Feedback control for fractional impulsive evolution systems

Cuie Xiao\textsuperscript{a}, Biao Zeng\textsuperscript{b}, Zhenhai Liu\textsuperscript{b,∗}

\textsuperscript{a}School of Mathematics and Computation Sciences, Hunan City University, Yiyang, 413000, China
\textsuperscript{b}Guangxi Key Laboratory of Universities Optimization Control and Engineering Calculation, and College of Sciences, Guangxi University for Nationalities, Nanning, 530006, Guangxi Province, PR China

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\textbf{Abstract}

This paper is mainly concerned with feedback control systems governed by fractional impulsive evolution equations involving Riemann–Liouville derivatives in reflexive Banach spaces. We firstly give an existence and uniqueness result of mild solutions for the equations by applying the Banach's fixed point theorem. Next, by using the Filippove theorem and the Cesari property, a new set of sufficient assumptions are formulated to guarantee the existence result of feasible pairs. We also present an existence result of optimal control pairs for the Lagrange problem.

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\section{1. Introduction}

In the last years, control theory has been greatly applied in engineering, economies, computers and ecology, especially towards system with feedback control and optimal control ([8,10,14,22,28,29]). Feedback control systems are ubiquitous around us, including trajectory planning of a robot manipulator, guidance of a tactical missile toward a moving target, regulation of room temperature, and control of string vibrations. In [14], Li and Yong considered feedback optimal control for evolution equations. In [29], Wang et al. considered feedback optimal control for fractional evolution equations with Caputo fractional derivative.

Recently, many authors are interested in the fractional differential equations with Riemann–Liouville derivatives. Riemann–Liouville fractional derivatives and integrals are strong tools to resolve some fractional differential problems in the real world. Heymans and Podlubny [9] have demonstrated that it is possible to attribute physical meaning to initial conditions expressed in terms of Riemann–Liouville fractional derivatives and integrals on the field of the viscoelasticity, and such initial conditions are more appropriate than physically interpretable initial conditions. For more details, we refer to [9,24,32]. However, there are still few literatures for the feedback optimal control of the fractional impulsive differential evolution with Riemann–Liouville fractional derivatives. This is one of the motivations of the present work.

For another, we shall consider the impulse response with Riemann–Liouville fractional derivatives, which is widely used in the fields of physics, such as viscoelasticity. The impulsive differential systems originate from the real world problems to describe the dynamics processes which are subjected to abrupt changes so that discontinuous jumps occur. Impulsive differential equations have become more and more important in various applications, such as control, physics, chemistry, population dynamics, aeronautics and engineering. For more details, we refer to [1–3,7,11,13,15–21,26,27].
Consider an interval $J = (0, T] (T > 0)$ and a finite set of points
\[ D = \{ t_i \in (0, T), i = 1, 2, \ldots, m \}, \quad 0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T. \]

In this paper, we firstly consider the following form:
\[
\begin{align*}
D^q_t x(t) &= A x(t) + f(t, x(t), u(t)), \quad t \in (0, T] - D, \\
\Delta I^q_k x(t_k) &= G(t_k, x(t_k)), \quad t_k \in D, \\
I^q_k x(t)|_{t=a} &= x_0 \in X,
\end{align*}
\]
where $D^q_t$ denotes the Riemann–Liouville fractional derivative of order $q (0 < q \leq 1)$ with the lower limit zero. $A : D(A) \subseteq X \to X$, is the infinitesimal generator of a uniformly bounded $C_0$-semigroup $\{ T(t) \}_{t \geq 0}$ on a reflexive Banach space $X$. Let $V$ be a separable Banach space. $J : f : J \times X \times V \to X$, $G : D \times X \times X$ are given functions to be specified later. $\Delta I^q_k x(t_k) = I^q_k x(t_k^+)^q - I^q_k x(t_k^-)^q$, where $I^q_k x(t_k^+)$ and $I^q_k x(t_k^-)$ denote the right and the left limits of $I^q_k x(t)$ (see the definition of $I^q_k x(t)$ below) at $t = t_k \in D$, respectively.

In this paper, by applying the Banach’s fixed point theorem, we obtain the existence result of mild solutions. We don’t use the hypothesis of compactness. But it is necessary to assume that $T(t)$ is compact for the existence of feasible pair and optimal pair. The compactness issue of $A$ is referred to [4–6].

Furthermore, we shall be concerned with the existence of feasible pairs of fractional impulsive evolution equations with feedback control and optimal control. But, for Riemann–Liouville fractional derivatives with impulsive term, we can’t simply study that $u(t) \in U(t, x(t))$ which is considered in [14,29] since in general $\| x(t) \| \to \infty$ as $t \to 0$. So, it is necessary to consider $u(t) \in U(t, (t - t_k)^{-q} x(t))$ for $t \in (t_k, t_{k+1}]$. More precisely, we will study the following problem:
\[
\begin{align*}
D^q_t x(t) &= A x(t) + f(t, x(t), u(t)), \quad t \in (0, T] - D, \\
u(t) &= U(t, (t - t_k)^{-q} x(t)), \quad \text{a.e. } t \in (t_k, t_{k+1}], \\
\Delta I^q_k x(t_k) &= G(t_k, x(t_k)), \quad t_k \in D, \\
I^q_k x(t)|_{t=a} &= x_0 \in X,
\end{align*}
\]
where $U : [0, T] \times X \to V$ is a feedback multifunction.

We consider the feedback optimal control problem (1.1) combining fractional impulsive evolution equations and feedback control system which is first studied. It is a promising topic since the great developments of impulsive equations and optimal control system. Our results obtained in this paper could be widely applied in many practical problems.

The rest of this paper is organized as follows. In Section 2, we will present some preliminaries which will be used to prove our main results. In Section 3, at the first, some sufficient conditions are established to guarantee the existence and uniqueness of mild solutions of the system (1.1). Next, we present some sufficient conditions for the existence of feasible pairs of problem (1.2). In Section 4, we will study the optimal control pairs for the Lagrange problem. In the last section, we give an example to illustrate our main results.

2. Preliminaries

In this section, we introduce some basic preliminaries which are used throughout this paper. The norm of a Banach space $X$ will be denoted by $\| \cdot \|_X$. For the uniformly bounded $C_0$-semigroup $T(t)(t \geq 0)$, we set $M := \sup_{t \in [0, \infty)} \| T(t) \| < +\infty$. Let $C(J, X)$ denote the Banach space of all $X$–value continuous functions from $J$ into $X$ with the norm $\| x \|_{C} = \sup_{t \in J} \| x(t) \|_X$. Let $AC(J, X)$ be the space of functions $f$ which are absolutely continuous on $J$ and $AC^m(J, X) = \{ f : J \to X \text{ and } f^{(m-1)} \in AC(J, X) \}$. Let $C_{1-q}(J, X) = \{ x : y(t) = t^{1-q} x(t), y \in C(J, X) \}$ with the norm $\| x \|_{C_{1-q}} = \sup_{t \in J} \| t^{1-q} \| x(t) \|_X : t \in J \}$. Obviously, the space $C_{1-q}(J, X)$ is a Banach space.

