Appeared in Geometriae Dedicata (2006) 117:1-9.
DOI: 10.1007/s10711-005-9004-5

THE MAPPING CLASS GROUP OF A NONORIENTABLE SURFACE IS GENERATED BY THREE ELEMENTS AND BY FOUR INVOLUTIONS

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Abstract. We prove that the mapping class group of a closed nonorientable surface is generated by three elements and by four involutions.

1. Introduction

Let X be a closed connected nonorientable surface of genus g. The mapping class group $\mathcal{M}(X)$ of X is defined to be the group of isotopy classes of all homeomorphisms $X \to X$. We show in this paper that $\mathcal{M}(X)$ is generated by 3 elements for every $g \geq 3$. We proved in [14] that $\mathcal{M}(X)$ is generated by a set of involutions whose cardinality depends linearly on g. In this paper we prove that $\mathcal{M}(X)$ is generated by 4 involutions for $g \geq 4$. If $g = 3$, then it follows easily from the work of Birman and Chillingworth [1] that $\mathcal{M}(X)$ is generated by three involutions (see section 4).

Lickorish was the first to consider the problem of finding generators of the group $\mathcal{M}(X)$. He proved in [9] that $\mathcal{M}(X)$ is generated by Dehn twists and the isotopy class of a homeomorphism he called the Y-homeomorphism. Chillingworth [3] determined a finite set of generators for $\mathcal{M}(X)$. The cardinality of this set depends on g. Korkmaz extended Chillingworth's result to the case of punctured surfaces [7] and also computed the first homology group $\mathrm{H}_1(\mathcal{M}(X))$ [6, 7].

The mapping class group $\mathcal{M}(X)$ of an orientable surface X is defined to be the group of isotopy classes of all orientation-preserving homeomorphisms $X \to X$. Wajnryb [15] proved that $\mathcal{M}(X)$ is generated by 2 elements, and Korkmaz [8] proved that it is generated by 2 elements of finite order. McCarthy and Papadopoulus [11] proved that $\mathcal{M}(X)$ is generated by involutions. Luo [10] described the first finite set of generating involutions for $\mathcal{M}(X)$. Brendle and Farb [2] provided the first universal upper bound on the minimal number of generating involutions by showing that $\mathcal{M}(X)$ is generated by 6 involutions for every $g \geq 3$. Their result was improved by Kassabov [5] who proved that $\mathcal{M}(X)$ is generated by 4 involutions if g is large enough. Stukow [12] proved that the extended mapping class group, i.e. the group of isotopy classes of all homeomorphisms $X \to X$, is generated by 3 orientation-reversing involutions.

1991 *Mathematics Subject Classification.* Primary 57N05; Secondary 20F38 , 20F05.
Key words and phrases. Mapping class group, nonorientable surfaces, involutions, generators.
Supported by KBN 1 P03A 024 26.

2. Preliminaries

Mapping class group $\mathcal{M}(X)$ of a closed nonorientable surface X is the group of isotopy classes of all homeomorphisms of X. By abuse of notation we do not distinguish a homeomorphism from its isotopy class. We use the functional notation for the composition of two homeomorphisms; if g and h are two homeomorphisms, then the composition gh means that h is applied first.

By a *circle* in X we mean a simple closed curve. We do not distinguish a circle from its isotopy class. We also identify a circle with its image in X, forgetting its orientation.

If a is a two-sided circle in X with oriented tubular neighborhood, then we can define the (right) Dehn twist A about a.

Recall the following property of Dehn twists. Let c and d be two circles in X with oriented tubular neighborhoods and let f be a homeomorphism of X such that $f(c) = d$. If C and D are the corresponding Dehn twists, then $fCf^{-1} = D^s$, where $s = \pm 1$ depending on whether f restricted to a neighborhood of c is orientation-preserving or orientation-reversing.

3. Mapping class group of a Klein bottle with a hole

Let us consider:

$$D = \{z \in \mathbb{C} \mid |z| \leq 4\},$$
$$D_1 = \{z \in \mathbb{C} \mid |z-2| < 1\},$$
$$D_2 = \{z \in \mathbb{C} \mid |z+2| < 1\}.$$

Let K be the space obtained by removing D_1 and D_2 from D and identifying the antipodal points on the boundary of each of the two removed discs. The space K is a Klein bottle with a hole. Denote by ∂K the boundary of K.

