Duality and Minors of Secondary Polyhedra

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Using Minkowski integration, we define the secondary polyhedron of a vector configuration $\mathcal{A}$ and study its behavior under the matroidal operations of duality, deletion, and contraction. A main tool is the identification of the regular polyhedral subdivisions of $\mathcal{A}$ with the cells in the dual chamber complex. As an application we construct a non-regular triangulation of a cyclic polytope. © 1993 Academic Press, Inc.

1. Introduction

Given a spanning set $\mathcal{A} = \{a_1, \ldots, a_n\}$ of non-zero vectors in $\mathbb{R}^d$, we are interested in the $(n-d)$-dimensional secondary polyhedron $\Sigma(\mathcal{A})$ whose faces correspond to the regular polyhedral subdivisions of the $(d-1)$-dimensional spherical polytope $P(\mathcal{A}) := \text{pos}(\mathcal{A}) \cap S^{d-1}$. The spherical polytope is thought of as a $(d-1)$-polytope in the usual affine sense whenever $\text{pos}(\mathcal{A})$ is a pointed cone, and in this case $\Sigma(\mathcal{A})$ is bounded and is normally equivalent to the secondary polytope defined in [5] (see also [2, 3, 6, 8]). On the other hand, if $\text{pos}(\mathcal{A}) = \mathbb{R}^d$, then $\Sigma(\mathcal{A})$ is unbounded and its vertices correspond to the regular triangulations of the $(d-1)$-sphere with vertices on the rays of $\mathcal{A}$. Our approach extends the work of Oda and Park [10], who have constructed the normal fan of $\Sigma(\mathcal{A})$ by means of linear transforms.
Each \( a_i \in \mathcal{A} \) gives rise to two \textit{minors}. The minor by \textit{deletion} of \( a_i \) is the configuration \( \mathcal{A} \setminus a_i = \{ a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n \} \) in \( \mathbb{R}^d \). The minor by \textit{contraction} of \( a_i \) is the configuration \( \mathcal{A} / a_i = \{ \pi_{a_i}(a_1), \ldots, \pi_{a_i}(a_{i-1}), \pi_{a_i}(a_{i+1}), \ldots, \pi_{a_i}(a_n) \} \) in \( \mathbb{R}^{d-1} \), where \( \pi_{a_i} : \mathbb{R}^d \rightarrow \mathbb{R}^{d-1} \) is any epimorphism with kernel \( \text{span}(a_i) \). It is our objective to relate the secondary polyhedron \( \Sigma(\mathcal{A}) \) of \( \mathcal{A} \) to the secondary polyhedra \( \Sigma(\mathcal{A} \setminus a_i) \) and \( \Sigma(\mathcal{A} / a_i) \) obtained by deletion and contraction of any \( a_i \in \mathcal{A} \).

In Section 2 we use an integral representation as in [3] to define \( \Sigma(\mathcal{A}) \), discuss its combinatorial interpretation in terms of polyhedral subdivisions, and give formulas for its vertices and the extreme rays of its recession cone. We use this integral representation to give a description of \( \Sigma(\mathcal{A} \setminus a_i) \) as a facet of \( \Sigma(\mathcal{A}) \). Section 3 is concerned with the behavior of the secondary polyhedron \( \Sigma(\mathcal{A}) \) under duality and under minors by contraction. We show that the boundary complex of \( \Sigma(\mathcal{A}) \) is anti-isomorphic to the \textit{chamber complex} of a linear transform \( \mathcal{B} \) of \( \mathcal{A} \), and we show that \( \Sigma(\mathcal{A} / a_i) \) either is a Minkowski summand of \( \Sigma(\mathcal{A}) \) or can be obtained from one by removing a single facet. In Section 4 we answer a question raised by Kapranov and Voevodsky [8] by presenting an example of a non-regular triangulation of a cyclic polytope.

2. The Secondary Polyhedron and Deletions

Let \( \mathcal{A} = \{ a_1, \ldots, a_n \} \) be a set of \( n \) non-zero vectors spanning \( \mathbb{R}^d \). The polyhedral cone \( \text{pos}(\mathcal{A}) \) is the image of the non-negative orthant \( \mathbb{R}^d_+ = \text{pos}(\{ e_1, \ldots, e_n \}) \) in \( n \)-space under the linear map \( \pi : e_i \mapsto a_i \). The fiber of a point \( x \in \text{pos}(\mathcal{A}) \) is the polyhedron

\[
\pi^{-1}(x) = \{ (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n_+ : \lambda_1 a_1 + \cdots + \lambda_n a_n = x \}
\]

