OPTIMAL CONTROL COMPUTATION FOR DISCRETE TIME
TIME-DELAYED OPTIMAL CONTROL PROBLEM WITH
ALL-TIME-STEP INEQUALITY CONSTRAINTS

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ABSTRACT. In this paper, we consider a class of discrete time optimal control problems
with time delay and subject to nonlinear all-time-step inequality constraints on both the
state and control. By using a constraint transcription technique in conjunction with
a local smoothing method, the problem is approximated by a sequence of discrete time
optimal control problems with time delay and subject to nonlinear inequality constraints in
canonical form. Rigorous analysis is carried out, showing the convergence of the optimal
solutions of the approximate problems to the optimal solution of the original problem.
We then move on to consider a general class of discrete time optimal control problems
with time delay and subject to nonlinear constraints in canonical form. A computational
method is developed based on the sequential quadratic programming (SQP) approximation
scheme with active set strategy. It solves the discrete time optimal control problem with
time delay and subject to canonical constraints as a nonlinear optimization problem.
As an application, we consider a tactical logistic decision analysis problem, which is
formulated as a discrete time optimal control problems with time delay and subject to all-
time-step inequality constraints. Using the computational method proposed, this practical
problem is solved effectively, producing much better results than those obtained in existing
literature.

Keywords: Time delayed system, Discrete time system, Optimal control, All-time-step
inequality constraints, Constraint transcription, Tactical logistic

1. Introduction. For many natural and man-made systems, inherent delays exist during
the transmission of information between different parts of the systems. As a consequence,
it gives rise to time delayed systems for which the evolution of current states depends
on the past and present values of states and controls. Optimal control of time delayed
systems has been an active research area since 1960s. For problems involving continuous
time systems with time delay, many papers are now available. See, for example, [1-16].
Amongst these references, several computational methods (see [3-6,8-19]) are suggested.
For problems involving discrete time systems with time delay, there are much less pa-
ers available in the literature. In [20], Kuhn-Tucker theorem of nonlinear programming
(see [22]) is used to derive a discrete maximum principle similar to Pontryagin maximum
principle for an optimal discrete time system with a pure delay. However, no efficient com-
putational algorithm is proposed using this discrete maximum principle. In [23], optimal
tracking control for discrete time delay systems affected by persistent disturbances with quadric cost function is considered. However no constraints either on states or controls are involved. In fact, a continuous time optimal control problem can be appropriately discretised to become a discrete time optimal control problem. Thus, computational methods for discrete time optimal control problems with time delay, especially those subject to constraints, are important. In [24], computational methods are proposed for several classes of optimal control problems subject to constraints on states and/or controls. These include a discrete time optimal control problem subject to constraints on states and controls at each time point. However, no time delay is involved in the problem considered. In [17], motivated by a tactical logistic decision analysis problem, a class of discrete time optimal control problems subject to constraints at each time point with time delay appearing in the control is considered. First, the optimal control problem with the constraints ignored is solved by using the method suggested in [26], yielding the unconstrained optimal control. Then the control and the state are saturated when they violate their respective constraints. Clearly, such a control is, in general, not an optimal control for the discrete time optimal control problems with time delay and subject to constraints at each time point under consideration. More importantly, it is, in reality, impossible to saturate the states when their constraints are violated. It is thus clear that the problem considered in [17] has not yet been solved satisfactory.

In this paper, we consider a class of discrete time nonlinear optimal control problems with time delay and subject to constraints on states and controls at each time point. These constraints are called all-time-step constraints. This class of optimal control problems covers the problem considered in [17] as a special case. It has not been studied before in the literature. Thus, no efficient computational methods exist for solving this complex discrete time optimal control problem with time delay and subject to all-time-step constraints. We shall devise an efficient computational method for solving this constrained discrete time optimal control problem with time delay. More specifically, by using the constraint transcription technique introduced in [18] in conjunction with a local smoothing method, we construct a sequence of approximate discrete time optimal control problems involving time delay in states and controls and subject to nonlinear inequality constraints in canonical form. Rigorous analysis is carried out, showing the convergence of the optimal solutions of the approximate problems to the optimal solution of the original optimal control problem. On this basis, we see that solving optimal discrete time nonlinear optimal control problems with time delay and subject to constraints on states and controls at each time point, we need to solve a sequence of approximate discrete time optimal control problems involving time delay in states and controls and subject to nonlinear inequality constraints in canonical form. Thus, we will devise an efficient computational method for solving a general class of discrete time optimal control problems with time delay appearing in states and controls and subject to nonlinear inequality constraints in canonical form. For this, we will derive the gradient formulas for the cost and the canonical constraint functions. These gradient formulas are obtained the first time in this paper. With these gradient formulas, the discrete time optimal control problem with time delay appearing in states and controls and subject to nonlinear inequality constraints in canonical form can be solved as the sequential quadratic programming approximation scheme (see [24]).

As an application, we consider a realistic tactical logistic decision analysis problem formulated in [17], where tactical logistic decision analysis problem is formulated as a discrete time optimal control problem with time delay appearing in the control variable. The physical limitations in capacity at locations and the requirements for reserve stock
are formulated as all-time-step inequality constraints. This problem formulation is realistic. However, in view of the discussion given above, the solution method with the way of handling the all-time-step constraints proposed in [17] does not really solve the problem rigorously. In this paper, the constraints are considered explicitly in the development of our algorithm. Thus, it is not surprising that the optimal control cost obtained using our algorithm is much less than that reported in [17]. More importantly, the optimal control obtained using our algorithm is such that the all-time-step constraints are satisfied at each time. This is unlike the approach used in [17] by artificially saturating the state and control when the constraints are violated. Although saturating the control is possible, the task of saturating the state is really not realistic in practice. Finally, we consider the problem for a much longer planning horizon, the case which cannot be solved using the solution method proposed in [17]. For the solution obtained using our algorithm, we observe some interesting phenomena at the equilibrium state.

