The attracting set for impulsive stochastic difference equations with continuous time

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ABSTRACT

In this letter, an impulsive stochastic difference equation with continuous time is considered. By constructing an improved time-varying difference inequality, some sufficient criteria for the global attracting set and exponential stability in mean square are obtained. A numerical example is given to demonstrate the efficiency of the proposed methods.

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1. Introduction

Difference equations with continuous time are difference equations in which the unknown function is a function of a continuous variable. These equations appear as natural descriptions of observed evolution phenomena in many branches of the natural sciences (see [1]). Difference equations with continuous time have attracted more and more attention from many researchers because of their plentiful dynamic behaviors and extensive applications.


However, to our knowledge, few studies have been done on the global attracting set of impulsive stochastic difference equations with continuous time. Motivated by this lack, our main aim in this letter is to present a new method for studying the global attracting set and exponential stability in mean square. After constructing an improved time-varying difference inequality, we derive several sufficient criteria for the global attracting set and exponential stability in mean square of...
impulsive stochastic difference equations with continuous time. The techniques presented in this paper are still applicable to impulsive stochastic difference equations with discrete time. A numerical example is given to show the power of the proposed methods.

2. The model description and preliminaries

Throughout this letter, let $R$ and $R^+$ be the sets of real numbers and nonnegative real numbers, respectively. $R^n$ is the space of $n$-dimensional real column vectors. $PC[I, R] = \{\psi : I \mapsto R | \psi(s) \text{ is continuous for all but at most countably many points } s \in J \subset R \}$ and at these $s \in J$. $\psi(s^+)$ and $\psi(s^-)$ exist and $\psi(s) = \psi(s^+) = \psi(s^-)$ denote the right-hand and left-hand limits of the function, respectively. Let $E$ and $\lim_{s \to \pm \infty}$ denote the expectation of the stochastic process.

**Definition 1.** There exist several constants $a_j(t)$, $b_j(t)$, $p(t)$ and $q(t)$ such that

$$
|f(t, x(t), x(t - \tau_1), \ldots, x(t - \tau_m))| \leq \sum_{j=0}^{m} a_j(t)|x(t - t_j)| + p(t)
$$

and

$$
|g(t, x(t), x(t - \tau_1), \ldots, x(t - \tau_m))| \leq \sum_{j=0}^{m} b_j(t)|x(t - t_j)| + q(t).
$$

Here $t_0 = 0$, $p(t)$ and $q(t)$ are bounded.

**H1.** There exist several constants $l_k \geq 1$ such that

$$
|x(t_k)| = |h_k(x(t_k^-))| \leq l_k|x(t_k^-)|, \quad k = 1, 2, \ldots
$$

Some definitions are employed in this letter.

**Definition 1.** A set $D \subset PC_{R^1}([-\tau - \sigma, 0], R)$ is called a global attracting set of (1) if for any solution $x(t, \phi)$ with initial function $\phi \in PC_{R^1}([-\tau - \sigma, 0], R)$,

$$
d(x(t, \phi), D) \to 0, \quad \text{as } t \to +\infty,
$$

in which $d(x, D) = \inf_{y \in D} d(x, y)$, and $d(x, y)$ is the distance from $x$ to $y$ in $PC_{R^1}([-\tau - \sigma, 0], R)$.

**Definition 2.** The null solution of (1) is called global exponential stability in mean square if for any initial function $\phi \in PC_{R^1}([-\tau - \sigma, 0], R)$ there exists a pair of positive numbers $K$ and $\gamma$ such that

$$
E|x(t, \phi)|^2 \leq Ke^{-\gamma t}, \quad t \geq 0.
$$

3. The main results

In order to study global attracting set, we need to make bounded estimations for all solutions of model (1). But it is very difficult to derive the estimations from previous results. Therefore, we introduce an improved time-varying difference inequality as follows.