In order to define the mild solutions of problems (1.1), we also consider the Banach space $PC_{1-q}(J, X) = \{ x : y(t) = (t - t_k)^{1-q} x(t), y \in C((t_k, t_{k+1}], X) \text{ and } \lim_{t \to t_k} (t - t_k)^{1-q} x(t) \text{ exists}, k = 0, 1, 2, \ldots, m \}$ with the norm $\| x \|_{PC_{1-q}} = \max_{k = 0, 1, 2, \ldots, m} \left\{ \sup_{t \in (t_k, t_{k+1}]} (t - t_k)^{1-q} \| x(t) \|_X : k = 0, 1, 2, \ldots, m \right\}$.

Firstly, let us recall the following basic definitions from fractional calculus (cf. [12,25]):

**Definition 2.1.** The integral
\[
I^q_t f(t) = \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} f(s) \, ds, \quad q > 0,
\]
is called Riemann–Liouville fractional integral of order $q$, where $\Gamma$ is the gamma function.
Definition 2.2. For a function $f(t)$ given in the interval $[0, \infty)$, the expression
\[
D^q_t f(t) = \frac{1}{\Gamma(n-q)} \left( \frac{d}{dt} \right)^n \int_0^t (t-s)^{n-q-1} f(s) ds,
\]
where $n = \lfloor q \rfloor + 1$, $\lfloor q \rfloor$ denotes the integer part of number $q$, is called the Riemann–Liouville fractional derivative of order $q$.

In order to study the $PC_{1-q}$—mild solutions of problem (1.1), we need:

Lemma 2.3 ([20]). Let $0 < q \leq 1$, and $x_{1-q}(t) = I_{t}^{1-q}x(t)$ be the fractional integral of order $1 - q$. If $x \in PC_{1-q}(J, X)$ and $x_{1-q} \in AC(J, X)$, then we have the following equation
\[
x_{1-q}(t) = x(0) + \int_0^t (t-s)^{q-1} \frac{d}{d(t')} x(t') dt',
\]
where $x_{1-q}(t) = x_{1-q}(t^+), \quad x_{1-q}(t^-), \quad k = 1, 2, \ldots, m.$

Lemma 2.4 ([17]). Let $0 < q \leq 1$ and $h \in L^p(J, X)$ ($p > \frac{1}{q}$). If $x \in PC_{1-q}(J, X)$ and $x_{1-q} \in AC(J, X)$ and $x$ is a solution of the following problem
\[
\begin{align*}
D^q_t x(t) &= A x(t) + h(t), \quad t \in (0, T], t \neq t_k, \quad k = 1, 2, \ldots, m, \\
\Delta I_{t_k}^{1-q} x(t_k) &= x_k, \quad k = 1, 2, \ldots, m, \\
I_{t_0}^{1-q} x(t)|_{t=0} &= x_0 \in X,
\end{align*}
\]
then, $x$ satisfies the following equation
\[
x(t) = \begin{cases} 
I_{t}^{q-1} T_q(t)x_0 + \int_0^t (t-s)^{q-1} T_q(t-s) h(s) ds, & t \in [0, t_1], \\
I_{t}^{q-1} T_q(t)x_0 + \sum_{i=1}^{k} T_q(t-t_i) (t-t_i)^{q-1} x_i + \int_0^t (t-s)^{q-1} T_q(t-s) h(s) ds, & t \in (t_k, t_{k+1}].
\end{cases}
\]
where
\[
T_q(t) = q \int_0^\infty \xi_q(\theta) T(t^\theta) d\theta,
\]
\[
\xi_q(\theta) = \frac{1}{q} (\theta - 1)^{-\frac{1}{q}} \sigma_q(\theta^{-\frac{1}{q}}),
\]
\[
\sigma_q(\theta) = \frac{1}{\Gamma(nq+1)} \sum_{n=0}^\infty (-1)^{n-1} \theta^{-nq-1} \sin(n\pi q), \quad \theta \in (0, \infty).
\]
$\xi_q$ is a probability density function defined on $(0, \infty)$, that is
\[
\xi_q(\theta) \geq 0, \quad \theta \in (0, \infty), \quad \text{and} \int_0^\infty \xi_q(\theta) d\theta = 1.
\]

According to Lemma 2.4, we give the following definition.

Definition 2.5. A function $x \in PC_{1-q}(J, X)$ is called a mild solution of problem (1.1) if it satisfies the following fractional integral equation
\[
x(t) = \begin{cases} 
I_{t}^{q-1} T_q(t)x_0 + \int_0^t (t-s)^{q-1} T_q(t-s) f(s, x(s), u(s)) ds, & t \in [0, t_1], \\
I_{t}^{q-1} T_q(t)x_0 + \sum_{i=1}^{k} T_q(t-t_i) (t-t_i)^{q-1} G(t, x(t^-)) + \int_0^t (t-s)^{q-1} T_q(t-s) f(s, x(s), u(s)) ds, & t \in (t_k, t_{k+1}].
\end{cases}
\]

From the work of the paper [23,31], we have the following result:

Lemma 2.6. If the $C_{0}$-semigroup $T(t)(t \geq 0)$ is uniformly bounded (i.e. $\sup_{t \geq 0} \|T(t)\| \leq M < \infty$), then the operator $T_q(t)$ has the following properties:
(i) for any fixed $t \geq 0$, $T_q(t)$ is linear and bounded operators, i.e., for any $x \in X$,
\[
\|T_q(t)x\| \leq \frac{M}{\Gamma(q)} \|x\|
\]
(ii) $T_q(t)(t \geq 0)$ is strongly continuous.

To obtain our main results, we need the following. See [14] for more details.

**Definition 2.7** ([14]). Let $X$ and $Y$ be two metric spaces. A multifunction $F : X \rightrightarrows Y$ is said to be pseudo-continuous at $x \in X$ if
\[
\bigcap_{\varepsilon > 0} F(O_x(\varepsilon)) = F(x),
\]
where $O_x(\varepsilon) = \{y \in X ||y - x|| \leq \varepsilon\}$. We say that $F$ is pseudo-continuous on $X$ if it is pseudo-continuous at each point $x \in X$.

**Remark 2.8.**

(i) Let $F : x \rightrightarrows Y$ be a multifunction taking closed set values. Then $F$ is pseudo-continuous if and only if the graph
\[
\mathcal{G} = \{(x, y) \in X \times Y | y \in F(x)\}
\]
is closed in $X \times Y$.

(ii) If $F : X \rightrightarrows Y$ is a multifunction taking closed set values and upper semicontinuous, then it is pseudo-continuous.

**Definition 2.9** ([14]). Let $X$ be a Banach space and $Y$ be a metric space. Let $F : X \rightrightarrows Y$ be a multifunction. We say $F$ possesses the Cesari property at $x_0 \in X$, if
\[
\bigcap_{\varepsilon > 0} \overline{\Delta}(O_\delta(x_0)) = F(x_0),
\]
where $\overline{\Delta}$ is the closed convex hull of $D$, $O_\delta(x)$ is the $\delta$-neighborhood of $x$. If $F$ has the Cesari property at every point $x \in Z \subset X$, we simply say that $F$ has the Cesari property on $Z$.

