Why Brouwer was justified in his objection to Hilbert’s unqualified interpretation of quantification

Bhupinder Singh Anand

This paper appeared—minus Appendix B—in the proceedings of the 2008 International Conference on Foundations of Computer Science, July 14-17, 2008, Las Vegas, USA.

(Do not cite this update of September 8, 2010; it may differ in formatting and other inessential details from the formal paper.)

Abstract

We define a finitary model of first-order Peano Arithmetic in which quantification is interpreted constructively in terms of Turing-computability, and show that it is inconsistent with the standard interpretation of PA.

1 Hilbert’s interpretation of quantification

Hilbert interpreted quantification in terms of his $\varepsilon$-function as follows [Hi27]:

IV. The logical $\varepsilon$-axiom

13. $A(a)$ $\rightarrow$ $A(\varepsilon(A))$

Here $\varepsilon(A)$ stands for an object of which the proposition $A(a)$ certainly holds if it holds of any object at all; let us call $\varepsilon$ the logical $\varepsilon$-function.

1. By means of $\varepsilon$, “all” and “there exists” can be defined, namely, as follows:

(i) $(\forall a)A(a) \leftrightarrow A(\varepsilon(\neg A))$

(ii) $(\exists a)A(a) \leftrightarrow A(\varepsilon(A))$

On the basis of this definition the $\varepsilon$-axiom IV(13) yields the logical relations that hold for the universal and the existential quantifier, such as:

$(\forall a)A(a) \rightarrow A(b)$ . . . (Aristotle’s dictum),

Keywords: Peano Arithmetic, Turing-computability, finitary model, soundness, $\omega$-consistency
and:
\[ \neg(\forall a)A(a) \rightarrow (\exists a)(\neg A(a)) \ldots \text{(principle of excluded middle).} \]

Thus, Hilbert’s interpretation of universal quantification — defined in (i) — is that the sentence \((\forall x)F(x)\) holds (under a consistent interpretation \(I\)) if, and only if, \(F(a)\) holds whenever \(\neg F(a)\) holds for any given \(a\) (in \(I\)); hence \(\neg F(a)\) does not hold for any \(a\) (since \(I\) is consistent), and so \(F(a)\) holds for any given \(a\) (in \(I\)).

Further, Hilbert’s interpretation of existential quantification — defined in (ii) — is that \((\exists x)F(x)\) holds (in \(I\)) if, and only if, \(F(a)\) holds for some \(a\) (in \(I\)).

Brouwer’s objection to such an unqualified interpretation of the existential quantifier was that, for the interpretation to be considered sound when the domain of the quantifiers under an interpretation is infinite, the decidability of the quantification under the interpretation must be constructively verifiable in some intuitively and mathematically acceptable sense of the term “constructive” [Br08].

Two questions arise:

(a) Is Brouwer’s objection relevant today?
(b) If so, can we interpret quantification ‘constructively’?

2 The standard interpretation \(M\) of PA

We consider the structure \([N]\), defined as \(\{N \text{ (the set of natural numbers)}; = \text{(equality)}; S \text{ (the successor function)}; + \text{(the addition function)}; \ast \text{(the product function)}; 0 \text{ (the null element)}\}\), that serves for a definition of today’s standard interpretation, say \(M\), of first-order Peano Arithmetic (PA).

Now, if \([(\forall x)F(x)]\) and \([(\exists x)F(x)]\) are PA-formulas, and the relation \(F(x)\) is the interpretation in \(M\) of the PA-formula \([F(x)]\), then, in current literature:

(1a) \([(\forall x)F(x)]\) is defined as true in \(M\) if, and only if, for any given natural number \(n\), \(F(n)\) holds in \(M\);
(1b) \([(\exists x)F(x)]\) is an abbreviation of \([\neg(\forall x)\neg F(x)]\), and is defined as true in \(M\) if, and only if, it is not the case that, for any given natural number \(n\), \(\neg F(n)\) holds in \(M\);
(1c) \(F(n)\) holds in \(M\) for some natural number \(n\) if, and only if, it is not the case that, for any given natural number \(n\), \(\neg F(n)\) holds in \(M\).