**Lemma 1.** Let $u(t)$ be a nonnegative function satisfying

$$
u(t + \sigma) \leq \sum_{j=0}^{m} a_j(t)u(t - h_j) + r(t), \quad t > 0,
$$

in which $\sigma > 0$, $h_j \geq 0$ and:

(i) $a_j(t) \in R^+$ for $j = 0, 1, \ldots, m$ and $\sup_{t \geq 0} \sum_{j=0}^{m} a_j(t) < 1$;

(ii) $r(t) \in R^+$ and $\sup_{t \geq 0} r(t) < +\infty$;
then
\[ u(t) \leq Ke^{-\lambda^* t} + T, \quad \forall t > 0, \] (5)

provided that the initial condition satisfies
\[ u(t) \leq Ke^{-\lambda^* t} + T, \quad \forall t \in [-h - \sigma, 0]. \] (6)

Here \( K > 0, \) \( T = \frac{\sup_{t \geq 0} r(t)}{1 - \sup_{t \geq 0} \sum_{j=0}^m q_j(t)} \) and \( h = \max_{0 \leq j \leq m} h_j; \) \( \lambda^* \) is determined by
\[ \lambda^* = \inf_{t \geq 0} \left\{ \lambda_t \mid \sum_{j=0}^m a_j(t) e^{\lambda_t (\sigma + h_j)} = 1 \right\}. \] (13)

**Proof.** Consider
\[ H(\lambda) = \sum_{j=0}^m a_j(t) e^{\lambda (\sigma + h_j)}. \] (7)

For any fixed \( t > 0, \) we can get
\[ H(0) = \sum_{j=0}^m a_j(t) < 1, \quad \lim_{\lambda \to +\infty} H(\lambda) = +\infty, \] (8)

and
\[ H'(\lambda) = \sum_{j=0}^m a_j(t) (\sigma + h_j) e^{\lambda (\sigma + h_j)} > 0. \] (9)

Hence, for any \( t > 0, \) by the continuity, (8) with (9) can yield that there is a unique real number \( \lambda_t > 0 \) satisfying
\[ H(\lambda_t) = \sum_{j=0}^m a_j(t) e^{\lambda_t (\sigma + h_j)} = 1. \] (10)

It is easy to obtain
\[ \lambda^* = \inf_{t \geq 0} \{ \lambda_t \mid \sum_{j=0}^m a_j(t) e^{\lambda_t (\sigma + h_j)} = 1 \} > 0. \] (11)

In order to prove that (5) holds, we will first show that for any small enough number \( \epsilon > 0, \)
\[ u(t) < (1 + \epsilon) (Ke^{-\lambda^* t} + T), \quad \forall t > 0. \] (12)

If (12) is not true, then there must be \( t^* \) such that
\[ u(t^* + \sigma) = (1 + \epsilon) (Ke^{-\lambda^* (t^* + \sigma)} + T) \] (13)

but
\[ u(t) < (1 + \epsilon) (Ke^{-\lambda^* t} + T), \quad -h - \sigma \leq t < t^* + \sigma. \] (14)

(4), with (14), can yield that
\[
\begin{align*}
    u(t^* + \sigma) &\leq \sum_{j=0}^m a_j(t^*) u(t^* - h_j) + r(t^*) \\
    &< \sum_{j=0}^m a_j(t^*) (1 + \epsilon) (Ke^{-\lambda^* (t^* - h_j)} + T) + r(t^*) \\
    &= (1 + \epsilon) \left( Ke^{-\lambda^* (t^*)} \sum_{j=0}^m a_j(t^*) e^{\lambda^* (\sigma + h_j)} + \sum_{j=0}^m a_j(t^*) T \right) + r(t^*) \\
    &< (1 + \epsilon) (Ke^{-\lambda^* (t^*)} + T)
\end{align*}
\] (15)

which contradicts (13). Therefore, (12) holds. Let \( \epsilon \to 0^+ \) in (12); we can get
\[ u(t) \leq Ke^{-\lambda^* t} + T, \quad \forall t > 0. \] (16)

We complete the proof. □
Theorem 1. Under hypotheses (H1) and (H2), we assume that model (1) satisfies assumptions as follows:

(A1) \( \sup_{t \geq 0} |\sum_{j=0}^{m} (p(t) a_j^2(t) + q(t) b_j^2(t)) + (\sum_{j=0}^{m} a_j(t))^2 + (\sum_{j=0}^{m} b_j(t))^2| \leq 1. \)