Define a homeomorphism $\tilde{u}\colon D \to D$ by

$$\tilde{u}(re^{i\theta}) = \begin{cases} re^{i(\theta-\pi)} & \text{if } 0 \leq r \leq 3, \\ re^{i(\theta-(r-2)\pi)} & \text{if } 3 \leq r \leq 4. \end{cases}$$

We have then that $\tilde{u}(D_1) = D_2$, $\tilde{u}(D_2) = D_1$. Also $\tilde{u}$ commutes with the identification and thus induces a homeomorphism $u\colon K \to K$. Note that u is the identity on ∂K. We also remark that u^2 is a Dehn twist about ∂K.

Let $\tilde{\tau}\colon D \to D$ be the reflection $\tilde{\tau}(z) = -\overline{z}$. We have $\tilde{\tau}\tilde{u}\tilde{\tau} = \tilde{u}^{-1}$. The reflection $\tilde{\tau}$ induces an involution $\tau\colon K \to K$ and $\tau u \tau = u^{-1}$.

Figure 1 shows the effect of the homeomorphism u on the interval d joining two boundary points of K and separating the removed discs from each other.

Let A be a Dehn twist about the two-sided circle a in Figure 1. With one of the two possible orientations of a neighborhood of a the effect of the homeomorphism $y = Au$ on the interval d is, up to isotopy, as in Figure 1. The homeomorphism y can be described as sliding a Möbius band once along the core of another one and keeping each point of the boundary of K fixed. It is called Y-homeomorphism and was introduced by Lickorish in [9]. We remark that $A^{-1}u$ is also a Y-homeomorphism, so the other choice of the orientation for a neighborhood of a also gives a Y-homoeomorphism. We also note that y^2 is a Dehn twist about ∂K.

If K is embedded in a surface X, we can extend y by the identity to a homeomorphism of X. We will call every such extension a Y-homeomorphism. Lickorish

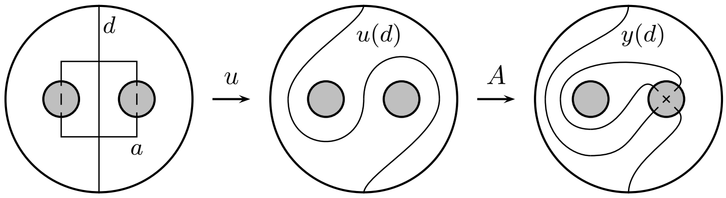

FIGURE 1. The homeomorphisms u and $y = Au$.

showed in [9] that the mapping class group of a nonorientable surface is generated by Dehn twists together with one Y-homeomorphism.

Let $\mathcal{M}(K)$ denote the mapping class group of K, i.e. the group of isotopy classes of those homeomorphisms of K which keep each point of ∂K fixed. The isotopies are also required to fix boundary points. The following lemma can be deduced from Theorem 4.9 of [7] (cf. Theorem A.7 of [13]).

Lemma 1. *The group $\mathcal{M}(K)$ is generated by A and y.* □

Corollary 2. *The group $\mathcal{M}(K)$ is generated by A and u.* □

4. GENERATORS OF THE MAPPING CLASS GROUP $\mathcal{M}(X)$

Let X be a closed nonorientable surface of genus g. If $g = 1$, then $\mathcal{M}(X)$ is trivial. If $g = 2$, then $\mathcal{M}(X) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$ (cf. [9]). If $g = 3$, then Birman and Chillingworth [1] proved that $\mathcal{M}(X)$ is generated by three elements a, b, and z which satisfy the relations $zaz^{-1} = a^{-1}$, $zbz^{-1} = b^{-1}$, $z^2 = 1$. It follows that $\mathcal{M}(X)$ is generated by three involutions $c_1 = za$, $c_2 = zb$ and $c_3 = z$.

For the rest of this section let X be a closed nonorientable surface of genus $g \geq 4$. We will prove that $\mathcal{M}(X)$ is generated by three elements and by four involutions.

We will consider two types of standard model for X as follows (see Figures 2 and 3):

(1) (a) If $g = 2r + 1$, then X is a closed orientable surface of genus r from which the interior of a disc is removed and the antipodal points on the boundary of the disc are identified.

(b) If $g = 2r + 2$, then X is a closed orientable surface of genus r from which the interiors of 2 disjoint discs are removed and the antipodal points on the boundary of each disc are identified.

(2) X is a 2-sphere from which the interiors of g disjoint discs are removed and the antipodal points on the boundary of each disc are identified.

There is a homeomorphism of X which takes the circles in Figure 2 to the circles with the same labels in Figure 3. For a description of such a homeomorphism see [3].