2. Problem Statement. Consider a process described by the following system of difference equations with time delay:

\[ x(k + 1) = f(k, x(k), x(k - h), u(k), u(k - h)), \quad k = 0, 1, \ldots, M - 1, \]  

where

\[ x = [x_1, \ldots, x_n]^\top \in \mathbb{R}^n, \quad u = [u_1, \ldots, u_r]^\top \in \mathbb{R}^r, \]

are, respectively, the state and control vectors, while

\[ f = [f_1, \ldots, f_n]^\top \in \mathbb{R}^n \]

is a given function and \( h \) is the delay time, which is an integer satisfying \( 0 < h < M \). Here, we consider the case where there is only one time delay. The extension to the case involving many time delays is straightforward but is more involved in terms of notation.

The initial functions for the state and control functions are:

\[ x(k) = \phi(k), \quad k = -h, -h + 1, \ldots, -1, \quad x(0) = x^0, \]

\[ u(k) = \gamma(k), \quad k = -h, -h + 1, \ldots, -1, \]

where

\[ \phi(k) = [\phi_1, \ldots, \phi_n]^\top, \quad \gamma(k) = [\gamma_1, \ldots, \gamma_r]^\top, \]

are given functions from \( k = -h, -h + 1, \ldots, -1 \) into \( \mathbb{R}^n \) and \( \mathbb{R}^r \), respectively, and \( x^0 \) is a given vector in \( \mathbb{R}^n \). Define

\[ U = \{ \nu = [v_1, \ldots, v_r]^\top \in \mathbb{R}^r : \alpha_i \leq v_i \leq \beta_i, \quad i = 1, \ldots, r \}, \]

where \( \alpha_i, i = 1, \ldots, r, \) and \( \beta_i, i = 1, \ldots, r, \) are given real numbers. Note that \( U \) is a compact and convex subset of \( \mathbb{R}^r \).

Consider the following all-time-step inequality constraints on the state and control variables given below:

\[ h_i(k, x(k), u(k)) \leq 0, \quad k = 0, 1, \ldots, M - 1; \quad i = 1, \ldots, N_2, \]

where \( h_i, i = 1, \ldots, N_2, \) are given real-valued functions.

A control sequence \( u = \{u(0), \ldots, u(M - 1)\} \) is said to be an admissible control if \( u(k) \in U, \quad k = 0, \ldots, M - 1 \), where \( U \) is defined by (2). Let \( \mathcal{U} \) be the class of all such admissible controls. If a \( u \in \mathcal{U} \) is such that the all-time-step inequality constraints (3) are satisfied, then it is called a feasible control. Let \( \mathcal{F} \) be the class of all such feasible controls.
We now state our problem formally as follows:

**Problem (Q)** Given system (1a, 1b, 1c), find a control \( u \in \mathcal{F} \) such that the cost function

\[
g_0(u) = \Phi_0(x(M)) + \sum_{k=0}^{M-1} \mathcal{L}_0(k, x(k), x(k-h), u(k), u(k-h))
\]

is minimized over \( \mathcal{F} \), where \( \Phi_0 \) and \( \mathcal{L}_0 \) are given real-valued functions.

### 3. Approximation.

Problem (Q) can, in principle, be solved as a nonlinear optimization problem, where the cost function (4) is minimized with respect to both the state and control variables subject to the all-time-step inequality constraints and the difference equations which are regarded as equality constraints. This approach will give rise to many nonlinear constraints. These include the equality constraints that arise from the difference equations. These equality constraints are nonlinear. It is acknowledged that nonlinear optimization problems with nonlinear equality constraints are difficult to solve, as the satisfaction of the nonlinear equality constraints are difficult to maintain during the optimization process. Another approach is to use the system of difference equations to calculate the state for a given control. In this way, only the control variables are decisions variables and the constraints contain only the all-time-step inequality constraints. Using this approach, Problem (Q) can also be regarded as a nonlinear optimization problem subject to nonlinear inequality constraints at each time point. To solve such a problem using a gradient-based method, such as SQP approximation scheme ([24]), we need the values and the gradients of the cost function and the all-time-step inequality constraint functions given by (3). There are a total of \((M - 1) \times N_2\) nonlinear inequality constraints. To calculate the gradient of each of these nonlinear inequality constraints, we need to introduce and solve \((M - 1) \times N_2\) associated co-state systems, one for each of these nonlinear inequality constraints. Clearly, the computational complexity is rather high. In this paper, we shall use the constraint transcription technique introduced in [24] to approximate each of the all-time-step inequality constraints by a sequence of inequality constraints in canonical form. In this way, we will obtain a sequence of discrete time optimal control problems with time delay and subject to canonical constraints. Therefore, to solve Problem (Q), it is required to solve a sequence of discrete time optimal control problems with time-delay and subject to canonical constraints. In this section, we shall construct these approximate optimal control problems and then show the convergence of these approximate optimal control problems to the original optimal control problem. In Section 4, a computational method for solving a general class of discrete time optimal control problems with time-delay and subject to canonical constraints.

To begin, we first note that the all-time-step inequality constraints (3) are equivalent to the following equality constraints:

\[
g_i(u) = \sum_{k=0}^{M-1} \max\{h_i(k, x(k), u(k)), 0\} = 0, \quad i = 1, \ldots, N_2.
\]

Thus, the set \( \mathcal{F} \) of feasible controls can be written as:

\[
\mathcal{F} = \{ u(k) \in U, k = 0, \ldots, M - 1 : g_i(u) = 0, \quad i = 1, \ldots, N_2, \}
\]

where \( U \) is defined by (2). However, the functions appeared in (5) are nonsmooth. Thus, for each \( i = 1, \ldots, N_2 \), we shall approximate the nonsmooth function \( \max\{h_i(k, x(k), u(k)), 0\} \)
by a smooth function $\mathcal{L}_{i,\varepsilon}(k, x(k), u(k))$ given by

$$
\mathcal{L}_{i,\varepsilon} = \begin{cases} 
0, & \text{if } h_i < -\varepsilon, \\
(h_i + \varepsilon)^2 / 4\varepsilon, & \text{if } -\varepsilon \leq h_i \leq \varepsilon, \\
h_i, & \text{if } h_i > \varepsilon,
\end{cases} 
$$

where $\varepsilon > 0$ is an adjustable constant with small value. Then, the all-time-step inequality constraints (3) are approximated by the inequality constraints in canonical form defined by

$$
-\frac{\varepsilon}{4} + g_{i,\varepsilon}(u) \leq 0, \ i = 1, \ldots, N_2, 
$$

where

$$
g_{i,\varepsilon}(u) = \sum_{i=1}^{N_2} \sum_{k=0}^{M-1} \mathcal{L}_{i,\varepsilon} k, x(k), u(k).
$$

Define

$$
\mathcal{F}_\varepsilon = \left\{ u(k) \in U, \ k = 0, \ldots, M - 1 : -\frac{\varepsilon}{4} + g_{i,\varepsilon}(u) \leq 0, \ i = 1, \ldots, N_2 \right\}.
$$

Now, we can define a sequence of approximate problems $Q(\varepsilon)$, where $\varepsilon > 0$, below.

