1})$$

and the first inequality (17) is proved.

Analogously as above, we obtain that the inequality

$$\gamma_{m+1} < \gamma_m$$

holds, i.e. the second inequality (17) is satisfied. In addition, considering (18) we have

$$0 < \gamma_{m+1} < \gamma_m < \gamma_{m-1} < ... < \gamma_0 \le 1/2L. \tag{21}$$

Hence the sequence $F(u_k)$ is a monotonically decreasing and bounded from below, and thus the limit

$$\lim_{k\to\infty} F(u_k) \ge 0.$$

exists. Therefore

$$\lim_{k\to\infty} \left(F(u_k) - F(u_{k+1})\right) = 0$$

and from (19) it follows that

$$\lim_{k\to\infty} \| \nabla F(u_k) \| = 0. \tag{22}$$

Now with the iterative process (7), we obtain

$$\| u_{k+1} - u_k \| = \gamma_m \cdot \| \nabla F(u_k) \|.$$

In view of (21), we have

$$\| u_{k+1} - u_k \| \le \gamma_0 \cdot \| \nabla F(u_k) \| \le \frac{1}{2L} \| \nabla F(u_k) \|$$

whence, taking into account (22)

$$\| u_{k+1} - u_k \| \underset{k\to\infty}{\rightarrow} 0 \tag{23}$$

is satisfied.

Observe that for any positive integer p we can write

$$\| u_{k+p} - u_k \| = \| u_{k+p} - u_{k+p-1} + u_{k+p-1} - u_{k+p-2} + u_{k+p-2} - u_{k+p-3} + \ldots$$
$$\ldots + u_{k+1} - u_k \| \le \| u_{k+p} - u_{k+p-1} \| + \| u_{k+p-1} - u_{k+p-2} \| + \ldots + \| u_{k+1} - u_k \| .$$

and since the relation (23) is true, we obtain that

$$\| u_{k+p} - u_k \| \underset{k\to\infty}{\to} 0 .$$

This means that the sequence $\{u_k\}$ is a fundamental one. As far as the Euclidean space is the Banach space, then $\{u_k\}$ converges to its limit, for example, y . But from (22) we have that

$$\lim_{k\to\infty} \| \nabla F(u_k) \| = \| \lim_{k\to\infty} \nabla F(u_k) \| = \| \nabla F(y) \| = 0 .$$

This means that $y = u^*$ is the stationary point of the functional $F(u)$,i.e.

$$\lim_{k\to\infty} u_k = u^* \in U_* . \tag{24}$$

Since the functional is continuous, then

$$\lim_{k\to\infty} F(u_k) = F(u^*) = 0 .$$

The estimate of (11) follows directly from the inequality (20).

Theorem is proved.

Note that if the functional is strongly convex, i.e. there is such a constant g that the inequality

$$F(u) - F(v) \ge (\nabla F(v), u - v) + \delta \|u - v\|^2 , \ u, v \in U , \tag{25}$$

is true. In this case such assertion holds.

Theorem 3. *Let the matrix T(λ) of the eigenvalue problem (1) be such that the functional (2) is strongly convex and its gradient satisfies the Lipschitz condition*

$$\|\nabla F(u) - \nabla F(z)\| \le L \|u - z\|, \quad \forall u, z \in U, \ L > 0 ,$$

where U is some closed convex set that contains the solution u. If for some initial approximation* $u_0 = (x_0, \lambda^{(0)}) \in U$ *the condition*

$$0 < \gamma_0 \equiv \gamma(u_0) \le 1/2L ,$$

is satisfied, then the iterative process (7) converges to the point of minimum of the functional (2) $u^* = \{x^*, \lambda^*\}$ *and, thus, to the eigenvector x* and the set of eigenvalues* $\lambda^* = (\lambda_1^*, \dots, \lambda_m^*)$ *of the problem (1), that is the relations (10), (11) and the estimation*

$$\| u_k - u^* \|^2 \le F(u_0) \cdot 2^{-2k} / \delta, \ k = 0,1, \dots , \tag{26}$$

where δ *is a constant from inequality (25), are true.*

Proof. Relations (10), (11) follow from the Theorem 2. We prove the estimate (26).

