IDENTIFIABILITY OF HOMOGENEOUS POLYNOMIAL SYSTEMS USING THE STATE ISOMORPHISM APPROACH

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We consider the local state isomorphism approach towards identifiability analysis of nonlinear state-space systems. It is shown that, under certain conditions, the local state isomorphism is linear for homogeneous polynomial systems. The ORC at the origin is shown to play a key role. Worked examples are given to illustrate the results.

1. Introduction

The identifiability problem for linear and nonlinear dynamical state-space systems can be approached in various different ways. In the case of linear state space systems, one can use the realization theory of Kalman, to conclude that two minimal state-space systems are equivalent if and only if there exists a linear state space transformation taking one state-space system into the other. When considering a parametrized family of minimal linear state-space systems of some fixed order, identifiability holds if and only if any nontrivial state space basis transformation takes any element of the family to a system outside the family. Here we are concerned with a generalization of such a scheme to nonlinear systems, in particular to a class of homogeneous polynomial systems. Our main result here presents conditions under which for two indistinguishable systems the state space transformation that links the two is always linear. Because of space limitations, proofs are given elsewhere; see [4]. Examples are given to illustrate how this linearity result can be used to analyze identifiability.
2. Problem statement

We consider the state isomorphism approach towards identifiability analysis of parametrized classes of nonlinear state-space systems with specified initial states, described by a set of equations of the form

\[
\begin{align*}
\dot{x} &= f(x, p) + g(x, p)u, \\
y &= h(x, p), \\
x(0) &= x_0(p).
\end{align*}
\]

(1) \hspace{1cm} (2) \hspace{1cm} (3)

Here, \( p \) denotes a parameter vector taken from some parameter set \( \Omega \subseteq \mathbb{R}^q \). For every \( p \), the vector fields \( f(\cdot, p), g(\cdot, p) : M_p \to \mathbb{R}^n \) are real analytic on an open connected state-space domain \( M_p \subseteq \mathbb{R}^n \) which contains the initial state \( x_0(p) \). The function \( h(\cdot, p) : M_p \to \mathbb{R} \) is also real analytic, and the input function \( u(\cdot) : [0, \tau) \to \mathbb{R} \) is taken from a suitable set of bounded measurable controls, with \( \tau > 0 \) some positive time horizon on which a solution to the set of differential equations exists.

For each \( p \in \Omega \), the map \( \Sigma_p : u(\cdot) \mapsto y(\cdot) \) which takes admissible input functions to their associated output functions as determined by the system \( (f(\cdot, p), g(\cdot, p), h(\cdot, p), x_0(p)) \) is called the input-output map. Two parameter vectors \( p, \tilde{p} \in \Omega \) are called indistinguishable if it holds that \( \Sigma_p = \Sigma_\tilde{p} \). The class of systems is called globally identifiable if the mapping \( p \mapsto \Sigma_p \) is injective, i.e., \( \Sigma_p = \Sigma_\tilde{p} \Rightarrow p = \tilde{p} \).

The state isomorphism approach uses the fact that, under certain technical conditions, indistinguishable state-space systems have locally isomorphic state spaces. (See, e.g., [2] for more background material and concepts used.) This is the basic content of the following theorem (cf. [5, 6]).

**Theorem 2.1. (Local state isomorphism theorem.)** Let \( p, \tilde{p} \in \Omega \) be two parameter vectors for which the system (1)–(3) is locally reduced at the initial state. Then \( p \) and \( \tilde{p} \) are indistinguishable if and only if there exists a real analytic diffeomorphism \( \lambda : \tilde{V} \to V \) with \( \tilde{V} \) an open neighborhood of \( x_0(\tilde{p}) \) and \( V \) an open neighborhood of \( x_0(p) \), satisfying:

(i) \( \text{rank}(\nabla \lambda(\tilde{x})) = n \), for all \( \tilde{x} \in \tilde{V} \),

(ii) \( \lambda(x_0(\tilde{p})) = x_0(p) \),

(iii) \( f(\lambda(\tilde{x}), p) = \nabla \lambda(\tilde{x}) \cdot f(\tilde{x}, \tilde{p}) \), for all \( \tilde{x} \in \tilde{V} \),

(iv) \( g(\lambda(\tilde{x}), p) = \nabla \lambda(\tilde{x}) \cdot g(\tilde{x}, \tilde{p}) \), for all \( \tilde{x} \in \tilde{V} \),

(v) \( h(\lambda(\tilde{x}), p) = h(\tilde{x}, \tilde{p}) \), for all \( \tilde{x} \in \tilde{V} \),

with \( \nabla \lambda(\tilde{x}) \) the Jacobian matrix of \( \lambda \) at \( \tilde{x} \). For a given choice of \( \tilde{V} \), this real analytic diffeomorphism \( \lambda \) is unique.
To use this theorem for identifiability analysis, first one must verify the systems to be locally reduced at the initial state, for every $p \in \Omega$. This requires the controllability rank condition (CRC) and the observability rank condition (ORC) to hold at $x_0(p)$. Next, consider a mapping $\lambda : \tilde{V} \rightarrow V$ which is imposed to satisfy the conditions (i)–(v) of the theorem. If these conditions imply that the trivial solution $p = \bar{p}$ and $\lambda = \text{identity}$ yields the only solution then global identifiability holds, otherwise unidentifiability holds.