**Lemma 2.10** ([14], Proposition 4.2). Let $X$ be a Banach space and $Y$ be a metric space. Let $F : X \rightrightarrows Y$ be u.s.c. with convex and closed valued. Then $F$ has the Cesari property on $X$.

### 3. Existence of mild solutions

In what follows, we will make the following hypotheses on the data of our problems.

- **H(1):** $T(t)$ is a compact operator for every $t > 0$.
- **H(2):** $f : f(x, t)$ is Borel measurable in $(t, x, u)$.
- **H(3):** There exist a function $\phi(\cdot) \in L^p(J, \mathbb{R}^+)$ ($p > \frac{1}{q}$) and constants $L, N > 0$ such that
\[
\|f(t, 0, 0)\| \leq \phi(t), \quad \text{a.e. } t \in J,
\]
\[
\|f(t, x_1, u_1) - f(t, x_2, u_2)\| \leq L(t - t_k)^{1-\delta}\|x - y\|_x + N\|u_1 - u_2\|_Y
\]
for a.e. $t \in (t_k, t_{k+1}]$, $k = 0, 1, \ldots, m$. for all $x_1, x_2 \in X, u_1, u_2 \in V$.

- **H(4):** There exist constants $d_k \geq 0$ with $2M \sum_{i=1}^{k} d_i(t_i - t_{i-1})^{q-1} < \Gamma(q)(k = 1, 2, \ldots, m + 1)$ such that
\[
\|G(t_k, x) - G(t, y)\| \leq d_k \|x - y\|_x, \quad t_k \in D, \quad \forall x, y \in X.
\]

Now, we are in the position to prove the following existence and uniqueness result of mild solutions for problem (1.1).

**Theorem 3.1.** Assume that the hypotheses H(2)–H(4) are satisfied. Then for each given control function $u \in L^p(J, V)$ ($p > \frac{1}{q}$), the problem (1.1) has a unique mild solution on $PC_{1-q}(J, X)$.

**Proof.** Define the operator $\mathcal{F} : PC_{1-q}(J, X) \rightarrow PC_{1-q}(J, X)$ by
\[
(\mathcal{F}x)(t) = \begin{cases}
  t^{q-1} T_q(t)x_0 + \int_0^t (t - s)^{q-1} T_q(t - s)f(s, x(s), u(s))ds, & t \in [0, t_1], \\
  t^{q-1} T_q(t)x_0 + \sum_{i=1}^{k} T_q(t - t_i)(t - t_i)^{q-1} G(t_i, x(t_{i-1}^-)) + \int_0^t (t - s)^{q-1} T_q(t - s)f(s, x(s), u(s))ds, & t \in (t_k, t_{k+1}], \quad k = 1, \ldots, m.
\end{cases}
\]

Then the problem of finding mild solutions for problem (1.1) is reduced to find the fixed point of $\mathcal{F}$. To prove this, we consider the operator $\mathcal{F}$ on the Banach space $PC_{1-q}(J, X)$ with a weighted norm
\[
\|x\| = \max\left\{\sup_{t \in (t_k, t_{k+1}]} (t - t_k)^{1-q}\|x(t)\|_x e^{-\pi t} : k = 0, 1, \ldots, m\right\}.
\]
where $r = \max \left\{ \left( \frac{2 \Gamma(q) M T^{1-q}}{\Gamma(q) - 2 M \sum_{i=1}^{\infty} |t_i - t_{i-1}|^{r-1}} \right)^{1/q}, k = 1, \ldots, m + 1 \right\}$. Now, set $B_r(R) = \{ x \in PC_{1-q}(J, X) : \| x \|_r \leq R \}$, where $R = 2\omega$
and
$$
\omega = \frac{M}{\Gamma(q)} \| x_0 \| + \frac{M}{\Gamma(q)} \sum_{i=1}^{m} \| G(t_i, 0) \|
+ \frac{MT^{1-q}}{\Gamma(q)} \left( \frac{p - 1}{pq - 1} \right)^{1-q} \| \| \phi \|_{L^p(J, R^+)} + N \| u \|_{L^p(J, V)} \|.
$$

Next, for the sake of convenience, we subdivide the proof into two steps.

Step 1: We shall prove that the operator $f$ maps $B_r(R)$ into itself.

Notice that

$$
\int_0^t (t - s)^{q-1} e^{rt} ds = -r^{-q} \int_0^t (t - s)^{q-1} e^{-r(t - s)} e^{r t} d[r(t - s)]
= r^{-q} e^{r t} \int_0^t z^{q-1} e^{-z} dz \quad (r(t - s) = z)
\leq r^{-q} e^{r t} \Gamma(q).
$$

If $t \in [0, t_1]$, then from the formula (3.1), the hypothesis $H(3)$ and the Hölder inequality, we obtain

$$
t^{1-q} \| (f x)(t) \|_X \leq \| T_q(t) x_0 \| + t^{1-q} \int_0^t (t - s)^{q-1} \| T_q(t - s) f(s, x(s), u(s)) \| ds
\leq \frac{M}{\Gamma(q)} \| x_0 \| + \frac{MT^{1-q}}{\Gamma(q)} \int_0^t (t - s)^{q-1} \| f(s, 0, 0) \| + L s^{1-q} \| x(s) \|_X + N \| u(s) \|_V ds
\leq \frac{M}{\Gamma(q)} \| x_0 \| + \frac{MT^{1-q}}{\Gamma(q)} \int_0^t (t - s)^{q-1} \| \phi(s) \| + N \| u(s) \|_V ds
+ \frac{MTL^{1-q} \| x \|_r}{\Gamma(q)} \int_0^t (t - s)^{q-1} e^{rt} ds
\leq \frac{M}{\Gamma(q)} \| x_0 \| + \frac{MT^{1-q}}{\Gamma(q)} \left( \frac{p - 1}{pq - 1} \right)^{1-q} \| \phi \|_{L^p(J, R^+)} + N \| u \|_{L^p(J, V)}
+ MTL^{1-q} r^{-q} \| x \|_r.
$$

Thus, we have

$$
\sup_{t \in [0, t_1]} t^{1-q} \| (f x)(t) \|_X e^{-rt} \leq \omega + MTL^{1-q} r^{-q} \| x \|_r \leq R.
$$