Since (1a), (1b) and (1c) together interpret \([(\forall x)F(x)]\) and \([(\exists x)F(x)]\) in \(M\) as intended by Hilbert’s \(\varepsilon\)-function, they attract Brouwer’s objection. This answers question (a).

3 A finitary model \(B\) of PA

Clearly, the specific target of Brouwer’s objection is (1c), which appeals to Platonically non-constructive, rather than intuitively constructive, plausibility.

We can thus re-phrase question (b) more specifically: Can we define an interpretation of PA over \([N]\) that does not appeal to (1c)?
Now, it follows from Turing’s seminal 1936 paper on computable numbers that every quantifier-free arithmetical function (or relation, when interpreted as a Boolean function) $F$ defines a Turing-machine $TM_F$ (cf. [Tu36], pp. 138-139). We can thus define another interpretation $B$ over the structure $[N]$ (cf. [An10], Section 5), where:

1. $[(\forall x)F(x)]$ is defined as true in $B$ if, and only if, the Turing-machine $TM_F$ computes $F(n)$ as always true (i.e., as true for any given natural number $n$) in $B$;
2. $[(\exists x)F(x)]$ is an abbreviation of $[\neg(\forall x)\neg F(x)]$, and is defined as true in $B$ if, and only if, it is not the case that the Turing-machine $TM_F$ computes $F(n)$ as always false (i.e., as false for any given natural number $n$) in $B$.

$B$ is a finitary model of PA since - when interpreted suitably - all theorems of first-order PA are constructively true in $B$ (cf. [An10], Section 6, Lemma 27).

This answers question (b).

4 Are both interpretations of PA over the structure $[N]$ sound?

The structure $[N]$ can thus be used to define both the standard interpretation $M$ and a finitary model $B$ for PA.

However, in the finitary model, from the PA-provability of $[\neg(\forall x)F(x)]$, we may only conclude that $TM_F$ does not compute $F(n)$ as always true in $B$.

We may not conclude further that $TM_F$ must compute $F(n)$ as false in $B$ for some natural number $n$, since $F(x)$ may be a Halting-type of function that is not Turing-computable (cf. [Tu36], pp. 132).

In other words, we may not conclude from the PA-provability of $[\neg(\forall x)F(x)]$ that $F(n)$ does not hold in $B$ for some natural number $n$.

The question arises: Are both the interpretations $M$ and $B$ of PA over the structure $[N]$ sound?

5 PA is $\omega$-inconsistent

Now, Gödel has constructed ([Go31], pp. 25(1)) an arithmetical formula, $[R(x)]$, such that, if PA is assumed simply consistent, then $[R(n)]$ is PA-provable for any given numeral $[n]$, but $[(\forall x)R(x)]$ is not PA-provable.

Further, he showed that ([Go31], pp. 26(2)), if PA is additionally assumed $\omega$-consistent, then $[\neg(\forall x)R(x)]$ too is not PA-provable.

Gödel defined ([Go31], pp. 23) PA as $\omega$-consistent if, and only if, there is no PA-formula $[F(x)]$ for which:

1. $[\neg(\forall x)F(x)]$ is PA-provable, and
2. $[F(n)]$ is PA-provable for any given numeral $[n]$ of PA.

In the general case, $TM_F$ is defined by the quantifier-free expression in the prenex normal form of $F$. 
However, as we show in Appendix A, if we apply an extension of the standard Deduction Theorem of first-order logic to Gödel’s reasoning, then it follows that \[\neg (\forall x) R(x)\] is PA-provable, and so PA is \(\omega\)-inconsistent!

### 6 The interpretation \(M\) of PA over the structure \([N]\) is not sound

Now, \(R(n)\) holds for any given natural number \(n\), since Gödel has defined \(R(x)\) ([Go31], pp. 24) such that \(R(n)\) is instantiationally equivalent to a primitive recursive relation \(Q(n)\) which is computable as true in \(B\) for any given natural number \(n\) by the Turing-machine \(TM_Q\).