(A2) there exists a nonnegative number \( \alpha \) such that

\[
\frac{\ln I_k}{t_k - t_{k-1}} \leq \alpha < \frac{\lambda^*}{2}, \quad k = 1, 2, \ldots,
\]

(A3) \( \sum_{k=1}^{\infty} \ln I_k \leq \mu, \) where \( \mu > 0 \) is a constant number;

then the set \( \Omega = \{ \phi \in PC_{[\tau, 0], R} \mid \| \phi(t) \| \leq e^{2\mu T} \} \) is the global attracting set in mean of model (1), in which

\[
\lambda^* = \inf_{t \geq 0} \left\{ \lambda \mid \sum_{j=0}^{m} \left[ (p(t) a_j^2(t) + q(t) b_j^2(t)) + (\sum_{j=0}^{m} a_j(t))^2 + (\sum_{j=0}^{m} b_j(t))^2 \right] e^{\lambda(\tau + t)} = 1 \right\}
\]

and

\[
T = \sup_{t \geq 0} \left( (m + 1)(p(t) + q(t) + p^2(t) + q^2(t)) \right)
\]

Proof. By virtue of the Holder inequality and mean value inequality, we obtain

\[
E x^2(t + \tau) = E f^2(t, x(t), x(t - \tau_1), \ldots, x(t - \tau_m)) + E g^2(t, x(t), x(t - \tau_1), \ldots, x(t - \tau_m)) \leq E \left[ \sum_{j=0}^{m} a_j(t) |x(t - \tau_j)| + p(t) \right]^2 + E \left[ \sum_{j=0}^{m} b_j(t) |x(t - \tau_j)| + q(t) \right]^2
\]

\[
= E \left[ \sum_{j=0}^{m} a_j(t) |x(t - \tau_j)| \right]^2 + 2E \left[ p(t) \sum_{j=0}^{m} a_j(t) |x(t - \tau_j)| \right] + p^2(t)
\]

\[
+ E \left[ \sum_{j=0}^{m} b_j(t) |x(t - \tau_j)| \right]^2 + 2E \left[ q(t) \sum_{j=0}^{m} b_j(t) |x(t - \tau_j)| \right] + q^2(t)
\]

\[
\leq \sum_{j=0}^{m} \left( (p(t) a_j^2(t) + q(t) b_j^2(t)) + (\sum_{j=0}^{m} a_j(t))^2 + (\sum_{j=0}^{m} b_j(t))^2 \right) E x^2(t - \tau_j)
\]

\[
+ (m + 1)(p(t) + q(t) + p^2(t) + q^2(t)), \quad t > 0, t \neq t_k.
\]

From (A1) and the proof of Lemma 1, we know that there exists a positive real number \( \lambda^* \) determined by (17). Thus, for any initial function \( \phi(t) \in PC_{[\tau, 0], R} \), the corresponding solution \( x(t, \phi) \) (simply \( x(t) \)) satisfies

\[
E x^2(t) \leq \| \phi(t) \| e^{-2\lambda^* (t - 2\alpha)T} + T, \quad -\tau - \sigma \leq t \leq 0.
\]

Using Lemma 1, from (19) and (20), we deduce that

\[
E x^2(t) \leq \| \phi(t) \| e^{-2\lambda^* (t - 2\alpha)T} + T, \quad 0 \leq t < t_1.
\]

Suppose that for any \( n = 1, 2, \ldots, k, \)

\[
E x^2(t) \leq I_{01}^2 \cdots I_{n-1}^2 Ke^{-(\lambda^* - 2\alpha)T} + I_{01}^2 \cdots I_{n-1}^2 T, \quad t_{n-1} \leq t < t_n
\]

with \( l_0 = 1. \) Next, we will show that (22) still holds when \( n = k + 1. \)