consisting of all positive representations of \( x \) with respect to \( \mathcal{A} \). Note that each \( k \)-face of the spherical polytope \( P(\mathcal{A}) = \text{pos}(\mathcal{A}) \cap S^{d-1} \) is of the form \( F \cap S^{d-1} \), where \( F \) is a \( (k+1) \)-face of \( \text{pos}(\mathcal{A}) \). We define the \textit{secondary polyhedron} of \( \mathcal{A} \) to be the Minkowski integral

\[
\Sigma(\mathcal{A}) := \int_{P(\mathcal{A})} \pi^{-1}(x) \, dx
\]

with respect to the rotation invariant probability measure on the unit sphere \( S^{d-1} \). This means that \( \Sigma(\mathcal{A}) \) is the set of all points \( \int_{P(\mathcal{A})} \gamma(x) \, dx \) in \( \mathbb{R}^n \), where \( \gamma : P(\mathcal{A}) \rightarrow \mathbb{R}^n_+ \) is a measurable function such that \( \pi \circ \gamma \) is the identity (see [3]).
We will now derive a description of the recession cone and the face lattice of $\Sigma(\mathscr{A})$. A circuit of $\mathscr{A}$ is any non-zero vector of the form
\[
C_v := \sum_{i=1}^{d+1} (-1)^i \det(a_{v_1}, ..., a_{v_{i-1}}, a_{v_{i+1}}, ..., a_{v_{d+1}}) e_{v_i},
\] (2.3)
where $v$ is a $(d+1)$-subset of $\{1, 2, ..., n\}$. We call $C_v$ a positive circuit if $C_v \in \mathbb{R}^d_+$. 

Given any basis $\tau = \{a_{v_1}, ..., a_{v_{d+1}}\}$ of $\mathscr{A}$, we define $L_{\tau,i}$ to be the unique linear functional on $\mathbb{R}^d$ with $L_{\tau,i}(a_{v_i}) = \delta_{vi}$ (Kronecker delta). For $x \in \text{pos}(\mathscr{A})$ let $\Omega_x$ denote the set of all bases $\tau$ of $\mathscr{A}$ with $x \in \text{pos}(\tau)$. The following straightforward lemma shows that all fibers have the same recession cone.

**Lemma 2.1.** (a) The zero fiber $\pi^{-1}(0)$ equals the positive hull of all positive circuits $C_v$ of $\mathscr{A}$.

(b) For all $x \in \text{pos}(\mathscr{A})$ we have $\pi^{-1}(x) = \pi^{-1}(0) + \text{conv} \{\sum_{i=1}^{d} L_{\tau,i}(x) \cdot e_{v_i} \mid \tau \in \Omega_x\}$.

Note that $\text{pos}(\mathscr{A})$ is pointed if and only if $\pi^{-1}(0) = \{0\}$. Since the integral in (2.2) is additive with respect to the Minkowski sum in Lemma 2.1(b), we get the following result.

**Corollary 2.2.** The secondary polyhedron $\Sigma(A)$ has the recession cone $\pi^{-1}(0)$. Thus $\Sigma(A)$ is a polytope if and only if $\text{pos}(\mathscr{A})$ is pointed.

A subdivision of $\mathscr{A}$ is a collection $\Pi$ of subsets of $\mathscr{A}$ such that the polyhedral cones $\{\text{pos}(\sigma) \mid \sigma \in \Pi\}$ form a fan (i.e., a complex of cones) which covers $\text{pos}(\mathscr{A})$. Equivalently, $\Pi$ can be viewed as a subdivision of $P(\mathscr{A})$ into spherical polytopes $\text{pos}(\sigma) \cap S^{d-1}$. A triangulation of $\mathscr{A}$ is a subdivision into simplicial cones (respectively, spherical simplices). Given polyhedral subdivisions $\Pi_1$ and $\Pi_2$, we say $\Pi_1$ refines $\Pi_2$, written $\Pi_2 \prec \Pi_1$, if every face of $\Pi_2$ is a subset of some face of $\Pi_1$.