If $t \in [t_k, t_{k+1}](k = 1, \ldots, m)$, from the formula (3.1), the hypotheses $H(3), H(4)$ and the Hölder inequality, we have

$$
(t - t_k)^{1-q} \| (f x)(t) \|_X \leq (t - t_k)^{1-q} \| T_q(t) x_0 \|
+ (t - t_k)^{1-q} \| \sum_{i=1}^{k} T_q(t - t_i)(t - t_j)^{q-1} G(t_i, x(t^-_i)) \|
+ (t - t_k)^{1-q} \int_0^t (t - s)^{q-1} \| T_q(t - s) f(s, x(s), u(s)) \| ds
\leq \frac{M}{\Gamma(q)} \| x_0 \| + \frac{M}{\Gamma(q)} \sum_{i=1}^{k} d_i(t_i - t_{i-1})^{q-1} (t_i - t_{i-1})^{1-q} \| x(t^-_i) \|_X
+ \frac{M}{\Gamma(q)} \sum_{i=1}^{k} \| G(t_i, 0) \| + \frac{MTL^{1-q}}{\Gamma(q)} \int_0^t (t - s)^{q-1} \| f(s, 0, 0) \| ds
+ L(s - t_k)^{1-q} \| x(s) \|_X + N \| u(s) \|_V ds
\leq \frac{M}{\Gamma(q)} \| x_0 \| + \frac{M}{\Gamma(q)} \sum_{i=1}^{k} \| G(t_i, 0) \| + MTL^{1-q} \left( \frac{p - 1}{pq - 1} \right)^{1-q} \| \phi \|_{L^p(J, R^+)} + N \| u \|_{L^p(J, V)}
+ \left( \frac{M \sum_{i=1}^{k} d_i(t_i - t_{i-1})^{q-1}}{\Gamma(q)} + MTL^{1-q} r^{-q} \right) e^{rt} \| x \|_r.
Lemma 3.3. Impulsive Riemann–Liouville fractional neutral evolution equations on the literature \([20]\) under much weaker conditions.

Remark 3.2.

Definition 3.4.

Now, we study the existence result of feasible pairs for system (1.2).

**Theorem 3.1.** Let \(H(1)\) holds. Then operator \(\pi : L^p(J, X) \to C(J, X)\) for some \(p > \frac{1}{q}\), given by

\[
(\pi h) (\cdot) = \int_{0}^{\cdot} (\cdot - s)^{q-1} T_q (\cdot - s) h(s) \, ds,
\]

is compact for \(h \in L^p(J, X)\).

To the readers’ convenience, we give the definition as follows.

**Definition 3.4.** A pair \((x, u)\) is said to be feasible if \((x, u)\) satisfies (1.2).

Set

\[
\begin{align*}
V[0, T] &= \{ u : J \to V | u(\cdot) \text{ is measurable} \}, \\
H[0, T] &= \{ (x, u) \in PC_{1-q}(J, X) \times V[0, T] | (x, u) \text{ feasible} \}.
\end{align*}
\]

Now, we study the existence result of feasible pairs for system (1.2).
We assume that the feedback multifunction $U : J \times X \to V$ satisfies the following hypotheses:

$H(5)$: $U$ is pseudo-continuous and there exist a function $\phi_1 \in L^p(J, \mathbb{R}^\delta) (p > \frac{1}{\delta})$ and a constant $L_1 > 0$, such that

$$\|U(t, x)\| = \sup_{z \in U(t, x)} \|z\|_V \leq \phi_1(t) + L_1 \|x\|_X$$

for all $(t, x) \in J \times X$;

$H(6)$: for almost all $t \in (t_k, t_{k+1}) (k = 0, 1, \ldots, m)$ and all $x \in X$, the set $f(t, x, U(t, (t - t_k)^{1-q}x))$ satisfies the following

$$\bigcap_{\delta > 0} \overline{\delta f}(t, O_\delta(x), U(O_\delta(t, (t - t_k)^{1-q}x))) = f(t, x, U(t, (t - t_k)^{1-q}x)).$$

We are in the position to present the main result of this section.

**Theorem 3.5.** Assume that $H(1) – H(6)$ are satisfied. Then the set $H(0, T)$ is nonempty.

**Proof.** **Case 1.** Let $t \in [0, t_1]$. For any $n > 0$, let $t_{0,j} = \frac{j}{n} t_1, 0 \leq j \leq n - 1$. We set

$$u_{0,n}(t) = \sum_{j=0}^{n-1} u^{0,j} \chi_{[t_{0,j}, t_{0,j+1})}(t), \quad t \in [0, t_1].$$

where $\chi_{[t_{0,j}, t_{0,j+1})}$ is the characteristic function of interval $[t_{0,j}, t_{0,j+1})$. The sequence $(u^{0,j})$ is constructed as follows.

Firstly, we take $u^{0,0} \in U(0, \Gamma(q))$ since $x_0 = \Gamma(q)t^{1-q}x(t)|_{t=0}$. By Theorem 3.1, there exists a unique $x_{0,n}(\cdot)$ which is given by

$$x_{0,n}(t) = t^{q-1}T_q(t)x_0 + \int_0^t (t-s)^{q-1}T_q(t-s)f(s, x_{0,n}(s), u^{0,0}(s))ds, \quad t \in [0, t_{0,1}].$$

Then take $u^{0,1} \in U(t_{0,1}, \tau_{0,1}^{1-q}x_{0,n}(t_{0,1}))$. We can repeat this procedure to obtain $x_{0,n}$ on $[t_{0,1}, t_{0,2}]$, etc. By induction, we end up with the following:

$$\begin{cases}
  x_{0,n}(t) = t^{q-1}T_q(t)x_0 + \int_0^t (t-s)^{q-1}T_q(t-s)f(s, x_{0,n}(s), u_{0,n}(s))ds, & t \in [0, t_1], \\
  u_{0,n}(t) \in U(t_{0,j}, \tau_{0,j}^{1-q}x_{0,n}(t_{0,j})), & t \in [t_{0,j}, t_{0,j+1}), 0 \leq j \leq n - 1. 
\end{cases} \tag{3.2}$$

By Corollary 2 in [30] and $H(3)$, there exists $r_{0,0} > 0$ such that

$$\|x_{0,n}\|_{C_{1-q}(0, t_1, X)} \leq r_{0,0}.$$ 

Moreover, it comes from $H(3)$ and $H(5)$ that there exists $r_{0,1} > 0$ such that

$$\|f(\cdot, x_{0,n}(\cdot), u_{0,n}(\cdot))\|_{L^p(0, t_1, X)} \leq r_{0,1}.$$ 

Since $L^p$ is a reflexive Banach space, there is a subsequence of $(f(\cdot, x_{0,n}(\cdot), u_{0,n}(\cdot)))$, denoted by $(f(\cdot, x_{0,n}(\cdot), u_{0,n}(\cdot)))$ again, such that

$$f(\cdot, x_{0,n}(\cdot), u_{0,n}(\cdot)) \to f_0(\cdot) \text{ in } L^p([0, t_1], X). \tag{3.3}$$

From $H(1)$ and Lemma 3.3 we have

$$\int_0^t (t-s)^{q-1}T_q(t-s)f(s, x_{0,n}(s), u_{0,n}(s))ds \to \int_0^t (t-s)^{q-1}T_q(t-s)f_0(s)ds, \quad t \in [0, t_1].$$

Let

$$\overline{x}_0(t) = t^{q-1}T_q(t)x_0 + \int_0^t (t-s)^{q-1}T_q(t-s)f_0(s)ds, \quad t \in [0, t_1].$$

Then

$$t^{1-q}x_{0,n}(t) \to t^{1-q}\overline{x}_0(t), \quad n \to \infty$$

uniformly in $t \in [0, t_1]$, i.e.

$$x_{0,n}(\cdot) \to \overline{x}(\cdot) \text{ in } C_{1-q}(0, t_1, X). \tag{3.4}$$

On the other hand, by the definition of $u_{0,n}(\cdot)$ for $n$ large enough, we have

$$u_{0,n}(t) \in U(t_{0,j}, \tau_{0,j}^{1-q}x_{0,n}(t_{0,j}))) \subset U(O_\delta(t, t^{1-q}\overline{x}_0(t))), \tag{3.5}$$

for all $t \in [t_{0,j}, t_{0,j+1}), 0 \leq j \leq n - 1$.