Until the end of the next subsection, let us fix the model in Figure 2 for X. Let X' be the orientable sub-manifold of X bounded by the circle f and let us fix the orientation of X' induced by the standard orientation of the plane of Figure 2. For even g let us also fix the orientation of a tubular neighborhood of the circle a_{g-1} which agrees with the orientation of X'. Denote by A_i the right Dehn twist about

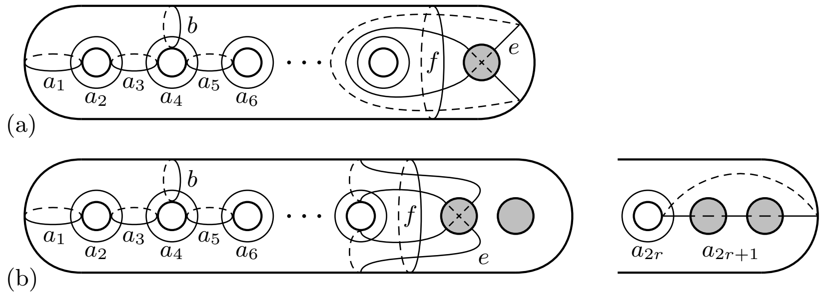

FIGURE 2. The surface X — model 1.

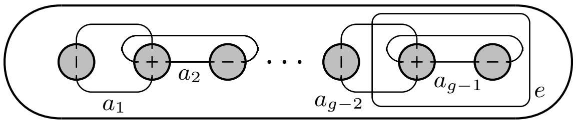

FIGURE 3. The surface X — model 2.

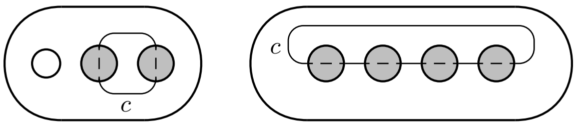

FIGURE 4. $g = 4$

the circle a_i for $1 \leq i \leq g-1$ and by B the right Dehn twist about b. Let S denote the product $A_{g-1}A_{g-2}\ldots A_2A_1$ of $g-1$ Dehn twists.

The circle e bounds a subspace of X which is homeomorphic to a Klein bottle with a hole. Denote by y a Y-homeomorphism of X such that y^2 is a Dehn twist about e.

Theorem 3. *Let X be a closed nonorientable surface of genus $g \geq 4$. The mapping class group $\mathcal{M}(X)$ is generated by the collection $\mathcal{C}$ of homeomorphisms, where:*

- $\mathcal{C} = \{y, B, A_i \mid 1 \leq i \leq g-1\}$ *if* $g > 4$,
- $\mathcal{C} = \{y, C, A_i \mid 1 \leq i \leq 3\}$ *if* $g = 4$,

where C is a Dehn twist about the circle c in Figure 4.

Proof. Lickorish showed in [9] that $\mathcal{M}(X)$ is generated by Dehn twists together with one Y-homeomorphism. Chillingworth established in [3] a finite collection of generators for $\mathcal{M}(X)$. If $g = 4$, then Chillingworth's set of generators coincides with $\mathcal{C}$. If $g > 4$, then Chillingworth's set of generators contains $\mathcal{C}$ and some more Dehn twists. However, all of these Dehn twists can be written as a product of B and A_i for $1 \leq i \leq g-1$ by using the method of Humphries [4]. Since each of the Chillingworth's generators can be obtained using the elements of $\mathcal{C}$, the collection $\mathcal{C}$ generates $\mathcal{M}(X)$. $\square$

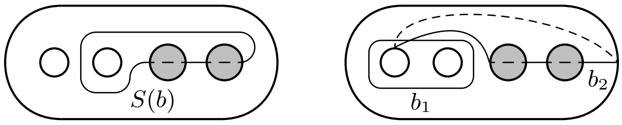

FIGURE 5. $g = 6$

4.1. **Generating mapping class group by three elements.** Wajnryb proved in [15] that the mapping class group of an orientable surface is generated by two elements. Korkmaz showed in [8] that it is generated by two elements, one of which is a Dehn twist. We will follow the outline of the proof of Theorem 5 of [8] to prove that the mapping class group of a nonorientable surface is generated by three elements.

Theorem 4. *Let X be a closed nonorientable surface of genus $g \geq 4$. The mapping class group $\mathcal{M}(X)$ is generated by the collection $\mathcal{C}_1$, where:*

- $\mathcal{C}_1 = \{B, S, \phi\}$ *if g is odd,*
- $\mathcal{C}_1 = \{B, S, \phi A_4\}$ *if g is even and $g \geq 8$,*
- $\mathcal{C}_1 = \{B, S, \phi A_2\}$ *if $g = 6$,*
- $\mathcal{C}_1 = \{A_1, SC, \phi\}$ *if $g = 4$,*

where ϕ can be taken as any homeomorphism supported in the holed Klein bottle bounded by the circle e and such that y can be written as a product of ϕ and A_{g-1}. In particular we may take $\phi = y$.