The identifiability analysis may simplify considerably if additional properties of $\lambda$ are known to hold in advance. For instance, under the conditions of this theorem $\lambda$ is well-known to be linear for the class of linear systems (with $f$ and $h$ linear and $g$ constant) and for the class of bilinear systems too (with $f$, $g$, and $h$ all linear), see [1]. This motivates some of our interest to study properties of $\lambda$ for other subclasses of systems. The class of homogeneous polynomial systems is of particular importance, because it features when nonlinear systems are approximated using the first non-vanishing term of each of the Taylor series expansions of $f$, $g$ and $h$ around a working point, where essential nonlinearities must be retained. Care should be taken with some of the earlier literature on this subject, since some papers have recently been recognized to be flawed (cf. [3]).

3. New results for homogeneous polynomial systems

A real vector field $v : D \rightarrow D$ on an open connected domain $D \subseteq \mathbb{R}^n$ is called homogeneous of degree $k$, if for every $x \in D$ and for every scalar $\phi \geq 0$ in some open connected interval containing 1 such that $\phi x \in D$, it holds that $v(\phi x) = \phi^k v(x)$. According to Euler’s Theorem, if $v$ is differentiable on $D$ then $v$ is homogeneous of degree $k$ if and only if $\nabla v(x) \cdot x = kv(x)$ for all $x \in D$. If $v$ is real analytic at the origin and homogeneous of degree $k$ on a domain $D$ containing the origin, then $v$ is a homogeneous polynomial of total degree $k$. A system in the class (1)–(3) is called homogeneous if $f(\cdot, p)$, $g(\cdot, p)$ and $h(\cdot, p)$ are homogeneous.

**Theorem 3.1.** Consider a parametrized subclass of (real analytic) homogeneous systems (1)–(3) which are locally reduced at $x_0(p)$. Let $\lambda$ be a local state isomorphism for two indistinguishable parameters $p, \bar{p} \in \Omega$. Then $\lambda$ is homogeneous of degree 1.

This theorem aims to correct a false claim in the literature (see Lemma 1 in [7]) which incorrectly suggests $\lambda$ to be linear under such circumstances. See also Example 1.
The main result of this paper applies to homogeneous polynomial systems and establishes that requiring the ORC to hold at the origin forces λ to be linear.

**Theorem 3.2.** Consider a parametrized subclass of homogeneous polynomial systems (1)–(3) which are locally reduced at \( x_0(p) \). Assume that the origin is contained in \( M_p \) and that the ORC holds at the origin. Let \( \lambda \) be a local state isomorphism for two indistinguishable parameters \( p, \bar{p} \in \Omega \). Then \( \lambda \) is linear.

Requiring the CRC to hold at the origin is a stronger condition than requiring the ORC to hold at the origin. This is demonstrated by the following result.

**Lemma 3.1.** For a homogeneous polynomial system, if the ORC holds at some state \( x_0 \) and the CRC holds at the origin, then the ORC and the CRC both hold everywhere.

Example 2 presents a homogeneous polynomial system for which the ORC holds at the origin, while the CRC does not. Thus, an obvious converse to this lemma does not hold.

If the function \( h(\cdot, p) \) is no longer required to be a homogeneous polynomial but only real analytic, then requiring the CRC to hold at the origin still allows one to conclude that \( \lambda \) is affine.

**Theorem 3.3.** Consider a parametrized subclass of systems (1)–(3) which are locally reduced at \( x_0(p) \), with \( f \) and \( g \) homogeneous polynomial vector fields. Assume that the origin is contained in \( M_p \) and that the CRC holds at the origin. Let \( \lambda \) be a local state isomorphism for two indistinguishable parameters \( p, \bar{p} \in \Omega \). Then \( \lambda \) is affine.