Secondly, by (3.3) and the Mazur theorem (Chapter 2, Corollary 2.8, [14]), let $q^0_{1,\delta} \geq 0$ and $\sum_{i=1}^{\infty} q^0_{1,\delta} = 1$ such that

$$\psi_{0,1}(\cdot) = \sum_{i=1}^{\infty} q^0_{1,\delta} f(\cdot, x_{0,i+1}(\cdot), u_{0,i+1}(\cdot)) \to f_0(\cdot) \text{ in } L^p([0, t_1], X).$$
Then, there is a subsequence of \( \{ \psi_{0,j} \} \), denoted by \( \{ \psi_{0,j} \} \) again, such that
\[
\psi_{0,j}(t) \to \bar{f}_0(t) \text{ in } X, \quad \text{a.e. } t \in [0, t_1].
\]
Hence, from (3.4) and (3.5), for \( l \) large enough,
\[
\psi_{0,j}(t) \in \mathcal{C}(t, O_\delta(\bar{x}_0(t)), U(O_\delta(t, t^{1-\delta}\bar{x}_0(t)))) \quad \text{a.e. } t \in [0, t_1].
\]
Thus, for any \( \delta > 0 \),
\[
\bar{f}_0(t) \in \mathcal{C}(t, \bar{x}_0(t), U(t, t^{1-\delta}\bar{x}_0(t))) \quad \text{a.e. } t \in [0, t_1].
\]
By (6), we have
\[
\bar{f}_0(t) \in f(t, \bar{x}_0(t), U(t, t^{1-\delta}\bar{x}_0(t))) \quad \text{a.e. } t \in [0, t_1].
\]
By (3) and Corollary 2.18 of [14], we have that \( U(\cdot, t^{1-\delta}\bar{x}_0(\cdot)) \) is Souslin measurable. By the Fillippove theorem (Chapter 2, Corollary 2.26, [14]), there exists a measurable function \( \bar{u}_0 \) on \([0, t_1]\) such that
\[
\bar{u}_0(t) \in U(t, t^{1-\delta}\bar{x}_0(t)), \quad t \in [0, t_1],
\]
and
\[
\bar{f}_0(t) = f(t, \bar{x}_0(t), \bar{u}_0(t)), \quad t \in [0, t_1].
\]

**Case 2.** Let \( t \in (t_1, t_2] \). For any \( n > 0 \), let \( \tau_{1,j} = t_1 + \frac{j}{n}(t_2 - t_1), \; 0 \leq j \leq n - 1 \). We set
\[
u_{1,j}(t) = \sum_{j=0}^{n-1} u_{1,j}(\tau_{1,j,\tau_{1,j+1}}(t)), \quad t \in (t_1, t_2].
\]
The sequence \( \{ u_{1,j} \} \) is constructed as follows.

Take \( u^{1,0} \in U(t_1, \cdot - t_1)1^{1-\delta}\bar{x}(\cdot)|_{t=t_1} \). By Theorem 3.1, there exists a unique \( x_{1,n}(\cdot) \) which is given by
\[
x_{1,n}(t) = \begin{cases} \bar{x}_0(t), & t \in [0, t_1], \\ t^{1-\delta}T_q(t)x_0 + T_q(t - t_1)(t - t_1)^{\delta-1}G_1(t_1, x_{1,n}(t_1)) + \int_0^t (t - s)^{\delta-1}T_q(t - s)f(s, x_{1,n}(s), u_{1,0}(s))ds, & t \in (t_1, t_1]. 
\end{cases}
\]
Then take \( u^{1,1} \in U(\tau_{1,1}, (\tau_{1,1} - t_1)^{1-\delta}x_{1,n}(\tau_{1,1})) \). We can repeat this procedure to obtain \( x_{1,n} \) on \((\tau_{1,1}, \tau_{1,2}]\), etc. By induction, we end up with the following:
\[
x_{1,n}(t) = \begin{cases} \bar{x}_0(t), & t \in [0, t_1], \\ t^{1-\delta}T_q(t)x_0 + T_q(t - t_1)(t - t_1)^{\delta-1}G_1(t_1, x_{1,n}(t_1)) + \int_0^t (t - s)^{\delta-1}T_q(t - s)f(s, x_{1,n}(s), u_{1,n}(s))ds, & t \in (t_1, t_1]. 
\end{cases}
\]
By the proof of Theorem 3.1 and (3), there exists \( r_{1,0} > 0 \) such that
\[
\|x_{1,n}\|_{C_{1-\delta}([0,t_1])} \leq r_{1,0}.
\]
Moreover, it comes from (3) and (5) that there exists \( r_{1,1} > 0 \) such that
\[
\|f(\cdot, x_{1,n}(\cdot), u_{1,n}(\cdot))\|_{L^p([0,t_2], X)} \leq r_{1,1}.
\]
Similar to Case 1, there is a subsequence \( \{ f(\cdot, x_{1,n}(\cdot), u_{1,n}(\cdot)) \} \) such that
\[
f(\cdot, x_{1,n}(\cdot), u_{1,n}(\cdot)) \to \bar{f}_1(\cdot) \quad \text{in } L^p([0,t_2], X).
\]
It is clear that \( \bar{f}_1|_{[0,t_2]} = \bar{f}_0 \). From (1) and Lemma 3.3 we have
\[
\int_0^t (t - s)^{1-\delta}T_q(t - s)f(s, x_{1,n}(s), u_{1,n}(s))ds \to \int_0^t (t - s)^{\delta-1}T_q(t - s)\bar{f}_1(s)ds, \quad t \in (t_1, t_2].
\]
Let
\[
\bar{x}_1(t) = \begin{cases} \bar{x}(t), & t \in [0, t_1], \\ t^{1-\delta}T_q(t)x_0 + T_q(t - t_1)(t - t_1)^{\delta-1}G_1(t_1, x_{1}(t_1)) + \int_0^t (t - s)^{\delta-1}T_q(t - s)\bar{f}_1(s)ds, & t \in (t_1, t_2]. 
\end{cases}
\]
Therefore, for any \( t \in (t_1, t_2] \) we have
\[(t - t_1)^{1-q}||x_{1,n}(t) - \bar{x}_1(t)|| \leq T_0(t - t_1)\|G_1(t_1, x_{1,n}(t_1^-)) - G_1(t_1, \bar{x}_1(t_1^-))\|
+ (t - t_1)^{1-q}\int_0^t (t - s)^{q-1}\|T_q(t - s)(f(s, x_{1,n}(s), u_{1,n}(s)) - \bar{T}_1(s))\|ds
\leq \frac{Md_1}{\Gamma(q)}\|x_{1,n}(t_1^-) - \bar{x}_1(t_1^-)\|
+ (t - t_1)^{1-q}\int_0^t (t - s)^{q-1}\|T_q(t - s)(f(s, x_{1,n}(s), u_{1,n}(s)) - \bar{T}_1(s))\|ds.
\]