From inequality (25) for $u = u_k$, $v = u^*$, we have

$$F(u_k) \ge (\nabla F(u^*), u_k - v^*) + \delta \left\| u_k - u^* \right\|^2 = \delta \left\| u_k - u^* \right\|^2, \ k = 0,1, \dots .$$

Hence, considering (11), we obtain the estimate (26).

Theorem is proved.

4. NUMERICAL RESULTS

The proposed algorithm we test on examples of two-parameter eigenvalue problems. Note that to problem (1) in a real Euclidean space E^n one can reduce the problem

$$Ax = \lambda Bx \tag{27}$$

with complex matrices A and B, which is sufficiently well studied in the literature. Therefore, it seems very appropriate to test the proposed algorithm and its software implementation.

To perform calculations in a real space, reformulate the problem (27).

Let $A = A_R + iA_I$, $B = B_R + iB_I$, $\lambda = \lambda_1 + i\lambda_2$, $x = x_R + ix_I$, $i^2 = -1$. Easy to see that problem (27) is equivalent to the real two-parameter eigen value problem

$$\mathbf{Ax} = \lambda_1 \mathbf{B_1 x} + \lambda_2 \mathbf{B_2 x}, \qquad \mathbf{x} = (x_R, x_I) \in \mathbf{E} = E^n \oplus E^n,$$

Where

$$\mathbf{A} = \begin{pmatrix} A_R & -A_I \\ A_I & A_R \end{pmatrix}, \quad \mathbf{B_1} = \begin{pmatrix} B_R & -B_I \\ B_I & B_R \end{pmatrix}, \quad \mathbf{B_2} = \begin{pmatrix} -B_I & -B_R \\ B_R & -B_I \end{pmatrix}.$$

If A, B are real: $A = A_R$, $B = B_R$, than

$$\mathbf{A} = \begin{pmatrix} A_R & 0 \\ 0 & A_R \end{pmatrix}, \quad \mathbf{B_1} = \begin{pmatrix} B_R & 0 \\ 0 & B_R \end{pmatrix}, \qquad \mathbf{B_2} = \begin{pmatrix} 0 & -B_R \\ B_R & 0 \end{pmatrix}.$$

Further are some calculations that give the concept about the number of iterations and convergence of the algorithm.

Example 1. Let A be the matrix with a complex coefficients, and B be the identity matrix:

$$A = \frac{1}{4}\begin{pmatrix} 3+i & 3-i \\ -3+i & 3+i \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

The eigenvalues and eigenvectors of the problem are known and are equal to:

a). $\lambda = 1+i$, $x = (x^1, x^2) = (1, i)$;

b). $\lambda = 0.5(1-i)$, $x = (x^1, x^2) = (1, -i)$.

The results are show in Table. 1 and Table. 2. The condition for stopping the iterative process is the inequality $\| u_{k+1} - u_k \| \le \varepsilon$ where $\varepsilon = 10^{-6}$, and the vector u has the following structure

$$u = (\mathbf{x}, \boldsymbol{\lambda}) = ((x_R, x_I), (\lambda_1, \lambda_2)) = (x_R^1, x_R^2, x_I^1, x_I^2, \lambda_1, \lambda_2) .$$

Table 1. Numerical results for Example 1, case a)

u	**Initial approximation.** u_0	**Approximate solution**	**Exact solution** u
x_R^1	1.5	1.0000000	1
x_R^2	0.5	0.0000000	0
x_I^1	0.5	0.0000000	0
x_I^2	1.5	1.0000000	1
λ_1	1.2	1.0000003	1
λ_2	1.2	0.9999999	1
F(u)		6.17139333e-013	
Number of iteration		20	

Table 2. Numerical results for Example 1, case b)

u	**Initial approximation.** u_0	**Approximate solution**	**Exact solution** u
x_R^1	1.7	1.00000000	1
x_R^2	0.2	0.00000000	0
x_I^1	-0.2	0.00000000	0
x_I^2	-1.5	-1.0000000	-1
λ_1	0.9	0.5000004	0.5
λ_2	-0.7	-0.5000001	-0.5
F(u)		3.00133411e-013	
Number of iteration		22	