For a polyhedron $Q \subset \mathbb{R}^n$ and a vector $\psi \in \mathbb{R}^n$, we say that $Q$ is bounded in direction $\psi$ if the linear functional $\langle \psi, \cdot \rangle$ attains a finite minimum over $Q$. In this case $\psi$ defines a proper face $Q^\psi := \{y \in Q \mid \langle \psi, y \rangle \leq \langle \psi, Q \rangle\}$, having an inward pointing normal $\psi$. We call $\psi \in \mathbb{R}^n$ feasible if $\Sigma(\mathscr{A})$ is bounded in direction $\psi$. By Corollary 2.2, the cone of feasible vectors is the polar of $\pi^{-1}(0)$. Note also that $\psi = (\psi_1, ..., \psi_n)$ is feasible if and only if $(0, ..., 0, -1)$ does not lie in
\[
\text{pos}\{(a_1, \psi_1), (a_2, \psi_2), ..., (a_n, \psi_n)\} \subseteq \mathbb{R}^{d+1}.
\] (2.4)

In this case the projection of the “bottom” faces of the polyhedron in (2.4)
onto the first $d$ coordinates defines a subdivision $\Pi(\psi)$ of $\mathcal{A}$. This method of obtaining subdivisions goes back to Walkup and Wets [12]. We call a subdivision $\Pi$ regular if it arises in this way. If $\Pi$ is a regular subdivision, then the set

$$\mathcal{F}_\mathcal{A}(\Pi) := \{ \psi \in \mathbb{R}^n | \Pi(\psi) = \Pi \}$$

(2.5)

is a non-empty relatively open convex polyhedral cone. These cones define a polyhedral fan $\mathcal{F}_\mathcal{A}$, which we call the secondary fan. The face lattice of the secondary fan is isomorphic via (2.5) to the poset of regular subdivisions, ordered by refinement. Here the maximal cells of $\mathcal{F}_\mathcal{A}$ correspond to regular triangulations of $\mathcal{A}$.

We remark that if $\text{pos}(\mathcal{A})$ fails to be pointed, then already four vectors in the plane can have non-regular subdivisions. For example, $\Pi = \{ \{1, 2\}, \{1, 4\}, \{2, 3, 4\} \}$ is a non-regular subdivision of $\mathcal{A} = \{a_1, a_2, a_3, a_4\} = \{(1, 1), (1, 0), (1, -1), (-1, 0)\} \subset \mathbb{R}^2$.

Every triangulation $\Delta$ of $\mathcal{A}$ gives rise to a piecewise linear section $\gamma : \text{pos}(\mathcal{A}) \to \mathbb{R}_+^n$, via $\gamma_\Delta(x) := \sum_{i=1}^d L_{c_i}(x) e_i$, whenever $x \in \tau \in \Delta$. If $\Delta$ is a regular triangulation of $\mathcal{A}$, then any vector $\psi \in \mathcal{F}_\mathcal{A}(\Delta)$ satisfies $\pi^{-1}(\psi') = \{\gamma_\Delta(x)\}$ for all points $x$ in the interior of $P(\mathcal{A})$. By [3, Proposition 1.2], integration over $P(\mathcal{A})$ yields the following result.

**Proposition 2.3.** For any regular triangulation $\Delta$ the vector $\phi_\Delta := \int_{\text{pos}(\mathcal{A})} \gamma_\Delta(x) \, dx$ is a vertex of $\Sigma(\mathcal{A})$, and all vertices are of this form. The inner normal cone of $\Sigma(\mathcal{A})$ at $\phi_\Delta$ equals $\mathcal{F}_\mathcal{A}(\Delta)$.

Proposition 2.3 states in other words that the maximal cells of the secondary fan are precisely the maximal cells of the normal fan $\mathcal{N}(\Sigma(\mathcal{A}))$ of the secondary polyhedron. Since all cells of a polyhedral cell complex are obtained by intersecting closures of maximal cells, this implies that

$$\mathcal{N}(\Sigma(\mathcal{A})) = \mathcal{F}_\mathcal{A}.$$

**Theorem 2.4.** The face lattice of $\Sigma(\mathcal{A})$ is antiisomorphic to the poset of regular subdivisions of $\mathcal{A}$, ordered by refinement.

We now prove the following direct geometric description for the secondary polyhedron by deletion $\Sigma(\mathcal{A} \setminus a_i)$.

**Theorem 2.5.** The face $\Sigma(\mathcal{A})^{\bullet}_i$ of the secondary polyhedron in the direction of the $i$th unit vector is a facet, which is a translate of the secondary polyhedron $\Sigma(\mathcal{A} \setminus a_i)$ by deletion.
Proof. The \((d-1)\)-polyhedron \(\Sigma(\mathcal{A} \setminus a_i)\) is defined as \(\int_{P(\mathcal{A} \setminus a_i)} \theta_i^{-1}(x) \, dx\), where

\[
\theta_i(\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_n) \mapsto \lambda_i a_i + \cdots + \lambda_{i-1} a_{i-1} + \lambda_{i+1} a_{i+1} + \cdots + \lambda_n a_n.
\] (2.6)