Hence,
\[(t - t_1)^{1-q}x_{1,n}(t) \to (t - t_1)^{1-q}\bar{x}_1(t), \quad n \to \infty\]
uniformly in \(t \in (t_1, t_2].\) Therefore,
\[x_{1,n}(\cdot) \to \bar{x}_1(\cdot) \quad \text{in} \quad PC_{1-q}([0, t_2], X).
\tag{3.7}
\]

On the other hand, by the definition of \(u_{1,n}(\cdot)\) for \(n\) large enough, we have
\[u_{1,n}(t) \in U(\tau_{1,j}, (\tau_{1,j} - t_1)^{1-q}x_{1,n}(\tau_{1,j})) \subset U(O_q(t_2/k, (t_1 - t_1)^{1-q}\bar{x}_1(t_1))),
\tag{3.8}
\]
for all \(t \in (\tau_{1,j}, \tau_{1,j+1}], 0 \leq j \leq n - 1.\)

Secondly, by (3.6) and the Mazur theorem again, let \(q^*_n \geq 0\) and \(\sum_{i=1}^{q^*_n} = 1\) such that
\[\psi_{1,l}(\cdot) = \sum_{i=1}^{q^*_n} \int f(\cdot, x_{1,i+l}(\cdot), u_{1,i+l}(\cdot)) \to \bar{T}_1(\cdot) \quad \text{in} \quad L^p([0, t_2], X).
\]

Then, there is a subsequence of \((\psi_{1,l})\), denoted by \((\psi_{1,l})\) again, such that
\[\psi_{1,l}(t) \to \bar{T}_1(t) \quad \text{in} \quad X, \quad \text{a.e.} \quad t \in [0, t_2].
\]

Hence, from (3.7) and (3.8), for \(l\) large enough,
\[\psi_{1,l}(t) \in \text{co}(f(t, O_q(\bar{x}_1(t)), U(O_q(t, (\tau_{1,j} - t_1)^{1-q}\bar{x}_1(\tau_{1,j}))))), \quad \text{a.e.} \quad t \in [0, t_2].
\]

Thus, for any \(\delta > 0,
\[\bar{T}_1(t) \in \text{co}(f(t, O_q(\bar{x}_1(t)), U(O_q(t, (\tau_{1,j} - t_1)^{1-q}\bar{x}_1(\tau_{1,j}))))), \quad \text{a.e.} \quad t \in [0, t_2].
\]

By H(6), we have
\[\bar{T}_1(t) \in f(t, \bar{x}_1(t), U(t, (\tau_{1,j} - t_1)^{1-q}\bar{x}_1(\tau_{1,j}))), \quad \text{a.e.} \quad t \in [0, t_2].
\]

By H(3) and Corollary 2.18 of [14] again, we have that \(U(\cdot, (\cdot - t_1)^{1-q}\bar{x}_1(\cdot))\) is Souslin measurable. By the Fillippov theorem again, there exists a measurable function \(\bar{u}_1\) on \([0, t_2]\) such that
\[
\begin{cases}
\bar{u}_1(t) = \bar{u}_0(t), & t \in [0, t_1], \\
\bar{u}_1(t) \in U(t, (t - t_1)^{1-q}\bar{x}_1(t)), & t \in (t_1, t_2],
\end{cases}
\]

and
\[\bar{T}_1(t) = f(t, \bar{x}_1(t), \bar{u}_1(t)), \quad t \in [0, t_2].
\]

**Case 3.** Let \(t \in (t_k, t_{k+1}) (k = 2, \ldots, m).\) For any \(n > 0,\) let \(\tau_{k,j} = t_k + \frac{j}{n}(t_{k+1} - t_k), 0 \leq j \leq n - 1.\) We set
\[u_{k,n}(t) = \begin{cases}
\bar{u}_{k-1}(t), & t \in [0, t_k], \\
\sum_{j=0}^{n-1} u^{k,j} x_{(\tau_{k,j}, \tau_{k,j+1})}(t), & t \in (t_k, t_{k+1}].
\end{cases}
\]

By induction, we end up with the following:
\[x_{k,n}(t) = \begin{cases}
\bar{x}_{k-1}(t), & t \in [0, t_k], \\
\int_0^q (t - s)^{q-1}T_q(t - s)f(s, x_{k,n}(s), u_{k,n}(s))ds, & t \in (t_k, t_{k+1}].
\end{cases}
\]

\[u_{k,n}(t) \in U(\tau_{k,j}, (\tau_{k,j} - t_1)^{1-q}\bar{x}_1(\tau_{k,j})), \quad t \in (\tau_{k,j}, \tau_{k,j+1}], 0 \leq j \leq n - 1.\]

By the proof of Theorem 3.1 and H(3), there exists \(r_{k,0} > 0\) such that
\[\|x_{k,n}\|_{PC_{1-q}([0, t_{k+1}], X)} \leq r_{k,0}.\]
Moreover, it comes from $H(3)$ and $H(5)$ that there exists $t_{k,1} > 0$ such that
\[
\|f(\cdot, x_{k,n}(\cdot), u_{k,n}(\cdot))\|_{L^p([0,t_{k,1}],X)} \leq f_{k,1}.
\]

Similar to Case 2, there is a subsequence $\{f(\cdot, x_{k,n}(\cdot), u_{k,n}(\cdot))\}$ such that
\[
f(\cdot, x_{k,n}(\cdot), u_{k,n}(\cdot)) \rightharpoonup \overline{f}(\cdot) \quad \text{in } L^p([0,t_{k,1}],X).
\]

It is clear that $\overline{f}_{k[i]} \rightharpoonup \overline{f}_{k-1}$. From $H(1)$ and Lemma 3.3 we have
\[
\int_0^t (t-s)^{q-1} T_q(t-s) f(s, x_{k,n}(s), u_{k,n}(s))ds \rightharpoonup \int_0^t (t-s)^{q-1} T_q(t-s) \overline{f}_{k}(s)ds, \quad t \in (t_k, t_{k+1}].
\]

Let
\[
\overline{x}_{k-1}(t) = \begin{cases} 
\overline{x}_{k-1}(t), & t \in [0, t_k], \\
t^{q-1} \overline{T}_q(t)x_0 + \sum_{i=1}^k t_q(t-t_i)(t-t_i)^{q-1} G_i(t_i^-, \overline{x}_i(t_i^-)) \
+ \int_0^t (t-s)^{q-1} T_q(t-s) \overline{f}_k(s)ds, & t \in (t_k, t_{k+1}].
\end{cases}
\]