**Problem (Q(\varepsilon))** Problem (Q) with (5) replaced by

$$
G_\varepsilon(u) = -\frac{\varepsilon}{4} + g_\varepsilon(u) \leq 0, \ i = 1, \ldots, N_2.
$$

In Problem (Q(\varepsilon)), our aim is to find a control $u$ in $\mathcal{F}_\varepsilon$ such that the cost function (4) is minimized over $\mathcal{F}_\varepsilon$. For each $\varepsilon > 0$, Problem (Q(\varepsilon)) is a special case of a general class of discrete time optimal control problems with time-delay and subject to canonical constraints defined below.

**Problem (P)** Given system (2.1), find an admissible control $u \in \mathcal{U}$ such that the cost function

$$
g_0(u) = \Phi_0(x(M)) + \sum_{k=0}^{M-1} \mathcal{L}_0(k, x(k), x(k - h), u(k), u(k - h))
$$

is minimized over $\mathcal{U}$ subject to the following constraints in canonical form:

$$
g_i(u) = 0, \ i = 1, 2, \ldots, N_e, \tag{12a}
$$

$$
g_i(u) \leq 0, \ i = N_e + 1, \ldots, N, \tag{12b}
$$

where

$$
g_i(u) = \Phi_i(x(M)) + \sum_{k=0}^{M-1} \mathcal{L}_i(k, x(k), x(k - h), u(k), u(k - h)).
$$

We shall develop an efficient computational method for solving Problem (P) in the next section. In the rest of this section, our aim is to establish the required convergence properties of Problems (Q(\varepsilon)) to Problem (Q). We assume that the following conditions are satisfied. These conditions are now quite standard in optimal control algorithms.

**A1** For each $k = 0, 1, \ldots, M - 1$, $f(k, \cdot, \cdot, \cdot, \cdot)$ is continuously differentiable on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^r$.

**A2** For each $i = 1, \ldots, N_2$, and for each $k = 0, 1, \ldots, M - 1$, $h_i(k, \cdot, \cdot, \cdot)$ is continuously differentiable on $\mathbb{R}^n \times \mathbb{R}^r$.

**A3** $\Phi_0$ is continuously differentiable on $\mathbb{R}^n$.

**A4** For each $k = 0, 1, \ldots, M - 1$, $\mathcal{L}_0(k, \cdot, \cdot, \cdot, \cdot)$ is continuously differentiable on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^r$.

**A5** For any control $u$ in $\mathcal{F}$, there exists a control $\bar{u} \in \mathcal{F}^0$ such that $\alpha \bar{u} + (1 - \alpha)u \in \mathcal{F}^0$ for all $\alpha \in (0, 1]$.
Remark 3.1. Under (A5), it can be shown that for any $u$ in $F$ and $\delta > 0$, there exists a $\bar{u} \in F^0$ such that

$$\max_{0 \leq k \leq M-1} |u(k) - \bar{u}(k)| \leq \delta,$$

where $F^0$ is the interior of $F$, meaning that if $u \in F^0$, then $h_i(k, x(k), u(k)) < 0$, $i = 1, ..., N_2$, for all $k = 0, 1, ..., M - 1$.

In what follows, we shall present an algorithm for solving Problem (Q) as a sequence of Problems (Q($\varepsilon$)).

**Algorithm 1**

1. Set $\varepsilon = \varepsilon_0$.

2. Solve Problem Q($\varepsilon$) as a nonlinear programming problem, obtaining an optimal solution.

3. Set $\varepsilon = \varepsilon/10$, and go to Step 2.

**Remark 3.2.** $\varepsilon_0$ is usually set as $1.0 \times 10^{-2}$; and the algorithm is terminated as "successful exit" when $\varepsilon < 10^{-7}$.

**Remark 3.3.** In Step 2, we need to solve Problem Q($\varepsilon$) for each $\varepsilon > 0$, which is a special case of Problem (P). Thus, we will develop an efficient computational method for solving Problem (P) in Section 4.

To establish the convergence properties of Problems (Q($\varepsilon$)) to Problem (Q), we need

**Lemma 3.1.** If $u_\varepsilon$ is a feasible control of Problem (Q($\varepsilon$)), then it is also a feasible control of Problem (Q).

**Proof:** Suppose $u_\varepsilon$ is not a feasible control of Problem (Q). Then, there exit some $i \in \{1, \ldots, N_2\}$, and $k \in \{0, 1, \ldots, M - 1\}$ such that

$$h_i(k, x(k | u_\varepsilon), u_\varepsilon(k)) > 0.$$

This, in turn, implies that

$$\mathcal{L}_{i, \varepsilon}(k, x(k | u_\varepsilon), u_\varepsilon(k)) > \frac{\varepsilon}{4},$$

and hence,

$$g_\varepsilon(u_\varepsilon) > \frac{MN_2\varepsilon}{4} > \frac{\varepsilon}{4}.$$ 

That is,

$$-\frac{\varepsilon}{4} + g_\varepsilon(u_\varepsilon) > 0.$$ 

This is a contradiction to the constraints specified in (10). This completes the proof.