Firstly, when \( t = t_k, \) the second equation in model (1), together with (H1) (22), can yield

\[
E x^2(t_k) = E h_k(x(t_k)) \leq I_k^2 E x^2(t_k)
\]

\[
\leq I_{01}^2 \cdots I_{k-1}^2 Ke^{-(\lambda^* - 2\alpha)T} + I_{01}^2 \cdots I_{k-1}^2 T.
\]

Considering \( l_k \geq 1, \) we obtain

\[
E x^2(t) \leq I_{01}^2 \cdots I_{k-1}^2 Ke^{-(\lambda^* - 2\alpha)T} + I_{01}^2 \cdots I_{k-1}^2 T, \quad t_k - \tau - \sigma \leq t \leq t_k.
\]
and inequality (19) can be transformed into
\[
\begin{align*}
\mathbb{E}x^2(t + \sigma) & \leq \sum_{j=0}^{n} \left( p(t) a_j^2(t) + q(t)b_j^2(t) \right) + \left( \sum_{j=0}^{m} a_j(t) \right) a_j(t) + \left( \sum_{j=0}^{m} b_j(t) \right) b_j(t) \\
& \quad + I_{k_1}^2 \cdots I_{k_{n-1}}^2 I_{k_n}^2 (m + 1)(p(t) + q(t)) + p^2(t) + q^2(t), \quad t > 0, \ t \neq t_k.
\end{align*}
\]

(24) and (25), together with Lemma 1, can yield
\[
\mathbb{E}^2(t) \leq I_{k_1}^2 \cdots I_{k_{n-1}}^2 I_{k_n}^2 Ke^{-(\lambda^* - 2\alpha)t} + I_{k_1}^2 \cdots I_{k_{n-1}}^2 T, \quad t_n \leq t < t_{n+1}.
\]

(26) indicates that (22) still holds when \( n = k + 1 \). By mathematical induction, we conclude that for any \( k = 1, 2, \ldots \) and any \( t \in [t_{k-1}, t_k) \),
\[
\mathbb{E}^2(t) \leq I_{k_1}^2 \cdots I_{k_{n-1}}^2 Ke^{-(\lambda^* - 2\alpha)t} + I_{k_1}^2 \cdots I_{k_{n-1}}^2 T.
\]

Thus, it follows from (A2) and (A3) that
\[
\begin{align*}
\mathbb{E}^2(t) & \leq I_{k_1}^2 \cdots I_{k_{n-1}}^2 Ke^{-(\lambda^* - 2\alpha)t} + I_{k_1}^2 \cdots I_{k_{n-1}}^2 T \\
& \leq Ke^{2\alpha t}e^{-(\lambda^* - 2\alpha)t} + e^{2\mu T} \\
& \leq Ke^{-(\lambda^* - 2\alpha)t} + e^{2\mu T}, \quad t \in [t_{k-1}, t_k), \ k = 1, 2, \ldots.
\end{align*}
\]

Let \( t \to +\infty \) in both sides of (28); we obtain
\[
\lim_{t \to +\infty} \mathbb{E}^2(t) \leq e^{2\mu T}.
\]

That is to say, \( \Omega = \{ \phi \in PC\mathbb{R}([-\tau - \sigma, 0], \mathbb{R}) \mid \| \phi(t) \| \leq e^{2\mu T} \} \) is the global attracting set in mean square. The proof is complete. \( \square \)

In particular, if \( f(t, 0, \ldots, 0) = g(t, 0, \ldots, 0) = h(t, 0) = 0 \), then model (1) has a null solution. Assuming \( p(t) = q(t) = 0 \) in (H1), we know that the global attracting set \( \Omega = \{ 0 \} \). By a proof similar to that of Theorem 1, we derive the following corollary.

**Corollary.** Suppose \( f(t, 0, \ldots, 0) = g(t, 0, \ldots, 0) = h(t, 0) = 0 \) and that (H1) and (H2) with \( p(t) = q(t) = 0 \) hold. Let model (1) satisfy:

(A1) \( \sup_{t \geq 0} \left[ (\sum_{j=0}^{m} a_j(t))^2 + (\sum_{j=0}^{m} b_j(t))^2 \right] < 1 \),

(A2) there exists a nonnegative number \( \alpha \) such that
\[
\frac{\ln b_k}{t_k - t_{k-1}} \leq \alpha < \frac{\lambda^*}{2}, \quad k = 1, 2, \ldots.
\]

where \( \lambda^* \) is defined by (17).