Example 2. Let A be a nonsymmetrical matrix with real coefficients, and B be the identity matrix:

$$A=\begin{pmatrix} 8 & -1 & -5 \\ -4 & 4 & -2 \\ 18 & -5 & -7 \end{pmatrix}, \qquad B=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The eigenvalues and eigenvectors of the problem are known and are equal to:

a). $\lambda = 2+4i$, $x=(x^1,x^2,x^3)=(1-i,2,-2i)$;

b). $\lambda = 2-4i$, $x=(x^1,x^2,x^3)=(1+i,2,2i)$.

c). $\lambda = 1$, $x=(x^1,x^2,x^3)=(1,2,1)$.

The results are show in Table. 3 - Table. 5. The conditions to stop the iterative process and structure of the vector u are the same as in previous example/

Table 3. Numerical results for Example 2, case a)

u	**Initial approximation.** u_0	**Approximate solution**	**Exact solution** u
x_R^1	1	1.0000000	1
x_R^2	2	2.0000000	2
x_R^3	3	0.0000000	0
x_I^1	2	-1.0000000	-1
x_I^2	1	0.0000000	0
x_I^3	2	-1.9999999	-2
λ_1	-0.695	2.0000000	2
λ_2	3.739	3.9999999	4
F(u)		5.112161e-15	
Number of iteration		972	

Table 4. Numerical results for Example 2, case b)

u	**Initial approximation.** u_0	**Approximate solution**	**Exact solution** u
x_R^1	0	0.9999999	1
x_R^2	0	2.0000000	2
x_R^3	1	0.0000000	0
x_I^1	-1	1.0000000	1
x_I^2	0	0.0000000	0
x_I^3	-1	2.0000000	2
λ_1	2.334	2.0000000	2
λ_2	-7.667	-4.0000000	-4
F(u)		5.112161e-15	
Number of iteration		527	

Table 5. Numerical results for Example 2, case c)

u	Initial approximation. u_0	Approximate solution	Exact solution u
x_R^1	1	0.9999999	1
x_R^2	1	2	2
x_R^3	1	0.9999999	1
x_I^1	-1	0	0
x_I^2	-1	0	0
x_I^3	-1	0	0
λ_1	2	1.0000000	1
λ_2	0	0	0
F(u)		3.69045e-015	
Number of iteration		889	

5. CONCLUSION

In the work which was announced in [2] a similar approach was considered where it was proposed to determine the value γ_k by means of the relation

$$\gamma_k = F(u_k)/\|\nabla F(u_0)\|^2,$$

but the choice γ_k by means of formula (6) as is proposed in this paper, significantly improves the convergence of the gradient method (7).

Note also that the difference between the proposed algorithm and similar algorithms in [9,10,11,13] is not only in different approaches to calculation γ_k, but also that the gradient procedure is applied to the extended problem in the space of direct sum of the Euclidean spaces, where the algorithm simultaneously finds the eigenvector and a set of eigenvalues but not separately as in [9,10,11,13]. However, this requires an initial approximation both of the vector and a set of eigenvalues, in contrast to mentioned studies, in which the initial approximation is the eigenvector only. But this disadvantage can be overcome by calculating (once at the first step) the set of eigenvalues $\lambda = \{\lambda_1, \dots, \lambda_m\}$ from the system of linear equations, which are present in (4), namely

$$f_i(\lambda, x_0) = 0, \quad i = 1,2,\dots, m,$$

giving only an initial approximation to eigenvector x_0, and next to take them as an initial approximation (with the initial approximation to eigenvector) $u_0 = \{x_0, \lambda^0\}$ for the iterative process (7). It is realized for Example 2.

Finally, the applicability of the proposed method has been shown on several examples. Although these examples are quite simple, they illustrate how the method can be applied to the typical multi-parameter spectral problems.

Note, that the studies of the discussed iterative procedure are still ongoing. It is needed to analyse in details the stability of the method as well as its convergence rate. Another subject of the further investigation is the aposteriori evaluation of the solution, which can indicate the exactness of the approximate solution provided by the iterative process.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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