**Theorem 3.1.** Let $u^*$ be an optimal control of Problem (Q) and let $u_\varepsilon^*$ be an optimal control of Problem (Q($\varepsilon$)). Then,

$$\lim_{\varepsilon \to 0} g_0(u_\varepsilon^*) = g_0(u^*).$$
**Proof:** By (A5), there exists a \( \bar{u} \in \mathcal{F}^0 \) such that
\[
u \equiv \alpha\bar{u} + (1 + \alpha)u^* \in \mathcal{F}^0, \quad \forall \alpha \in (0, 1].
\]
Thus, for any \( \delta_1 > 0 \), \( \exists \) an \( \alpha_1 \in (0, 1] \) such that
\[
g_0(u^*) \leq g_0(u_\alpha) \leq g_0(u^*) + \delta_1, \quad \forall \alpha \in (0, 1].
\]
Choose \( \alpha_2 = \alpha_1/2 \). Then, it is clear that \( \alpha_2 \in \mathcal{F}^0 \). Thus, there exists a \( \delta_2 > 0 \) such that
\[
h_i(k, x(k | u_{\alpha_2}), u_{\alpha_2}) < -\delta_2, \quad i = 1, \ldots, N_2,
\]
for all \( k, \ 0 \leq k \leq M - 1 \). Let \( \varepsilon = \delta_2 \). Then, it follows from the definition of \( \mathcal{L}_{i, \varepsilon} \) given by (7) that \( \mathcal{L}_{i, \varepsilon} = 0 \). Thus, (10) is satisfied and hence \( u_{\alpha_2} \in \mathcal{F}_\varepsilon \). Let \( u^*_\varepsilon \) be an optimal control of Problem \((Q(\varepsilon))\). Clearly, \( u^*_\varepsilon \in \mathcal{F}_\varepsilon \) and
\[
g_0(u^*_\varepsilon) \leq g_0(u_{\alpha_2}).
\]
However,
\[
g_0(u^*) \leq g_0(u^*_\varepsilon).
\]
Thus, if follows from (14), (15) and (16) that
\[
g_0(u^*) \leq g_0(u^*_\varepsilon) \leq g_0(u_{\alpha_2}) \leq g_0(u^*) + \delta_1.
\]
Letting \( \varepsilon \to 0 \) and noting that \( \delta_1 > 0 \) is arbitrary, the conclusion of the theorem follows readily. This completes the proof.

**Theorem 3.2.** Let \( u^*_\varepsilon \) and \( u^* \) be optimal controls of Problems \((Q(\varepsilon))\) and \((Q)\), respectively. Then, there exists a subsequence of \( \{u^*_\varepsilon\} \), which is again denoted by the original sequence, and a control \( \bar{u} \in \mathcal{F} \) such that, for each \( k = 0, 1, \ldots, M - 1 \),
\[
\lim_{\varepsilon \to 0} |u^*_\varepsilon(k) - \bar{u}(k)| = 0.
\]
Furthermore, \( \bar{u} \) is an optimal control of Problem \((Q)\).

**Proof:** Since \( U \) is a compact subset of \( \mathbb{R}^r \), and \( \{u^*_\varepsilon\} \), as a sequence in \( \varepsilon \), is such that \( u^*_\varepsilon(k) \in U \), for \( k = 0, 1, \ldots, M - 1 \), it is clear that there exists a subsequence, which is again denoted by the original sequence, and a control parameter vector \( \bar{u} \in U \) such that, for each \( k = 0, 1, \ldots, M - 1 \),
\[
\lim_{\varepsilon \to 0} |u^*_\varepsilon(k) - \bar{u}(k)| = 0.
\]
By induction, we can show, by using (A1) and (18), that, for each \( k = 0, 1, \ldots, M \),
\[
\lim_{\varepsilon \to 0} |x(k | u^*_\varepsilon) - x(k | \bar{u})| = 0.
\]
Thus, by (A2), we have, for each \( k = 0, 1, \ldots, M \),
\[
\lim_{\varepsilon \to 0} h_i(k, x(k | u^*_\varepsilon), u^*_\varepsilon(k)) = h_i(k, x(k | \bar{u}), \bar{u}(k)),
\]
\[
i = 1, \ldots, N_2.
\]
By Lemma 1, \( u^*_\varepsilon \in \mathcal{F} \) for all \( \varepsilon > 0 \). Thus, it follows from (20) that \( \bar{u} \in \mathcal{F} \). Next, by (A1), we deduce from (18) and (19) that
\[
\lim_{\varepsilon \to 0} g_0(u^*_\varepsilon) = g_0(\bar{u}).
\]
For any \( \delta_1 > 0 \), it follows from Remark 3.1 that there exists a \( \hat{u} \in \mathcal{F}^0 \) such that, for each \( k = 0, 1, \ldots, M - 1 \),
\[
|u^*(k) - \hat{u}(k)| \leq \delta_1.
\]
By (A1) and induction, we can show that, for any $\rho_1 > 0$, there exists a $\delta_1 > 0$ such that for each $k = 0, 1, \ldots, M$,

$$
|x(k|u^*) - x(k|\hat{u})| \leq \rho_1,
$$

whenever (22) is satisfied. Using (22), (23) and (A4), it follows that, for any $\rho_2 > 0$, there exists a $\hat{u} \in F^0$ such that

$$
g_0(u^*) \leq g_0(\hat{u}) \leq g_0(u^*) + \rho_2.
$$

Since $\hat{u} \in F^0$, we have, for each $k = 0, 1, \ldots, M$,

$$
h_i(k, x(k|\hat{u}), \hat{u}(k)) < 0, \quad i = 1, \ldots, N_2,
$$

and hence there exists a $\delta > 0$ such that, for each $k = 0, 1, \ldots, M$,

$$
h_i(k, x(k|\hat{u}), \hat{u}(k)) \leq -\delta, \quad i = 1, \ldots, N_2.
$$

Thus, in view of (9), we see that

$$
\hat{u} \in F_\varepsilon,
$$

for all $\varepsilon$, $0 \leq \varepsilon \leq \delta$. Therefore,

$$
g_0(u^*_\varepsilon) \leq g_0(\hat{u}).
$$

Using (24) and (26), and noting that $u^*_\varepsilon \in F$, we obtain

$$
g_0(u^*) \leq g_0(u^*_\varepsilon) \leq g_0(u^*) + \rho_2.
$$

Since $\rho_2 > 0$ is arbitrary, it follows that

$$
\lim_{\varepsilon \to 0} g_0(u^*_\varepsilon) = g_0(u^*).
$$

Combining (21) and (28), we conclude that $\hat{u}$ is an optimal control of Problem (Q). This completes the proof.