4. **Examples**

4.1. **Example 1: the state isomorphism \( \lambda \) can be nonlinear**

Consider the parametrized class of homogeneous polynomial state-space systems (1)–(3) given by

\[
\begin{align*}
    f &= \left( \begin{array}{c} px_1^3 - x_2^3 + x_1 x_2 x_3 \\ 2px_1^2 x_2 + 2x_2^2 x_3 \\ 2px_1 x_2^2 \end{array} \right), \\
    g &= \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right), \\
    h &= x_1, \\
    x_0 &= \left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right).
\end{align*}
\]

It can be verified that the ORC and the CRC both hold at all states with \( x_2 = 0 \), see [4] for details. Now consider the mapping \( \lambda \), defined for arbitrary
parameters $p$ and $\tilde{p}$ by

$$
\lambda : \begin{cases}
    x_1 = \bar{x}_1 \\
    x_2 = \bar{x}_2 \\
    x_3 = \bar{x}_3 + (\tilde{p} - p)\bar{x}_2^2
\end{cases}
$$

It is easily verified that the mapping $\lambda$ is real analytic on any open neighborhood $\bar{V}$ of the initial state $x_0(\tilde{p})$ not containing points with $x_2 = 0$, and that it satisfies the properties (i)-(v) of Theorem 2.1. It thus constitutes a local state isomorphism and it shows that all parameters $p$ are indistinguishable. Therefore, the model class is clearly unidentifiable. The mapping $\lambda$ is homogeneous of degree 1 in agreement with Theorem 3.1, but it is truly nonlinear. Indeed, Theorem 3.2 does not apply since the ORC does not hold at the origin. (This demonstrates the invalidity of Lemma 1 of [7].)

4.2. Example 2: no obvious converse to Lemma 3.1 exists

Consider the state-space system $(f, g, h, x_0)$ given by

$$
f = \begin{pmatrix} x_1 x_2 \\ x_2^2 \end{pmatrix}, \quad g = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad h = x_1, \quad x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
$$

To investigate the ORC, we compute repeated Lie derivatives of the output function $h$ along the vector fields $f$ and $g$. This yields the functions $\omega_1 := h = x_1$ and $\omega_2 := L_g L_f h = x_1 + x_2$, for which the gradients are $\nabla \omega_1 = (1, 0)$ and $\nabla \omega_2 = (1, 1)$. Thus, the ORC holds at every point $(x_1, x_2)$, in particular at the origin. To investigate the CRC, we compute repeated Lie brackets of the vector fields $f$ and $g$. It is found that

$$
[f, g] = \begin{pmatrix} -x_1 - x_2 \\ -2x_2 \end{pmatrix}, \quad [f, [f, g]] = 2f, \quad [g, [f, g]] = -2g.
$$

Consequently, the CRC holds everywhere except at points for which $x_1 = x_2$. Thus, the CRC does hold at the initial state $x_0$ so that the system is locally reduced, but the CRC does not hold at the origin.

4.3. Example 3: improved identifiability analysis

Consider the class of homogeneous polynomial systems (1)-(3) given by

$$
f = \begin{pmatrix} -px_1^2 x_3 + x_2 x_3^2 \\ -2px^2_3 x_3 + 2px_1 x_2 x_3 + x_1 x_3^2 + x_3^3 \\ 0 \end{pmatrix}, \quad g = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad h = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix},
$$
and an initial state $x_0(p) = (0, 0, 1)^T$. Although so far we have been addressing single-input single-output systems, all the theorems above admit straightforward generalizations to the multi-input multi-output case; see again [4]. To perform identifiability analysis one may first compute the functions $\omega_1 = h_1 = x_1$, $\omega_2 = L_f h_1 = -px_1^2 x_3 + x_2 x_3^2$, $\omega_3 = h_2 = x_3$. The determinant of the associated Jacobian matrix is equal to $x_2^3$, so the ORC holds at $x_0(p)$. Next, one may compute repeated Lie brackets of $f$ and $g$ to verify that the CRC holds everywhere. An explicit representation of $\lambda$ can be obtained from the functions $\omega_1, \omega_2, \omega_3$, yielding the identities: $\bar{x}_1 = x_1$, $-\bar{x}_1^2 \bar{x}_3 + \bar{x}_2 \bar{x}_3^2 = -px_1^2 x_3 + x_2 x_3^2$ and $\bar{x}_3 = x_3$. This gives:

$$\lambda : \begin{cases} x_1 = \bar{x}_1 \\
    x_2 = \bar{x}_2 + (p - \bar{p}) \frac{x_2^2}{x_3} \\
    x_3 = \bar{x}_3 \end{cases}$$

Observe that $\lambda$ is homogeneous of degree 1, but truly nonlinear for $p \neq \bar{p}$. However, the CRC holds at the origin and the ORC holds at $x_0(p)$, so that by Lemma 3.1 the ORC holds everywhere. Since the ORC holds at the origin too, $\lambda$ is linear by Theorem 3.2. Hence $p = \bar{p}$ and $\lambda = \text{identity}$. Global identifiability therefore holds. Note that the classical state isomorphism approach would normally require the conditions (i)-(v) to be imposed on $\lambda$ instead, which has now been avoided altogether.

References