Then, the null solution of model (1) is exponential stable in mean square and the exponential convergence rate equals \( \lambda^* - 2\alpha \).

**Remark 1.** The corollary above is exactly the main result, Theorem 3.1, in [17]. In other words, our Theorem 1 makes some extension to the main result in [17].

### 4. An illustrative example

**Example.** Consider an impulsive stochastic difference equation with continuous time as follows:
\[
\begin{align*}
\dot{x}(t+1) &= \sin t \cdot x(t) - \cos t \cdot x(t-1) + \cos t \cdot \frac{x(t-1) + \sin 2t}{2} + \frac{1}{4} \xi(t+1), \quad t \geq 0, \ t \neq t_k, \\
x(t) &= e^{(\pi k)^2} x(-t), \quad t = t_k, \ t_{k+1} = t_k + k.
\end{align*}
\]

Define
\[
\begin{align*}
f(x, x(t-1)) &= \frac{\sin t}{4} x(t) - \frac{\cos t}{3} x(t-1) + \frac{\cos t}{2}, \\
g(x, x(t-1)) &= \frac{1}{4} x(t) + \frac{\sin 2t}{2}, \ h(x(t_k^+)) = e^{(\pi k)^2} x(t_k^-).
\end{align*}
\]
Thus, we can deduce
\[
|f(x, x(t - 1))| \leq \frac{1}{4} |\sin t| |x(t)| + \frac{1}{3} |\cos t| |x(t) - 1| + \frac{1}{2} |\cos t|.
\]
\[
|g(x, x(t - 1))| \leq \frac{1}{4} |x(t)| + \frac{1}{2} |\sin 2t| |h(x(t^-))| = \frac{1}{4} e^{\frac{1}{2} |x(t^-)|}.
\]
That is, \(a_0(t) = \frac{\sin t}{4}, a_1(t) = \frac{\cos t}{3}, p(t) = \frac{\cos t}{2}, b_0(t) = \frac{1}{4}, b_1(t) = 0, q(t) = \frac{\sin 2t}{2},\) and \(I_k = e^{\frac{1}{4} |x|}.\)

It is easy to compute that
\[
\sup_{t \geq 0} \left\{ p(t)(a_0^2(t) + a_1^2(t)) + q(t)(b_0^2(t) + b_1^2(t)) + (a_0(t) + a_1(t))^2 + (b_0(t) + b_1(t))^2 \right\} \leq \frac{150}{288} < 1.
\]

We can choose \(\lambda = 0.328\) such that (17) holds and
\[
\ln l_k \leq \frac{\ln e^{\frac{1}{4} |x|}}{k} \leq \frac{1}{\pi^2} < \frac{0.328}{2}.
\]
Meanwhile, we deduce
\[
\lim_{k \to \infty} l_k = \lim_{k \to \infty} \ln e^{\frac{1}{4} |x|} = \frac{1}{6} = \mu, \quad T = 5.2174.
\]
Hence, according to Theorem 1, we can derive that
\[
E|x(t)|^2 \leq \|\phi(t)\|e^{-0.125t} + e^{ \frac{1}{2} \mu T} < 1.
\]
and \(\Omega = \{ \phi \in PC_{[\mu]}([-2, \mu]) \mid \|\phi(t)\| \leq e^{ \frac{1}{2} \mu T} \}\) is the global attracting set in mean square of (30).

**Remark 2.** Clearly, solutions of (30) are unstable because of \(p(t) = \frac{\cos t}{3}\) and \(q(t) = \frac{\sin 2t}{2}\). Hence, all results in [15, 17] are invalid. Besides these, the results in [16] are not appropriate for model (30) because of the stochastic process \(\xi(t + 1)\). However, by Theorem 1 proposed in this letter, we can derive an effective bounded estimation for all solutions. So, our method is applicable to a wider range than those in [15–17].

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**References**