4. Computational Method. In this section, we shall develop an efficient computational method for solving Problem (P) as a nonlinear mathematical programming problem, where the SQP approximation scheme is used together with the active set strategy. There are several efficient implementations of SQP available (see, for example, the subroutines NLPQL and NLPQLP written by Schittkowski [25]). For doing this, it is required to calculate, for each control sequence $u = \{u(0), \ldots, u(M-1)\}$, the values of the cost function $g_0(u)$ and the constraints functions $g_{i,\varepsilon}(u)$, $i = 1, \ldots, N$, as well as their gradients. The calculation of the values of the cost function (11) and the canonical constraint functions given by (12a) and (12b) corresponding to each $u \in U$ can be done as follows. For each $u = \{u(0), \ldots, u(M-1)\}$, where $u(k) \in U$, $k = 0, 1, \ldots, M-1$, with $U$ being defined by (2), we solve system (1a), (1b), (1c)) to obtain the corresponding solution sequence $x(k|u)$, $k = 0, 1, \ldots, M-1$, is obtained. Then, the value of the cost function (11) and the values of the canonical constraint functions given by (12a) and (12b) are calculated.

To calculate the gradients of the cost and constraint functions, we will derive the required gradient formulas corresponding to each control sequence $u = \{u(0), \ldots, u(M-1)\}$ as follows.

For each $i = 0, 1, \ldots, N$, let

$$
H_i(k, x(k), y(k), z(k), u(k), v(k), w(k), \lambda^i(k+1), \bar{\lambda}^i(k))
$$
be the corresponding Hamiltonian sequence defined by

\[
H_i(k, x(k), y(k), z(k), u(k), v(k), w(k), \lambda^i(k + 1), \ddot{\lambda}^i(k)) = \mathcal{L}_i(k, x(k), y(k), u(k), v(k))
\]

\[
+ \mathcal{L}_i(k + h, z(k), x(k), w(k), u(k)) e(M - k - h)
\]

\[
+ (\lambda^i(k + 1))^\top f(k, x(k), y(k), u(k), v(k))
\]

\[
+ (\ddot{\lambda}^i(k))^\top f(k + h, z(k), x(k), w(k), u(k)) e(M - k - h),
\]

where \(e(\cdot)\) denotes the unit step function defined by

\[
e(k) = \begin{cases} 
1, & k \geq 0 \\
0, & k < 0,
\end{cases}
\]

and

\[
y(k) = x(k - h),
\]

\[
z(k) = x(k + h),
\]

\[
v(k) = u(k - h),
\]

\[
w(k) = u(k + h),
\]

\[
\ddot{\lambda}^i(k) = \lambda^i(k + h + 1).
\]

For each control \(u\), \(\lambda^i\) is the solution of the following co-state system

\[
(\lambda^i(k))^\top = \frac{\partial H_i(k)}{\partial x(k)}, \quad k = M - 1, M - 2, \ldots, 0,
\]

with boundary conditions

\[
(\lambda^i(M))^\top = \frac{\partial \Phi_i(x(M))}{\partial x(M)};
\]

\[
\lambda^i(k) = 0, \quad k > M.
\]

We set

\[
z(k) = 0, \quad \forall k = M - h + 1, M - h + 2, \ldots, M,
\]

and

\[
w(k) = 0, \quad \forall k = M - h, M - h + 1, \ldots, M.
\]

Then, the gradient formulas for the cost functions (for \(i = 0\)) and constraint functions (for \(i = 1, \ldots, N\)) are given in the following theorem.

**Theorem 4.1.** Let \(g_i(u)\), \(i = 0, 1, \ldots, N\), be defined by (11) (the cost function for \(i = 0\)) and (13) (the constraint functions for \(i = 1, \ldots, N\)). Then, for each \(i = 0, 1, \ldots, N\), the gradient of the function \(g_i(u)\) is given by

\[
\frac{\partial g_i(u)}{\partial u} = \left[\frac{\partial H_i(0)}{\partial u(0)}, \frac{\partial H_i(1)}{\partial u(1)}, \ldots, \frac{\partial H_i(M - 1)}{\partial u(M - 1)}\right],
\]

where

\[
H_i(k) = H_i(k, x(k), y(k), z(k), u(k), v(k), w(k), \lambda^i(k + 1), \ddot{\lambda}^i(k)),
\]

\(k = 0, 1, \ldots, M - 1\).
**Proof:** Define
\[ u = [(u(0))^T, (u(1))^T, \ldots, (u(M - 1))^T]^T. \] (37)

Let the control \( u \) be perturbed by \( \varepsilon \hat{u} \), where \( \varepsilon > 0 \) is a small real number and \( \hat{u} \) is an arbitrary but fixed perturbation of \( u \) given by
\[ \hat{u} = [(\hat{u}(0))^T, (\hat{u}(0))^T, \ldots, (\hat{u}(M - 1))^T]^T. \] (38)

Then, we have
\[ u_\varepsilon = u + \varepsilon \hat{u} = [(u(0, \varepsilon))^T, (u(1, \varepsilon))^T, \ldots, (u(M - 1, \varepsilon))^T]^T, \] (39)

where
\[ u(k, \varepsilon) = u(k) + \varepsilon \hat{u}(k), \ k = 0, 1, \ldots, M - 1. \] (40)

Let the perturbed solution be denoted by
\[ x(k, \varepsilon) = x(k | u_\varepsilon), \ k = 1, 2, \ldots, M. \] (41)

Then,
\[ x(k + 1, \varepsilon) = f(k, x(k, \varepsilon), y(k, \varepsilon), u(k, \varepsilon), v(k, \varepsilon)). \] (42)

The variation of the state for \( k = 0, 1, \ldots, M - 1 \) is:
\[ \triangle x(k + 1) = \left. \frac{dx(k + 1, \varepsilon)}{d\varepsilon} \right|_{\varepsilon=0} \]
\[ = \frac{\partial f(k, x(k), y(k), u(k), v(k))}{\partial x(k)} \triangle x(k) \]
\[ + \frac{\partial f(k, x(k), y(k), u(k), v(k))}{\partial y(k)} \triangle y(k) \]
\[ + \frac{\partial f(k, x(k), y(k), u(k), v(k))}{\partial u(k)} \hat{u}(k) \]
\[ + \frac{\partial f(k, x(k), y(k), u(k), v(k))}{\partial v(k)} \triangle v(k), \] (43a)

where
\[ \triangle x(k) = 0, \ k \leq 0, \] (43b)
\[ \triangle u(k) = 0, \ k < 0. \] (43c)