Each fiber \(\theta_i^{-1}(x)\) in \(\mathbb{R}^{n-1}\) of a point \(x \in \text{pos}(\mathcal{A} \setminus a_i)\) is a subset of \(\pi^{-1}(x)\) via the \(i\)th coordinate inclusion of \(\mathbb{R}^{n-1}\) in \(\mathbb{R}^n\). More precisely, \(\theta_i^{-1}(x) = \pi^{-1}(x)^{e_i}\) is the face of \(\pi^{-1}(x)\) on which the \(i\)-th coordinate function is zero and hence minimal. This face is a facet if \(x\) lies in the interior of \(\text{pos}(\mathcal{A} \setminus a_i)\). This implies

\[
\int_{P(\mathcal{A} \setminus a_i)} \theta_i^{-1}(x) \, dx = \int_{P(\mathcal{A} \setminus a_i)} \pi^{-1}(x)^{e_i} \, dx = \left(\int_{P(\mathcal{A} \setminus a_i)} \pi^{-1}(x) \, dx\right)^{e_i}.
\] (2.7)

If \(a_i\) is contained in \(\text{pos}(\mathcal{A} \setminus a_i)\), then the right-hand integral equals \(\Sigma(\mathcal{A})\) and we are done. Let us now assume that \(a_i \notin \text{pos}(\mathcal{A} \setminus a_i)\). Pick a sufficiently generic point \(x \in \text{pos}(\mathcal{A} \setminus \text{pos}(\mathcal{A} \setminus a_i))\). Then there exists a unique simplicial \((d-1)\)-cone \(\text{pos}(a_1, \ldots, a_{d-1})\) which spans a facet of \(\text{pos}(\mathcal{A} \setminus a_i)\) and such that \(x \in \text{pos}(a_1, a_{j_1}, \ldots, a_{j_{d-1}})\); say, \(x = \mu_1 a_1 + \mu_{j_1} a_{j_1} + \cdots + \mu_{j_{d-1}} a_{j_{d-1}}\), where \(\mu \in \mathbb{R}^n\) and \(\mu_k = 0\) for \(k \notin \{i, j_1, \ldots, j_{d-1}\}\). The point \(\mu\) gives the unique minimum of the \(i\)th coordinate function \(e_i\) over \(\pi^{-1}(x)\), i.e., \(\pi^{-1}(x)^{e_i} = \{\mu\}\). We have shown that the face \(\pi^{-1}(x)^{e_i}\) is a vertex of \(\pi^{-1}(x)\) for almost all \(x \in P(\mathcal{A}) \setminus P(\mathcal{A} \setminus a_i)\). Thus \(\int_{P(\mathcal{A} \setminus a_i)} \pi^{-1}(x)^{e_i} \, dx\) is a point, which completes the proof of Theorem 2.5. □

3. Duality and Contractions

Let \(\mathcal{A} = \{a_1, \ldots, a_n\} \subset \mathbb{R}^d\) as before. The circuit space of \(\mathcal{A}\) is the \((n-d)\)-dimensional subspace \(\mathcal{C}(\mathcal{A})\) of \(\mathbb{R}^n\) spanned by all circuits \(C_i\), as in (2.3). Equivalently,

\[
\mathcal{C}(\mathcal{A}) = \left\{ \beta = (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n \left| \sum_{i=1}^n \beta_i a_i = 0 \right. \right\}.
\] (3.1)

The cocircuit space of \(\mathcal{A}\) is the \(d\)-dimensional subspace \(\mathcal{D}(\mathcal{A})\) of \(\mathbb{R}^n\) spanned by the vectors

\[
D_i := \sum_{j=1}^n \text{det}(a_{j_1}, \ldots, a_{j_{d-1}}, a_i) e_j,
\] (3.2)
called cocircuits of $\mathcal{A}$, where $\mu$ ranges over all $(d-1)$-subsets of $\{1, \ldots, n\}$. Equivalently,

$$D(\mathcal{A}) = \{ (\phi(a_1), \ldots, \phi(a_n)) \in \mathbb{R}^n | \phi: \mathbb{R}^d \to \mathbb{R} \text{ any linear functional} \}. \quad (3.3)$$

Note that the vector spaces $C(\mathcal{A})$ and $D(\mathcal{A})$ are orthogonal complements in $\mathbb{R}^n$.