Therefore, for any $t \in (t_k, t_{k+1}]$ we have
\[
(t-t_k)^{1-q} \|x_{k,n}(t) - \overline{x}_{k}(t)\| \leq T_0(t-t_k) \sum_{i=1}^k \|G_i(t_i, x_{i,n}(t_i^-)) - G_i(t_i, \overline{x}_i(t_i^-))\| \\
+ (t-t_k)^{1-q} \int_0^t (t-s)^{q-1} \|T_q(t-s)[f(s, x_{k,n}(s), u_{k,n}(s)) - \overline{f}_k(s)]\|ds \\
\leq \frac{M}{(q-1)!} \sum_{i=1}^k d_i \|x_{i,n}(t_i^-) - \overline{x}_i(t_i^-)\| \\
+ (t-t_k)^{1-q} \int_0^t (t-s)^{q-1} \|T_q(t-s)[f(s, x_{k,n}(s), u_{k,n}(s)) - \overline{f}_k(s)]\|ds.
\]

Hence,
\[
(t-t_k)^{1-q} x_{k,n}(t) \rightharpoonup (t-t_k)^{1-q} \overline{x}_{k}(t), \quad n \to \infty
\]
uniformly in $t \in (t_k, t_{k+1}]$. Therefore,
\[
x_{k,n}(\cdot) \rightharpoonup \overline{x}_{k}(\cdot) \quad \text{in } \text{PC}_{1-q}([0, t_{k+1}], X).
\]

On the other hand, by the definition of $u_{k,n}(\cdot)$ for $n$ large enough, we have
\[
u_{k,n}(t) \in U(t_{k,j}, (t_{k,j} - t_k)^{1-q} x_{k,n}(t_{k,j})) \subset U(0, (t-t_k)^{1-q} \overline{x}_k(t)),
\]
for all $t \in [t_{k,j}, t_{k,j+1}]$, $0 \leq j \leq n-1$.

Secondly, by (3.9) and the Mazur theorem again, let $q_{ll}^k \geq 0$ and $\sum_{i=1}^k q_{ll}^k = 1$ such that
\[
\psi_{l,k}(\cdot) = \sum_{i=1}^k q_{ll}^k f(\cdot, x_{k,i+1}(\cdot), u_{k,i+1}(\cdot)) \rightharpoonup \overline{f}_k(\cdot) \quad \text{in } L^p([0, t_{k+1}], X).
\]

Then, there is a subsequence of $\{\psi_{l,k}\}$, denoted by $\{\psi_{l,k}\}$ again, such that
\[
\psi_{l,k}(t) \rightharpoonup \overline{f}_k(t) \quad \text{in } X, \quad \text{a.e. } t \in [0, t_{k+1}].
\]

Hence, from (3.10) and (3.11), for $l$ large enough,
\[
\psi_{l,k}(t) \in \text{co}f(t, O_{\delta}(\overline{x}_k(t)), U(0, (t-t_k)^{1-q} \overline{x}_k(t))), \quad \text{a.e. } t \in (t_k, t_{k+1}].
\]

Thus, for any $\delta > 0$,
\[
\overline{f}_k(t) \in \text{co}f(t, O_{\delta}(\overline{x}_k(t)), U(0, (t-t_k)^{1-q} \overline{x}_k(t))), \quad \text{a.e. } t \in (t_k, t_{k+1}].
\]

By $H(6)$, we have
\[
\overline{f}_k(t) \in f(t, \overline{x}_k(t), U(t, (t-t_k)^{1-q} \overline{x}_k(t))), \quad \text{a.e. } t \in (t_k, t_{k+1}].
\]

Similarly, there exists a $\overline{u}_k \in V[0, T]$ such that
\[
\begin{cases}
\Pi_k(t) = \overline{u}_k(t), \quad t \in [0, t_k], \\
\Pi_k(t) \in U(t, (t-t_k)^{1-q} \overline{x}_k(t)), \quad t \in (t_k, t_{k+1}].
\end{cases}
\]

and
\[ \overline{f}_k(t) = f(t, \overline{x}_k(t), \overline{u}_k(t)), \quad t \in [0, t_{k+1}]. \]

Let
\[ \overline{x}(t) = \begin{cases} \overline{x}_0(t), & t \in [0, t_1], \\ \overline{x}_k(t), & t \in (t_k, t_{k+1}], k = 1, \ldots, m. \end{cases} \]
\[ \overline{u}(t) = \overline{u}_{m+1}(t). \]

Therefore, \((\overline{x}, \overline{u}) \in H[0, T]\). The proof is complete. \(\square\)

### 4. Existence of optimal control pairs

In this section, we consider the following optimal control problem. Problem \((\varphi)\): find a pair \((\bar{x}, \bar{u}) \in H[0, T]\) such that
\[ \varphi(\bar{x}, \bar{u}) \leq \varphi(x, u), \quad \text{for all } (x, u) \in H[0, T], \]
where \(\varphi(x, u) = \int_0^T f_0(t, x(t), u(t)) dt\).

We will make the following assumptions on \(f_0\):
\[(f_01)\] The functional \(f_0 : X \times V \to \mathbb{R} \cup \{\pm \infty\}\) is Borel measurable in \((t, x, u);\)
\[(f_02)\] \(f_0(t, \cdot, \cdot)\) is lower semicontinuous on \(X \times V\) for a.e. \(t \in J\) (i.e., for all \(x \in X, u \in V\), \((x_n) \subset X, z(u_n) \subset V\) such that \(x_n \to x \in X\) and \(u_n \to u \in V\), we have lim inf \(f_0(t, x_n, u_n) \geq f_0(t, x, u)\)) and there exists a constant \(M_1 > 0\) such that
\[ f_0(t, x, u) \geq -M_1, \quad (t, x, u) \in J \times X \times V. \]

For any \((t, x) \in (t_k, t_{k+1}] \times X (k = 0, 1, \ldots, m)\), we set
\[ \varepsilon_k(t, x) = \{ (t^0, z) \in (\mathbb{R} \times X): t^0 \geq f_0(t, x, u), \text{ and } u \in U(t, \varepsilon f_0 - 1, x) \}. \]

To obtain the existence result of optimal control pairs for Problem \((\varphi)\), we assume that \((H_e)\) for a.e. \(t \in (t_k, t_{k+1}] \subset (0, 1, \ldots, m)\), the map \(\varepsilon_k(t, \cdot) : X \to \mathbb{R} \times X \times V\) has the Cesari property, i.e.,
\[ \bigcap_{\delta > 0} \varepsilon_{k, \delta} = +\infty, \]
for all \(x \in X\).

**Remark 4.1.** By Lemma 2.10, condition \((H_e)\) is fulfilled if \(\varepsilon_k\) is u.s.c. with convex and closed values.

**Theorem 4.2.** Assume that \(H(1) - H(6)\). \((f_01), (f_02), (H_e)\) are satisfied. Then Problem \((\varphi)\) admits at least one optimal control pair.