From (43b) and (43c), we obtain
\[ \triangle y(k) = 0, \ k = 0, 1, \ldots, h, \] (44a)

and
\[ \triangle v(k) = 0, \ k = 0, 1, \ldots, h - 1. \] (44b)

Define
\[ \breve{L}_i = L_i(k, x(k), y(k), u(k), v(k)), \] (45a)
\[ \breve{\bar{L}}_i = L_i(k + h, z(k), x(k), w(k), u(k)), \] (45b)
\[ \breve{f} = f(k, x(k), y(k), u(k), v(k)), \] (45c)
\[ \breve{\bar{f}} = f(k + h, z(k), x(k), w(k), u(k)), \] (45d)
\[ \breve{H}_i = H_i(k). \] (45e)
By chain rule and (45a), it follows that

$$
\frac{\partial g_i(u)}{\partial u} \hat{u} = \lim_{\varepsilon \to 0} \frac{g_i(u_\varepsilon) - g_i(u)}{\varepsilon} \equiv \frac{dg_i(u_\varepsilon)}{d\varepsilon} \bigg|_{\varepsilon=0} = \frac{\partial \Phi_i(x(M))}{\partial x(M)} \Delta x(M)
$$

$$
+ \sum_{k=0}^{M-1} \left[ \frac{\partial \hat{L}_i}{\partial x(k)} \Delta x(k) + \frac{\partial \hat{L}_i}{\partial y(k)} \Delta y(k) + \frac{\partial \hat{L}_i}{\partial u(k)} \hat{u}(k) \right].
$$

From (31a), (31c) and (45b), we have

$$
\sum_{k=0}^{M-1} \left\{ \left( \frac{\partial \hat{L}_i}{\partial y(k)} \right) \Delta y(k) + \left( \frac{\partial \hat{L}_i}{\partial v(k)} \right) \Delta v(k) \right\}
$$

$$
= \sum_{k=0}^{M-1} e(M-k-h) \left[ \left( \frac{\partial \hat{L}_i}{\partial x(k)} \right) \Delta x(k) + \left( \frac{\partial \hat{L}_i}{\partial u(k)} \right) \hat{u}(k) \right].
$$

Substituting (47) into (46), and then using (29) and (45a)-(45e), we obtain

$$
\frac{\partial g_i(u)}{\partial u} \hat{u} = \left( \frac{\partial \Phi_i(x(M))}{\partial x(k)} \right) \Delta x(M)
$$

$$
+ \sum_{k=0}^{M-1} \left[ \left( \frac{\partial H_i}{\partial x(k)} \right) \Delta x(k) + \left( \frac{\partial \hat{L}_i}{\partial u(k)} \right) \hat{u}(k) \right.
$$

$$
- \left. \left( \lambda^i(k+1) \right)^\top \frac{\partial \hat{f}}{\partial x} \Delta x(k) \right.
$$

$$
- \left. \left( \lambda^i(k) \right)^\top \frac{\partial \hat{f}}{\partial x} \Delta x(k)e(M-k-h) \right.
$$

$$
- \left. \left( \lambda^i(k+1) \right)^\top \frac{\partial \hat{f}}{\partial u} \hat{u}(k) \right.
$$

$$
- \left. \left( \lambda^i(k) \right)^\top \frac{\partial \hat{f}}{\partial u} \Delta u(k)e(M-k-h) \right].
$$

(48)

Using (33b) and the definition of \( e(\cdot) \), it follows that

$$
\sum_{k=0}^{M-1} \left( \lambda^i(k) \right)^T \left[ \frac{\partial \hat{f}}{\partial x(k)} \Delta x(k) + \frac{\partial \hat{f}}{\partial u(k)} \Delta u(k) \right] e(M-k-h)
$$

$$
= \sum_{k=0}^{M-h-1} \left( \lambda^i(k) \right)^T \left[ \frac{\partial \hat{f}}{\partial x(k)} \Delta x(k) + \frac{\partial \hat{f}}{\partial u(k)} \Delta u(k) \right] e(M-k-h)
$$

$$
= \sum_{k=h}^{M-1} \left( \lambda^i(k+1) \right)^T \left[ \frac{\partial \hat{f}}{\partial y(k)} \Delta y(k) + \frac{\partial \hat{f}}{\partial v(k)} \Delta v(k) \right].
$$

(49)
As $\triangle y(k) = 0$, for $0 \leq k \leq h$, and $\triangle v(k) = 0$, $0 < k \leq h$, we have
\[
\sum_{k=0}^{M-1} (\lambda^i(k+1))^\top \left[ \frac{\partial \tilde{f}}{\partial y(k)} \triangle y(k) + \frac{\partial \tilde{f}}{\partial v(k)} \triangle v(k) \right] = \sum_{k=0}^{M-1} (\lambda^i(k+1))^\top \left[ \frac{\partial \tilde{f}}{\partial y(k)} \triangle y(k) + \frac{\partial \tilde{f}}{\partial v(k)} \triangle v(k) \right].
\] (50)

Combining (49) and (50), we obtain
\[
\sum_{k=0}^{M-1} (\lambda^i(k+1))^\top \left[ \frac{\partial \hat{f}}{\partial x(k)} \triangle x(k) + \frac{\partial \hat{f}}{\partial u(k)} \triangle u(k) \right] e(M - k - h)
\]
\[
= \sum_{k=0}^{M-1} (\lambda^i(k+1))^\top \left[ \frac{\partial \hat{f}}{\partial y(k)} \triangle y(k) + \frac{\partial \hat{f}}{\partial v(k)} \triangle v(k) \right].
\] (51)

From (43a) and (51), it follows from (48) that
\[
\frac{\partial g_i(u)}{\partial u} \hat{u} = \left( \frac{\partial \Phi_i(x(M))}{\partial x(k)} \right) \triangle x(M)
\]
\[
+ \sum_{k=0}^{M-1} \left( \frac{\partial H_i}{\partial x(k)} \right) \triangle x(k)
\]
\[
+ \left( \frac{\partial \hat{H}_i}{\partial u(k)} \right) \hat{u}(k) - (\lambda^i(k+1))^\top \triangle x(k+1)
\]. (52)