Proof. Let ϕ be any homeomorphism supported inside the holed Klein bottle bounded by e and such that y can be written as a product of ϕ and A_{g-1}. Let G denote the subgroup of $\mathcal{M}(X)$ generated by $\mathcal{C}_1$. We will prove that $G = \mathcal{M}(X)$ by showing that the collection $\mathcal{C}$ in Theorem 3 is contained in G. It clearly suffices to prove that $A_i \in G$ for $1 \leq i \leq g - 1$.

It can be easily shown that $S(a_i) = a_{i-1}$ for $i > 1$. Hence $SA_iS^{-1} = A_{i-1}$. Thus, for $g > 4$, $A_{i-1} \in G$ if and only if $A_i \in G$ and to show that $G = \mathcal{M}(X)$ it suffices to show that $A_i \in G$ for some i.

If g is odd, then it follows from Theorem 5 of [8] that each A_i can be written as a product of B and S. Thus $\mathcal{M}(X) = G$.

Suppose that g is even and $g \geq 8$. We can write the braid relation between the Dehn twists B and A_4 as $A_4 = BA_4BA_4^{-1}B^{-1}$. Since ϕ commutes with B and A_4 we have $A_4 = B(\phi A_4)B(\phi A_4)^{-1}B^{-1}$. Hence $A_4 \in G$ and $\mathcal{M}(X) = G$.

Suppose that $g = 6$. Denote by b_1 the circle $S^{-1}(b)$ and by b_2 the circle $S^{-1}(b_1)$ (Figure 5). The corresponding Dehn twists B_1 and B_2 are in G, since $B_1 = S^{-1}BS$ and $B_2 = S^{-2}BS^2$. It can be checked that the homeomorphism $U = \phi A_2B_2B_1BS$ takes b to a_3. Since $U \in G$, we have $A_3 \in G$ and $\mathcal{M}(X) = G$.

Finally suppose that $g = 4$. Note, that C commutes with A_i for each i and so $(SC)A_i(SC)^{-1} = A_{i-1}$. Thus $A_i \in G$ for each i and $S \in G$. Thus $G = \mathcal{M}(X)$. □

Remark If $g = 4$ then the first homology group $\mathrm{H}_1(\mathcal{M}(X))$ is elementary abelian of order 8 (cf. [6]) and hence it is not generated by two elements. This means that $\mathcal{M}(X)$ is not generated by two elements. For $g > 4$ or $g = 3$ it is not known whether $\mathcal{M}(X)$ is generated by two elements.

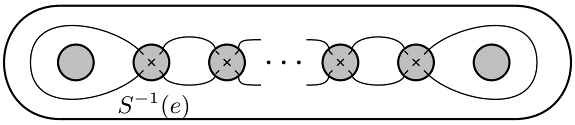

FIGURE 6. The circle $S^{-1}(e)$.

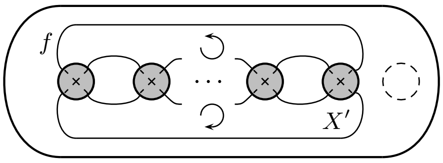

FIGURE 7. The orientation of X'.

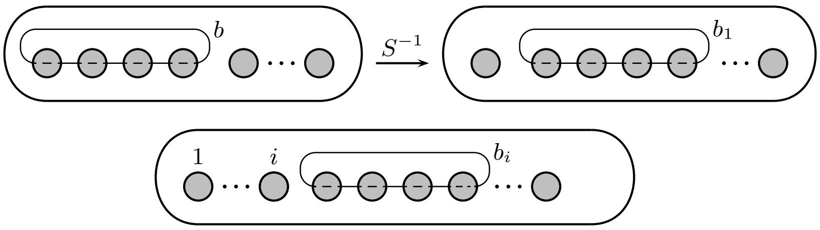

FIGURE 8. $b_i = S^{-i}(b)$

4.2. **Generating mapping class group by four involutions.** In this subsection we will use the model in Figure 3 for X . Now X is obtained by removing interiors of g open discs from a 2-sphere and by identifying antipodal points on the boundary of each disk. Let X' be the orientable submanifold of X bounded by the curve f (see Figure 7, the interior of the dashed circle should be replaced with a crosscap if g is even). As before, A_i and B denote the Dehn twists about about the circles a_i and b respectively, right with respect to the orientation of X' indicated in Figure 7. Observe that $SA_iS^{-1} = A_{i-1}$, where S is the product $A_{g-1}A_{g-2}\dots A_1$.