A spanning subset $\mathcal{B} = \{ b_1, \ldots, b_n \}$ of $\mathbb{R}^{n-d}$ is called a linear transform of $\mathcal{A}$ provided $C(\mathcal{A}) = D(\mathcal{B})$. This is equivalent to $D(\mathcal{A}) = C(\mathcal{B})$ and thus to $\mathcal{A}$ being a linear transform of $\mathcal{B}$. In this case the oriented matroids associated with $\mathcal{A}$ and $\mathcal{B}$ are dual. If we define $I_{\mathcal{A}} = \{ i | a_i \in \text{pos}(\mathcal{A}) \}$, then it is a consequence of oriented matroid duality that the sets $I_{\mathcal{A}}$ and $I_{\mathcal{B}}$ partition $\{1, \ldots, n\}$. Thus, for example, pos($\mathcal{A}$) is pointed (i.e., $I_{\mathcal{A}} = \emptyset$) if and only if pos($\mathcal{B}$) is $\mathbb{R}^{n-d}$ (i.e., $I_{\mathcal{B}} = \{1, \ldots, n\}$). See [4, 9, 11].

The chamber complex $\Gamma(\mathcal{A})$ of $\mathcal{A}$ is defined to be the coarsest polyhedral complex that covers pos($\mathcal{A}$) and that refines all triangulations of $\mathcal{A}$. Given $x_0 \in \text{pos}(\mathcal{A})$, the unique (relatively open) cell of $\Gamma(\mathcal{A})$ containing $x_0$ is

$$\Gamma(\mathcal{A}, x_0) = \bigcap \{ \text{rel int pos } \mathcal{A'} | \mathcal{A'} \subseteq \mathcal{A}, x_0 \in \text{rel int pos } \mathcal{A'} \}. \quad (3.4)$$

For a combinatorial study of chamber complexes we refer to [1]. We now relate the secondary polyhedra and the chamber complexes of $\mathcal{A}$ and $\mathcal{B}$. For the special case pos($\mathcal{A}$) = $\mathbb{R}^n$, Theorem 3.1 leads to an association between polytopes with normal vectors in $\mathcal{A}$ and cells in $\mathcal{B}$. This statement can also be inferred from [9, Theorem 5A2].

**Theorem 3.1.** Let $\mathcal{A} \subseteq \mathbb{R}^d$ and $\mathcal{B} \subseteq \mathbb{R}^{n-d}$ be linear transforms of each other. Then the boundary complex of the secondary polyhedron $\Sigma(\mathcal{A})$ is antiisomorphic to the chamber complex $\Gamma(\mathcal{B})$, and the boundary complex of $\Sigma(\mathcal{B})$ is antiisomorphic to $\Gamma(\mathcal{A})$.

For the proof of Theorem 3.1 we will need the following lemma.

**Lemma 3.2.** Given any subset $\sigma \subseteq \{1, \ldots, n\}$, then $\sigma \in \Pi(\psi)$ if and only if $\sum_{i=1}^n \psi_i b_i \in \text{rel int pos } \{ b_k | k \notin \sigma \}$.

**Proof.** The cone pos$\{ (a_i, \psi_i) | i \in \sigma \}$ is a bottom face of pos$\{ (a_i, \psi_i), \ldots, (a_i, \psi_n) \} \subseteq \mathbb{R}^{d+1}$ if and only if there is a linear functional $\phi: \mathbb{R}^d \to \mathbb{R}$ such that $\phi(a_i) + \psi_i = 0$ for $i \in \sigma$ and $\phi(a_i) + \psi_i > 0$ for $i \notin \sigma$. This is equivalent to the existence of a vector $v \in D(\mathcal{A}) = C(\mathcal{B})$ with $v_i + \psi_i = 0$ for $i \in \sigma$ and $v_i + \psi_i > 0$ for $i \notin \sigma$. In this case, $\sum_{i=1}^n \psi_i b_i = \sum_{k \in \sigma} (v_k + \psi_k) b_k$, which completes the proof.

**Proof of Theorem 3.1.** Consider the linear map $B: \mathbb{R}^n \to \mathbb{R}^{n-d}$, $\psi \mapsto \sum_{i=1}^n \psi_i b_i$. Fix a feasible vector $\psi \in \mathbb{R}^n$. It lies in $F_{\mathcal{A}}(\Pi)$ for some
regular subdivision $\Pi = \Pi(\psi)$ of $\mathcal{A}$. Applying Lemma 3.2 to any $\sigma \in \Pi$, we see that $B(\psi)$ lies in $\text{pos}(\mathcal{B})$ and thus lies in a unique cell $\Gamma(\mathcal{B}, B(\psi))$ of the chamber complex $\Gamma(\mathcal{B})$. Lemma 3.2 and (3.4) imply the relations

\[ \Gamma(\mathcal{B}, B(\psi)) = \bigcap_{\sigma \in \Pi(\psi)} \text{rel int pos}\{ b_k | k \notin \sigma \}. \tag{3.5} \]