Thus, by (32), (45c) and (52), we obtain
\[
\frac{\partial g_i(u)}{\partial u} \hat{u} = \left( \frac{\partial \Phi_i(x(M))}{\partial x(k)} \right) \triangle x(M)
\]
\[
+ \sum_{k=0}^{M-2} \left( \frac{\partial H_i}{\partial x(k)} \right) \triangle x(k)
\]
\[
- \left( \frac{\partial H_i(k+1)}{\partial x(k)} \right) \triangle x(k+1)
\]
\[
+ \frac{\partial \hat{H}_i(M-1)}{\partial x(k)} \triangle x(M-1) - (\lambda^i(M))^\top \triangle x(M)
\]
\[
+ \sum_{k=0}^{M-1} \left( \frac{\partial \hat{H}_i}{\partial u(k)} \right) \hat{u}(k)
\]. (53)

Therefore, by substituting (33a) and (43b) into (47), it follows that
\[
\frac{\partial g_i(u)}{\partial u} \hat{u} = \left[ \frac{\partial H_i(0)}{\partial u(0)}, \frac{\partial H_i(1)}{\partial u(1)}, \ldots, \frac{\partial H_i(M-1)}{\partial u(M-1)} \right] \hat{u}.
\]

Since $\hat{u}$ is arbitrary, we obtain
\[
\frac{\partial g_i(u)}{\partial u} = \left[ \frac{\partial H_i(0)}{\partial u(0)}, \frac{\partial H_i(1)}{\partial u(1)}, \ldots, \frac{\partial H_i(M-1)}{\partial u(M-1)} \right].
\]

This completes the proof.

The values of the cost and constraint functions as well as their gradients are calculated as described in following algorithm.

Algorithm 2
Step 1. For a given control sequence \( u = \{u(0), ..., u(M - 1)\} \) with \( u(k) \in U, k = 0, 1, ..., M - 1 \), compute the solution \( x(k), k = 0, 1, ..., M - 1 \), of system (1) by solving time delayed difference equations (1a) with initial conditions (1b) and (1c) forward from \( k = 0 \) to \( k = M \).

Step 2. Calculate the values of the cost function \( g_0(u) \) and the constraint functions \( g_i(x(u)), i = 1, ..., N \), by using the control sequence \( u = \{u(0), ..., u(M - 1)\} \), and the corresponding solution sequence \( x(k), k = 0, 1, ..., M - 1 \).

Step 3. For each \( i = 0, 1, ..., N \), compute the co-state solution \( \lambda^i(k), k = M - 1, M - 2, ..., 0 \), by solving co-state difference equations (32) with terminal conditions (33a), (33b), (34) and (35) backward, from \( k = M, M - 1, ..., 0 \). Thus, for each \( i = 1, ..., N \), \( \lambda^i(k), k = M - 1, M - 2, ..., 0 \), are obtained.

Step 4. Calculate the gradients of the cost function \( g_0(u) \) and the constraint functions \( g_i(u), i = 1, ..., N \), according to the formulas given in Theorem 3.

Based on Algorithm 2, Problem (P) can be solved as a nonlinear mathematical programming problem by using the SQP approximating scheme. The subroutines NLPQL and NLPQLP coded in [25] are two examples of efficient implementations of SQP.

5. A Tactical Logistic Decision Analysis Problem. We now consider a tactical logistic decision analysis problem studied in [17]. It is a problem of decision making for the distribution of resources within a network of support, where the network seeks to mimic how logistic support might be delivered in a military area of operations. The problem is formulated as a discrete time optimal control problem with a time delay appearing in the control, where the physical limitations in capacity at locations and the requirements for stock are formulated as all-time-step constraints. The objective is to minimize the combat power cost function. The procedure for constructing the "optimal" control reported in [17] is as follows. The optimal control is first obtained, by using the method reported in [26], for the optimal control problem with the constraints ignored. Then, the control and the state are forced to be saturated when they violated their constraints. Clearly, the control so obtained is not, in general, an optimal control. Furthermore, it is, in reality, impossible to saturate the state of the system. Thus, the problem has not yet been solved successfully in [17]. In this paper, both the control constraints and the all-time-step constraints are considered explicitly during the computation of our optimal control. Therefore, the solution obtained satisfies all the constraints.

We now recall the optimal control model of tactical logistic decision analysis problem formulated by [17] as follows. Let \( x(t) = [x_1(t), ..., x_5(t)]^T \) be the state vector, where \( x_i(t), i = 1, ..., 5 \), denote the stocks of logistic resources at the five locations at time \( t \). Let \( u(t) = [u_1(t), ..., u_8(t)]^T \) be the control vector, where \( u_i(t), i = 1, ..., 8 \), denote the stocks dispatched for supply along eight routes during the time period \( t \). It is assumed that there is a delay of one time period between the dispatch of material from a supply location and the receipt at a receiving location. \( A(t) \) denotes the proportions of stock at respective locations that are available for the next time period; \( B(t) \) denotes the proportions of stock along respective supply routes that are providing the supply. The bounds for \( x(t) \) and \( u(t) \) reflect, respectively, the physical limitations in capacity at locations and supply routes. Furthermore, the criteria are the footprint and the physical distribution effort, which are the average combat power spent in protecting logistic resources located in the network and the average combat power spent in maintaining and protecting distribution effort along supply routes. \( Q(t) \) and \( R(t) \) denote, respectively, the opportunity cost to combat power of protecting the logistic resources at all the locations and along all the supply
routes. Then, the model is
\[
x(t + 1) = Ax(t) + B_0u(t) + B_1u(t - 1), \quad \text{(54a)}
\]
\[
x(0) = x_0, u(-1) = 0, \quad \text{(54b)}
\]
\[
x_{\text{min}} \leq x(t) \leq x_{\text{max}}, \quad \text{(55a)}
\]
\[
u_{\text{min}} \leq u(t) \leq u_{\text{max}}, \quad \text{(55b)}
\]
where
\[
A = \begin{bmatrix}
0.95 & 0 & 0 & 0 & 0 \\
0 & 0.9 & 0 & 0 & 0 \\
0 & 0 & 0.75 & 0 & 0 \\
0 & 0 & 0 & 0.75 & 0 \\
0 & 0 & 0 & 0 & 0.85
\end{bmatrix},
\]
\[
B_0 = \begin{bmatrix}
0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{bmatrix},
\]
\[
B_1 = \begin{bmatrix}
0.95 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.87 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.75 & 0 & 0 & 0.7 & 0 \\
0 & 0 & 0 & 0.85 & 0 & 0 & 0 & 0
\end{bmatrix},
\]
\[
x_0 = \begin{bmatrix}
3500 \\
800 \\
400 \\
400 \\
200
\end{bmatrix}
\]
The cost function is
\[
G = \frac{1}{2}x^T(T)Qx(T) + \sum_{t=0}^{T-1} \frac{1}{2} \{x^T(t)Qx(t) + u^T(t)Ru(t)\},
\quad \text{(56)}
\]
where
\[
Q = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2.5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2
\end{bmatrix},
\]
\[
R = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2.5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2
\end{bmatrix}.
\]
The logistic network for the example is as shown in Figure 1. The constraints (55a) and
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Figure 1. Example for the logistic network