It follows that we cannot admit the standard (Hilbertian) interpretation of \(\neg (\forall x) R(x)\) in \(M\) as:

\[R(n)\] is false for some natural number \(n\).

In other words, the interpretation \(M\) of PA over the structure \([N]\) is not sound.

However, we can interpret \(\neg (\forall x) R(x)\) in \(B\) as:

It is not the case that the Turing-machine \(TM_R\) computes \(R(n)\) as true in \(B\) for any given natural number \(n\).

Moreover, the \(\omega\)-inconsistent PA is consistent with the finitary interpretation of quantification, as in (2a) and (2b) since the interpretation \(B\) of PA over the structure \([N]\) is sound ([An10], Section 6, Theorem 4).

### 7 Why the interpretation \(M\) of PA over \([N]\) is not sound

The reason why the interpretation \(M\) of PA over the structure \([N]\) is not sound lies in the fact that, whereas (1b) and (2b) preserve the logical properties of formal PA-negation under interpretation in \(M\) and \(B\) respectively, the further non-constructive inference in (1c) from (1b) — to the effect that \(F(n)\) must hold in \(M\) for some natural number \(n\) — does not, and is the one objected to by Brouwer [Br08].

### 8 Conclusion

Thus the interpretation \(M\) is not a model of PA, and Brouwer was justified in his objection to Hilbert’s unqualified interpretation of quantification.

It is implicit in the objection that, if we assume only simple consistency for Hilbert’s system, then we cannot unconditionally define:

\[[(\exists x) F(x)]\] is true in \(M\) if, and only if, \(F(n)\) holds for some natural number \(n\) in \(M\).

The above conclusion also follows independently of the above argument, since, if \([(\exists x) F(x)]\) is true in \(M\) if, and only if, \(F(n)\) holds for some natural
implies that, in any extension $K \vdash$ which $M$ 

Hence, the deduction $'A \rightarrow B'$ is vacuously true in $M$. 

(ii) If, however, $[A]$ is true under an interpretation $M$ of $K$, then the sequence $[B_1], [B_2], \ldots, [B_n]$, interprets as the deduction $'B$ follows from $A'$ in $M$. Hence $[B]$ is true in $M$, and the deduction $'A \rightarrow B'$ is true in $M$.

In other words, we cannot have $[A]$ true, and $[B]$ false, under an interpretation $M$ of $K$, as this would imply that there is some extension $K'$ of $K$ in which $\vdash_{K'} [A]$, but not $\vdash_{K'} [B]$; this would contradict our hypothesis, which implies that, in any extension $K'$ of $K$ in which we have $\vdash_{K'} [A]$, the sequence $[B_1], [B_2], \ldots, [B_n]$ yields $\vdash_{K'} [B]$.

Hence, the deduction $'A \rightarrow B'$, is true in all models of $K$. 

9 Appendix A

We show that classical theory does not admit an $\omega$-consistent first-order Peano Arithmetic (PA).

Author’s note (September 8, 2010): Wrong deduction. The above conclusion does not follow from the argument detailed below. However the conclusion does follow from the arguments in [An10, Section 7, Corollary 9].

Now, a first order theory $K$ is $\omega$-consistent if, and only if, for any well-formed formula $[F(x)]$ of $K$, if $\vdash_K [F(n)]$ for every numeral $[n]$, then it is not the case that $\vdash_K [\neg(\forall x)F(x)]$.

Here $\{[A]\} \vdash_K [B]$ interprets as: There is a finite deduction sequence of $K$-formulas, $[B_1], [B_2], \ldots, [B_n]$, such that $[B_n]$ is $[B]$ and, for $0 < i < n$, $[B_i]$ is either in the set of $K$-formulas $\{[A]\}$, or $[B_i]$ is an axiom of $K$, or $[B_i]$ is a consequence of the axioms of $K$ and the formulas preceding it in the sequence by the rules of inference of $K$.