We now define a map from the boundary complex of the secondary polyhedron $\Sigma(\mathcal{A})$ to the chamber complex $\Gamma(\mathcal{B})$ by

\[ \Sigma(\mathcal{A})^\psi \mapsto \Gamma(\mathcal{B}, B(\psi)). \tag{3.6} \]

This map is well-defined and order-reversing because if $\Sigma(\mathcal{A})^\psi \subseteq \Sigma(\mathcal{A})^\nu$, then $\Pi(\psi)$ refines $\Pi(\nu)$, by Theorem 2.4, and in this case $\Gamma(\mathcal{B}, B(\psi)) \subseteq \Gamma(\mathcal{B}, B(\psi))$ by (3.5). By Lemma 3.2, the assignment $\Gamma(\mathcal{B}, x) \mapsto \Sigma(\mathcal{A})^\psi$, for any $\psi$ such that $B(\psi) = x$, defines the inverse to (3.6).

The secondary polyhedron $\Sigma(\mathcal{A})$ as defined in (2.2) has codimension $d$ in $\mathbb{R}^n$. For the following discussion we will consider $\Sigma(\mathcal{A})$ and all fibers $\pi^{-1}(x)$ to be embedded in $\mathbb{R}^{n-d}$ via the map $B$, which is an isomorphism when restricted to translates of the kernel of $\pi$. In fact, the bijection (3.6) shows that under this embedding the normal fan of $\Sigma(\mathcal{A})$ equals $\Gamma(\mathcal{B})$. In order to describe minors by contraction we will now construct a polyhedron in $\mathbb{R}^{n-d}$ which is normally equivalent to (i.e., has the same normal fan as) $\Sigma(\mathcal{A})$. The support function of any such polyhedron is a strictly convex, piecewise linear function over the chamber complex $\Gamma(\mathcal{B})$.

First note that $\Gamma(\mathcal{B})$ is the coarsest polyhedral complex that refines all regular triangulations of $\mathcal{B}$, since every basis of $\mathcal{B}$ appears in some regular triangulation. A triangulation $\Lambda$ of $\mathcal{B}$ is regular if and only if there exists a vector $\psi \in \mathbb{R}^n$ that induces a convex piecewise linear function $g_\Lambda = g_\Lambda(x)$ over the fan defining $\Lambda$ (see [2] for the affine case). The function $g_\Lambda$ is the support function of a polyhedron $Q$, with normal fan $\Lambda$. Recalling that Minkowski addition of convex polyhedra corresponds both to addition of support functions and to intersection of normal fans (see [7, p. 309] or [2, Proposition 1.2.2]), we get the following.

**Proposition 3.3.** The function $\sum_\Lambda g_\Lambda$, the sum taken over all regular triangulations of $\mathcal{B}$, is the support function of a polyhedron $Q$ normally equivalent to the secondary polyhedron $\Sigma(\mathcal{A})$. In fact, $Q$ is the Minkowski sum $\Sigma_\Lambda Q_\Lambda$.

The summands of $\Sigma(\mathcal{A})$ in Proposition 3.3 may be taken to be $Q_\Lambda = \int_\pi^{-1}(x) dx$, where $\sigma$ is the maximal cell of $\Gamma(\mathcal{A}) \cap S^{d-1}$ corresponding to the regular triangulation $\Lambda$ of $\mathcal{B}$. In fact, the regular polyhedral subdivisions of $\mathcal{B}$ are precisely the normal fans of the fibers
\( \pi^{-1}(x), x \in \text{pos}(\mathcal{A}) \) (under the embedding \( \mathcal{B} \)). Thus each face of \( \Sigma(\mathcal{B}) \) gives rise to a Minkowski summand of \( \Sigma(\mathcal{A}) \).

Again, let \( \mathcal{A} = \{a_1, ..., a_n\} \) be a spanning subset of \( \mathbb{R}^d \), and let \( \mathcal{B} = \{b_1, ..., b_n\} \subset \mathbb{R}^{d-d} \) be a linear transform of \( \mathcal{A} \). We assume that \( a_i \) is neither a loop nor a coloop of \( \mathcal{A} \), which means that none of the \( a_i \) and none of the \( b_i \) are zero. The following lemma is a straightforward analogue to the matroidal duality of deletion and contraction.

**Lemma 3.4.** Deletion and contraction are dual in the sense that

(i) \( \mathcal{A}/a_i \) is a linear transform of \( \mathcal{B} \setminus b_i \), and

(ii) \( \mathcal{A} \setminus a_i \) is a linear transform of \( \mathcal{B}/b_i \).