(55b) can be rewritten as:

\[ g_i(u) = x_{i,\text{min}} - x_i(k) \leq 0, \quad k = 0, 1, \ldots, M - 1, \quad i = 1, 2, \ldots, 5, \quad (57) \]

\[ g_i(u) = x_i(k) - x_{i,\text{max}} \leq 0, \quad k = 0, 1, \ldots, M - 1, \quad i = 6, 7, \ldots, 10. \quad (58) \]

The constraints (57) and (58) are all-time-step inequality constraints. We now use the constraint transcription method described in Section 3 to approximate these all-time-step inequality constraints by a sequence (in \( \varepsilon > 0 \)) of the inequality constraints given below:

\[ -\frac{\varepsilon}{4} + g_{i,\varepsilon}(u) \leq 0, \quad i = 1, \ldots, 10, \quad (59) \]

where, for each \( i = 1, \ldots, 10 \), \( g_{i,\varepsilon}(u) \) is constructed from \( g_i(u) \) according to (7). In this way, we obtain a sequence (in \( \varepsilon > 0 \)) of discrete time optimal control problems with time delay and subject to canonical constraints (59). For each \( \varepsilon > 0 \), the corresponding discrete time optimal control problem with time delay and subject to canonical constraints (59) can be solved as an nonlinear optimization problem as explained in Section 4. The values of the cost function and the canonical constraint functions are calculated as mentioned in Step 1 to Step 2 of Algorithm 2. Their gradients are calculated as explained in Algorithm 2 using the gradient formulas obtained in Theorem 3. The initial value of \( \varepsilon \) is chosen as \( 1.0 \times 10^{-2} \). \( \varepsilon \) is reduced to \( \varepsilon /10 \) in each iteration. It is found that the change in the cost function value is negligible after \( \varepsilon \) is reduced to \( 1.0 \times 10^{-7} \). Thus, the corresponding optimal cost function value \( (1.68 \times 10^7) \) obtained is taken as the optimal cost function value. This value is much less than that obtained in [17], which is \( 3.5 \times 10^7 \).

The optimal control and the corresponding optimal state obtained using our method are depicted in Figures 2-4. By careful examination of these figures, we see that the constraints on the control and the all-time-step constraints are satisfied at each time point. From Figure 2, we see that \( u_1(k) = 0 \) for \( k = 0, 1, \ldots, 4 \), indicating no stock being dispatched along the supply route 1 to Node 1. This is because \( u_1(k) \) could only contribute extra stock to Node 1 through the supply route 1 from Node 0, and the initial stock in
**Figure 2.** Stock dispatched supply

**Figure 3.** Stock at each location

**Figure 4.** Stock at location 1
Node 1 is large, twice as large as those in the other nodes. Thus, it is clear that stock should be moved out of Node 1 to other nodes quickly through the supply routes 2 and 3 so as to decrease the cost of holding the stock in Node 1. Also from Figure 2, we see that $u_2(k)$ and $u_3(k)$ are very large at $k = 0$, meaning that a large amount of stock is
dispatched from Node 1 to the other nodes of the network at \( k = 0 \). From the structure of the network, it is clear that there is only one supply route to Node 5 with no supply route coming out of it. This means that Node 5 is a pure receiver of stock from other nodes. In view of the limits imposed on the maximum stock in various nodes, we see from Figure 3 that the amount of stock that is moved along the supply route 4 to Node 5 is low for \( k = 0, 1, \ldots, 4 \). The structure of the network depicted in Figure 1 clearly reveals that there are 4 supply routes (i.e., supply routes 3, 6 and 7) to Node 4 with only one supply route (i.e., supply route 8) coming out of it. For Node 3, there are 2 supply routes (i.e., supply routes 2 and 4) in and only 1 (i.e., supply route 7) out. By virtue of these observations, the amounts of stock along the supply routes for which the stocks are moved out should be large. This is confirmed in Figure 2 that \( u_6(k) \) and \( u_8(k) \), which denote, respectively, the amounts of stock being moved out from Node 3 along the supply route 7 and Node 4 along the supply route 8 are large for \( k = 1, \ldots, 4 \). Their values are quite low at \( k = 0 \). This is due to the appearance of time delay along the supply routes, which indicates that that nodes cannot receive stock instantaneously. The stocks arrive with delay. For Node 2, there are 3 supply routes (i.e., supply routes 4, 5 and 6) out but only 1 supply route (i.e., supply route 2) in. As shown in Figure 2, we see that the amounts of the stock being moved out along the supply routes 4, 5 and 6 are relatively small.

We also consider the situation when the time horizon is increased to 20. The optimal control and the corresponding optimal state obtained are shown in Figures 5, 6, 7, respectively. From these figures, we can see that the stock at each node reaches a balance state, i.e., the lower bounds for the stocks at each node, after \( t = 7 \). In conclusion, we see the solution obtained by using our method is highly effective.

6. Conclusions. This paper considered a class of discrete time optimal control problem with time delay and subject to all-time-state inequality constraints on both the state and control. It has been shown that, this problem can be approximated by a sequence of discrete time optimal control problems with time delay and subject to canonical constraints. A computational method was then proposed to solve a general class of discrete time optimal control problems with time delay and subject to canonical constraints as a nonlinear optimization problem. The computational method obtained was then applied to a tactical logistic decision analysis problem. The results obtained are much superior to those obtained in [17], showing the high effectiveness of the method proposed.

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