Further, for any first order theory $K$, we have the standard:

**Deduction Theorem:** If $\{[T]\}$ is a set of well-formed formulas of an arbitrary first order theory $K$, and if $[A]$ is a closed well-formed formula of $K$, and if $\{\{[T]\} \cup [A]\} \vdash_K [B]$, then $\{[T]\} \vdash_K [A \rightarrow B]$.

We prove, now, that:

**Theorem 1:** If $[A]$ is a closed well-formed formula of $K$, and if $\vdash_K [B]$ when we assume $\vdash_K [A]$, then $\vdash_K [A \rightarrow B]$.

**Proof:** (i) The case $\vdash_K [B]$ is straightforward, since the deduction $'A \rightarrow B'$ is, then, the interpretation of a well-formed $M$-formula which is true in any interpretation $M$ of $K$.

(ii) If not $\vdash_K [B]$, then, if $\vdash_K [B]$ when we assume $\vdash_K [A]$, then, by definition, there is a sequence $[B_1], [B_2], \ldots, [B_n]$, of well-formed $K$-formulas such that $[B_1]$ is $[A]$, $[B_n]$ is $[B]$ and, for each $i > 1$, either $[B_i]$ is an axiom of $K$ or $[B_i]$ is a direct consequence by some rules of inference of $K$ of the axioms of $K$ and some of the preceding well-formed formulas in the sequence.

(iia) If, now, $[A]$ is false under an interpretation $M$ of $K$, then the deduction $'A \rightarrow B'$ is vacuously true in $M$.

(iib) If, however, $[A]$ is true under an interpretation $M$ of $K$, then the sequence $[B_1], [B_2], \ldots, [B_n]$, interprets as the deduction $'B$ follows from $A'$ in $M$. Hence $[B]$ is true in $M$, and the deduction $'A \rightarrow B'$ is true in $M$.
By a consequence of Gödel’s Completeness Theorem for an arbitrary first order theory, it follows that \(\vdash_K [A \rightarrow B]. \square\)

Now, Gödel [Go31] defines a PA-proposition, \([\forall x)\!R(x)\)], such that if the Gödel-number of \([\forall x)\!R(x)\)] is \(17 Genr\), and if \([\forall x)\!R(x)\)] is PA-provable, then the PA-formula whose Gödel-number is \(Neg(17 Genr)\) is also PA-provable if PA is assumed simply consistent ([Go31], p25(1)).

i.e., if \(\vdash_P [\forall x)\!R(x)]\), then \(\vdash_P [\neg(\forall x)\!R(x)]\).

By applying Theorem 1, it follows that:
\[\vdash_P [\forall x)\!R(x) \rightarrow \neg(\forall x)\!R(x)]\]

Since:
\[\vdash_P [(A \rightarrow \neg A) \rightarrow A]\]

we conclude, by Modus Ponens, that:
\[\vdash_P [\neg(\forall x)\!R(x)]\]

Now, Gödel also proved ([Go31], p26(1)) that, if PA is assumed simply consistent, then \(\vdash_P [R(n)]\) for any, given, natural number \(n\).

Ergo, Theorem 1 implies that PA is \(\omega\)-inconsistent.

Author’s note (September 8, 2010): Wrong deduction. The above conclusion does not follow from the argument as detailed above. However the conclusion does follow from the arguments in [An10], Section 7, Corollary 9.

10 Appendix B

As part of his Program, Hilbert [Hi30] proposed an \(\omega\)-Rule as a finitary means of extending PA to a possible completion.

\(\omega\)-Rule: If it is proved that the formula \([F(n)]\) is a true numerical formula \(\text{[under interpretation]}\) for each given numeral \([n]\), then the formula \([\forall x)\!F(x)\] may be admitted as an initial formula.

However, Gödel’s Theorem VI ([Go31], p24) shows that it follows from Hilbert’s \(\omega\)-Rule that, if PA is consistent, then it is \(\omega\)-consistent and incomplete!

It now follows that, if PA is simply consistent, then it is \(\omega\)-inconsistent and Hilbert’s \(\omega\)-Rule cannot be applied to PA!

References