We have identified the normal fan of \( \Sigma(\mathcal{A}) \) with the chamber complex \( \Gamma(\mathcal{B}) \). Using Lemma 3.4, we can therefore identify the normal fan of \( \Sigma(\mathcal{A}/a_i) \) with the chamber complex \( \Gamma(\mathcal{B} \setminus b_i) \), and similarly the normal fan of \( \Sigma(\mathcal{A} \setminus a_i) \) with \( \Gamma(\mathcal{B}/b_i) \). Thus our problem is reduced to describing the behavior of the chamber complex under deletion and contraction. Using this point of view we now describe the relationship of the secondary polyhedron by contraction \( \Sigma(\mathcal{A}/a_i) \) to \( \Sigma(\mathcal{A}) \).

We say a point \( a_i \) is extreme in \( \mathcal{A} \) if \( a_i \notin \text{pos}(\mathcal{A} \setminus a_i) \). We note that \( a_i \) is extreme in \( \mathcal{A} \) if and only if \( b_i \) is not extreme in \( \mathcal{B} \). Every convex polyhedron \( Q \) can be written uniquely as a minimal intersection of halfspaces in its affine hull, each of which then defines a facet. If \( Q \) is a convex polyhedron and \( F \) is a facet of \( Q \), then we say \( \Sigma' \) results from \( \Sigma \) by removing \( F \) if \( \Sigma' \) is the intersection of all halfspaces in the minimal representation of \( Q \) except the one corresponding to \( F \).

**Theorem 3.5.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be as above. If \( a_i \) is extreme in \( \mathcal{A} \), then \( \Sigma(\mathcal{A}/a_i) \) is a Minkowski summand of \( \Sigma(\mathcal{A}) \). If \( a_i \) is not extreme in \( \mathcal{A} \), then \( \Sigma(\mathcal{A}/a_i) \) is unbounded and is obtained from a Minkowski summand of \( \Sigma(\mathcal{A}) \) by removing the facet with inner normal \( e_i \).

**Proof.** We consider the chamber complex \( \Gamma(\mathcal{B} \setminus b_i) \). If \( a_i \) is extreme in \( \mathcal{A} \), then since \( b_i \) is not extreme in \( \mathcal{B} \), regular triangulations of \( \mathcal{B} \setminus b_i \) are just those regular triangulations of \( \mathcal{B} \) that do not involve \( b_i \) as a vertex. Thus by Proposition 3.3

\[
\Sigma(\mathcal{A}) = \Sigma(\mathcal{A}/a_i) + \sum A, \tag{3.7}
\]

the summation over those regular triangulations of \( \mathcal{B} \) that do contain \( b_i \).

On the other hand, if \( a_i \) is not extreme in \( \mathcal{A} \), then since \( b_i \) is extreme in \( \mathcal{B} \), all regular triangulations of \( \mathcal{B} \) must contain \( b_i \). In this case, let \( R = \sum A \), where the sum here is over all regular triangulations \( A \) of \( \mathcal{B} \)
such that the complex $\Delta \setminus b_i$ triangulates $B \setminus b_i$. (By $\Delta \setminus b_i$ is meant the complex consisting of all simplices that do not contain $b_i$.) By Proposition 3.3 again, $R$ is a Minkowski summand of $\Sigma(\mathcal{A})$.

Each polytope $Q_i$ in the definition of $R$ has a facet with inner normal $h_i$. Removing this facet corresponds to restricting the support function $g_{\Delta}$ of $Q_i$ to $\text{pos}(B \setminus b_i)$, or equivalently, to replacing $g_{\Delta}$ by the new support function $g_{\Delta \setminus b_i}$ gotten by setting $\psi_i = +\infty$. The sum over these new support functions is the support function of the polyhedron $R'$ which results from $R$ by removing the facet with inner normal $h_i$. Thus the support function of $R'$ is strictly convex and piecewise linear over $\Gamma(B \setminus b_i)$, and therefore $R'$ is normally equivalent to $\Sigma(a/\alpha)$. Now Theorem 3.5 follows because the facet of $\Sigma(a/\alpha)$ (or of its summand $R'$) with inner normal $h_i$ becomes the facet with inner normal $e_i$ when these polyhedra are considered in the original embedding in $R^n$.

4. A Non-Regular Triangulation of a Cyclic Polytope

In this section we apply our duality results to triangulations of the cyclic 8-polytope $C(8, 12)$ with 12 vertices. In particular, we show that $C(8, 12)$ admits a non-regular triangulation; this proves a conjecture of Kapranov and Voevodsky [8, Remark 3.5].