**Proof.** Without considering the situation \(\inf\{\varphi(x, u) | (x, u) \in H[0, T]\} = +\infty\), we assume that \(\inf\{\varphi(x, u) | (x, u) \in H[0, T]\} = K < +\infty\). By \((f_02)\), we have \(\varphi(x, u) \geq K \geq -M_1T > -\infty\). So there exists a sequence \((x^n, u^n)_{n \geq 1} \subset H[0, T]\) such that
\[ \varphi(x^n, u^n) \to K. \]

By the proof of Theorem 3.5, without loss of generality, we obtain that
\[ u^n \rightharpoonup \bar{u} \ \text{ in } L^p(J, V), \quad f^n \to \overline{f} \ \text{ in } L^p(J, X) \]
and
\[ x^n \to \overline{x} \ \text{ in } PC_{1-q}(J, X), \]
where
\[ \overline{x}(t) = \begin{cases} t^{q-1} T_k(t) x_0 + \int_0^t (t - s)^{q-1} T_k(t - s) \overline{f}(s) ds, & t \in [0, t_1], \\ t^{q-1} T_k(t) x_0 + \sum_{i=1}^k T_k(t - t_i)(t - t_i)^{q-1} C_i(t, \overline{x}(t_i)) + \int_0^{t - t_k} (t - s)^{q-1} T_k(t - s) \overline{f}(s) ds, & t \in (t_k, t_{k+1}], k = 1, \ldots, m. \end{cases} \]

By Mazur Theorem again, let \(a_i, b_i \geq 0\) and \(\sum_{i \geq 1} a_i = \sum_{i \geq 1} b_i = 1\) such that
\[ \phi_i = \sum_{i \geq 1} a_i b_i \overline{f}^{i+1} \to \overline{u} \ \text{ in } L^p(J, V), \quad \psi_i = \sum_{i \geq 1} b_i f^{i+1} \to \overline{f} \ \text{ in } L^p(J, X). \]
Let
\[ \overline{y}_i(\cdot) = \sum_{k \geq 1} b_k f_0(\cdot, x^{k+1}(\cdot), u^{k+1}(\cdot)), \]
and
\[ \overline{f}_0(t) = \lim_{t \to +\infty} \overline{y}_i(t) \geq -M_1, \quad \text{a.e. } t \in J. \]
For any \( \delta > 0 \) and \( l \) large enough, by \((f_0.2)\) we have
\[
(\psi(t), \phi(t)) \in \varepsilon_l(\varepsilon, \mathcal{O}_k(X(t))), \quad \text{a.e. } t \in (t_k, t_{k+1}].
\]
By \((H_\varepsilon)\), we have
\[
(f_0(t), \bar{T}(t), \bar{u}(t)) \in \varepsilon_k(t, \bar{X}(t)), \quad \text{a.e. } t \in (t_k, t_{k+1}].
\]
i.e.,
\[
\begin{aligned}
& f_0(t) \geq f_0(t, \bar{X}(t), \bar{u}(t)), \quad t \in J, \\
& \bar{T}(t) = f(t, \bar{X}(t), \bar{u}(t)), \quad t \in J, \\
& \bar{u}(t) \in U(t, (t - t_k)\varepsilon - \bar{X}(t)), \quad (t_k, t_{k+1}].
\end{aligned}
\]
Therefore, \((\mathcal{X}, \mathcal{U}) \in H[0, T]\). By Fatou’s lemma, we obtain
\[
\int_0^T f_0(t) dt = \int_0^T \lim_{k \to +\infty} \psi_1(t) dt \leq \lim_{k \to +\infty} \int_0^T \psi_1(t) dt
\]
\[
= \lim_{k \to +\infty} \int_0^T \sum_{k \geq 1} b_{kl} f_0(t, x^{k+l}(t), u^{k+l}(t)) dt
\]
\[
= \lim_{k \to +\infty} \sum_{k \geq 1} b_{kl} \int_0^T f_0(t, x^{k+l}(t), u^{k+l}(t)) dt
\]
\[
= \lim_{k \to +\infty} \sum_{k \geq 1} b_{kl} \lim_{k \to +\infty} \int_0^T f_0(t, x^{k+l}(t), u^{k+l}(t)) dt
\]
= \( K \).

Then,
\[
K \leq \phi(\mathcal{X}, \mathcal{U}) = \int_0^T f_0(t, \bar{X}(t), \bar{u}(t)) dt \leq K,
\]
i.e.,
\[
\int_0^T f_0(t, \bar{X}(t), \bar{u}(t)) dt = K = \inf_{(x, u) \in H[0, T]} \phi(x, u).
\]
Hence, \((\mathcal{X}, \mathcal{U})\) is an optimal pair. The proof is complete. \( \square \)

5. An example

Consider the following initial-boundary value problem of fractional parabolic control system with Riemann–Liouville fractional derivatives:
\[
\begin{cases}
D_t^\alpha x(t, y) = \frac{\partial^2}{\partial y^2} x(t, y) + f(t, x(t, y), u(t)), & t \in [0, 1] \setminus \{1\}, \ y \in [0, \pi], \\
u(t) \in U\left(t, \left(1 - \frac{1}{2}\right)^{-q} x(t, y)\right), & \text{a.e. } t \in [0, 1] \setminus \{1\}, \ y \in [0, \pi], \\
\Delta I_1^{\alpha} x(t, y) = \frac{|x(t, y)|}{2 + |x(t, y)|}, & \ y \in [0, \pi], \\
x(t, 0) = x(t, \pi) = 0, & t \in J = [0, 1], \\
I_1^{\alpha} x(t, y)|_{t=0} = x_0(y), & t \in [0, 1], \ y \in [0, \pi].
\end{cases}
\]

Take \( X = V = L^2([0, \pi]) \) and the operator \( A: D(A) \subset X \to X \) is defined by
\[
A x = x'\prime',
\]
\[
D(A) = \{ x \in X : x, x' \text{ are absolutely continuous, } x'' \in X, x(0) = x(\pi) = 0 \}.
\]
Then, \( A \) can be written as
\[
A x = -\sum_{n=1}^{\infty} n^2 (x, x_n)x_n, \quad x \in D(A),
\]
where \( x_n(x) = \sqrt{2/\pi} \sin nx \) \((n = 1, 2, \ldots)\) is an orthonormal basis of \( X \). It is well known that \( A \) is the infinitesimal generator of a compact semigroup \( T(t)(t > 0) \) in \( X \) given by
\[
T(t)x = \sum_{n=1}^{\infty} \exp\left(-n^2 t\right) (x, x_n)x_n, \quad x \in X, \quad \text{and } \|T(t)\| \leq e^{-1} < 1 = M.
\]
Let \( U : [0, 1] \times X \rightarrow V \) be upper semicontinuous with closed values, \( f : [0, 1] \times X \times V \rightarrow X \) satisfy hypotheses \( H(2), H(3), H(6) \) (see [14] for more details). Then our main results could be applied to problem (5.1).

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References