We can assume that the centers of the removed discs are on the equator of the sphere and there is a reflection of the sphere across a plane perpendicular to the plane of the equator which induces an involution $\tau\colon X \to X$ such that $\tau(a_i) = a_{g-i}$ for $1 \leq i \leq g-1$. We have $\tau A_i \tau = A_{g-i}^{-1}$ and $\tau S \tau = S^{-1}$. Let σ denote the involution $S\tau$.

The circle $S^{-1}(e)$ bounds a Klein bottle with a hole (see Figure 6). We identify this submanifold with K defined in Section 3 in such a way that τ restricts to the involution of K denoted by the same symbol in Section 3. We extend the homeomorphism $u\colon K \to K$ to a homeomorphism $u\colon X \to X$ by the identity outside K. We have $\tau u \tau = u^{-1}$. The circle a in Figure 1 is isotopic, by an isotopy which is the identity outside K, to $S^{-1}(a_{g-1})$. By Corollary 2, $\mathcal{M}(K)$ is generated by u and $S^{-1}A_{g-1}S$. In particular, if $\phi = SuS^{-1}$ then y can be written as a product of A_{g-1} and ϕ.

Theorem 5. *Let X be a closed nonorientable surface of genus $g \geq 4$. The mapping class group $\mathcal{M}(X)$ is generated by four involutions.*

Proof. By Theorem 4, $\mathcal{M}(X)$ is generated by the collection $\mathcal{C}_1$ with $\phi = SuS^{-1}$. We will express each of the elements of $\mathcal{C}_1$ as a product of involutions.

By the considerations preceding Theorem 5 it is easy to write S and ϕ as products of involutions: $S = (S\tau)\tau = \sigma\tau$, $\phi = (S\tau)(\tau u)\tau(\tau S^{-1}) = \sigma(\tau u)\tau\sigma$.

Consider the circles $b_i = S^{-i}(b)$. It can be checked that, for $1 \leq i \leq g-4$, S^{-i} shifts b by i crosscaps to the right in Figure 8.

Suppose that $g = 2r+1$. Then the involution $\sigma = S\tau$ takes the circle b_{r-2} to itself reversing its neighborhood. Thus

$$\sigma(S^{2-r}BS^{r-2})\sigma = (S^{2-r}BS^{r-2})^{-1}.$$

Now we can write B as a product of two involutions:

$$B = (S^{r-2}\sigma S^{2-r})(S^{r-2}\sigma S^{2-r}B).$$

By Theorem 4, $\mathcal{M}(X)$ is generated by four involutions: τ, σ, τu and $S^{r-2}\sigma S^{2-r}B$.

Now suppose that $g = 2r$ and $r \geq 4$. Now τ takes b_{r-2} to itself reversing the neighborhood, $S^{r-2}\tau S^{2-r}B$ is an involution, and we have

$$B = (S^{r-2}\tau S^{2-r})(S^{r-2}\tau S^{2-r}B).$$

If $g \neq 10$, then it is clear from the proof of Theorem 4 that the generator ϕA_4 can be replaced with $A_{g-6}\phi A_4$ and hence $\mathcal{M}(X)$ is generated by B, S and V where

$$V = S^{-1}A_{g-6}\phi A_4 S = A_{g-5}uA_5.$$

We have $\tau V\tau = A_5^{-1}u^{-1}A_{g-5}^{-1} = V^{-1}$. Now $\mathcal{M}(X)$ is generated by the involutions τ, σ, $S^{r-2}\tau S^{2-r}B$, and τV. If $g = 10$ then $\mathcal{M}(X)$ is generated by the involutions τ, σ, $S^3\tau S^{-3}B$, and $\tau V'$, where $V' = S^{-1}\phi A_4 S = uA_5$.

If $g = 6$, then it is easy to check that $\tau S^{-1}\phi A_2 S = \tau u A_3$ has order 2 and, by Theorem 4, $\mathcal{M}(X)$ is generated by τ, σ, $S\tau S^{-1}B$, and $\tau u A_3$.

Finally suppose that $g = 4$. It follows from the proof of Theorem 4 that the generator A_1 can be replaced with any A_i. In particular $\mathcal{M}(X)$ is generated by A_2, SC and ϕ. Note that $\tau C\tau = C^{-1}$. Now $\mathcal{M}(X)$ is generated by the involutions τ, τSC, τA_2, and τu. $\square$

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