Let $\mathcal{A} := \{a_1, a_2, ..., a_{12}\} \subset R^3$, where $a_i := (1, i, i^2, ..., i^8)$ for $i = 1, 2, ..., 12$. The cyclic polytope $C(8, 12)$ is defined as $\text{conv}(\mathcal{A})$. It will here be identified with the spherical 8-polytope $P(\mathcal{A})$ or with its positive hull pos(\mathcal{A}). A linear transform of $\mathcal{A}$ is given by $B = \{b_1, b_2, ..., b_{12}\} \subset R^3$, where

\[
(\begin{array}{cccccccccc}
1 & 0 & 0 & -165 & 990 & -2772 & 4620 & -4950 & 3465 & -1540 & 396 & -45 \\
0 & -1 & 0 & 45 & -240 & 630 & -1008 & 1050 & -720 & 315 & -80 & 9 \\
0 & 0 & 1 & -9 & 36 & -84 & 126 & -126 & 84 & -36 & 9 & -1
\end{array})
\]

If we replace the vectors $b_2, b_4, b_6, b_8, b_{10}$, and $b_{12}$ by their negatives, then we obtain a pointed cone in 3-space, which can be represented by the 2-dimensional affine configuration depicted in Fig. 1. This diagram is an affine Gale diagram [11] of $C(8, 12)$. (See also [2, Fig. 2]).

By Theorem 3.1, the maximal cells of chamber complex $\Gamma(B)$ are in one-to-one correspondence with the regular triangulations of $C(8, 12)$. We now consider the specific maximal cell $\Gamma(B, x_0)$ which contains the vector $x_0 = (770, -159, 20)$. As is illustrated in Fig. 1, this cell is the intersection...
of the simplicial cones \( \text{pos}(\{b_i, b_j, b_k\}) \), where \( i, j, k \) ranges over the triples in the following list:

\[
\begin{array}{cccccccccccc}
 1 & 2 & 3 & 12 & 5 & 13 & 7 & 13 & 9 & 14 & 5 & 17 & 11 & 18 \\
 17 & 10 & 17 & 12 & 19 & 10 & 19 & 12 & 11 & 12 & 23 & 11 & 34 & 5 \\
 56 & 11 & 58 & 9 & 8 & 11 & 5 & 10 & 11 & 7 & 8 & 11 & 11 & 12 \\
\end{array}
\] (4.1)

The regular triangulation \( A_0 \) of \( C(8, 12) \) corresponding to the region \( \Gamma(\mathcal{A}, x_0) \) has as its maximal simplices the complements of all triples in (4.1), that is, the 35 maximal simplices of \( A_0 \) are 1 2 5 6 7 8 9 10 11 12, 3 4 6 7 8 9 10 11 12, 2 4 5 6 8 9 10 11 12, etc. We now replace the five underlined triples in (4.1) by

\[
2 \ 5 \ 11 \quad 4 \ 5 \ 11 \quad 5 \ 7 \ 11 \quad 5 \ 9 \ 11 \quad 5 \ 11 \ 12.
\] (4.2)

This corresponds to bending the line \( 5 \ 11 \) until it curves under the point \( x_0 \) in Fig. 1.

The new collection of complementary sets defines a simplicial 8-ball \( A_1 \) which is obtained from \( A_0 \) by a bistellar operation (see [2]) on \( \{5, 11\}^c = \{1, 2, 3, 4, 6, 7, 8, 9, 10, 12\} \). This bistellar operation can be carried out geometrically for the triangulation \( A_0 \) of \( C(8, 12) \) because \( \{1, 3, 6, 8, 10\}, \{2, 4, 7, 9, 12\} \) is a Radon partition of \( C(8, 12) \). For, we can see in Fig. 1 that \((-1,-,+,+,0,-,+,-,+,0,+1\) is a signed cocircuit of \( \mathcal{A} \) and hence is a signed circuit of \( \mathcal{A} \).

**Proposition 4.1.** The triangulation \( A_1 \) of \( C(8, 12) \) is not regular.

**Proof.** If the triangulation \( A_1 \) were regular, then there would exist a
vector \( x_i \) in the interior of all simplicial cones \( \text{pos}(\{b_i, b_j, b_k\}) \), where \( ijk \) ranges over all triples in (4.2) and all non-underlined triples in (4.1). However, as can be seen in Fig. 1, the intersection of these simplicial cones (or spherical triangles) is empty. This shows that \( A_1 \) is a non-regular triangulation of the cyclic polytope \( C(8, 12) \).

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