

# On Unitary Quasi-Equivalence of Operators

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**Abstract:** In this paper we investigate unitary quasi-equivalence of operators in Hilbert spaces. We characterize operators that are unitarily quasi-equivalent. We also investigate equivalence relations closely related to unitary quasi-equivalence. We give and prove conditions under which unitary quasi-equivalence coincides with other operator equivalence relations.

**Keywords:** Unitary quasi-equivalence, near-equivalence, isometric equivalence, metric equivalence, skew-normal.

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## 1. Introduction

In this paper  $H$  will denote a complex separable Hilbert space and  $B(H)$  will denote the Banach algebra of bounded linear operators. If  $T \in B(H)$ , then  $T^*$  denotes the adjoint of  $T$ , while  $\text{Ker}(T)$ ,  $\text{Ran}(T)$ ,  $\overline{M}$  and  $M^\perp$  stands for the kernel of  $T$ , range of  $T$ , closure of  $M$  and orthogonal complement of a closed subspace  $M$  of  $H$ , respectively. Recall that an operator  $T \in B(H)$  is normal if  $T^*T = TT^*$ , an isometry if  $T^*T = I$ , unitary if  $T^*T = TT^* = I$ , a symmetry if  $T = T^* = T^{-1}$ , a projection (or idempotent) if  $T^2 = T$ , an orthogonal projection if  $T^2 = T$  and  $T^* = T$ . An operator  $T \in B(H)$  is said to be scalar if it is a scalar multiple of the identity operator. That is, if  $T = \alpha I$ , where  $\alpha \in \mathbb{C}$  and  $I$  is the identity operator on  $H$ . An operator  $T \in B(H)$  is quasinormal if  $T(T^*T) = (T^*T)T$ , binormal if  $(T^*T)(TT^*) = (TT^*)(T^*T)$ , hyponormal if  $T^*T \geq TT^*$ .

An operator  $T \in B(H)$  is said to be positive if it is self-adjoint and  $\langle Tx, x \rangle \geq 0$ . The spectrum of  $T \in B(H)$  is given by  $\sigma(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not invertible}\}$ . The numerical range of  $T$  is  $W(T) = \{\langle Tx, x \rangle : x \in H, \|x\| = 1\}$ .

Let  $H_1$  and  $H_2$  be Hilbert spaces. An operator  $V : H_1 \rightarrow H_2$  is called a partial isometry if there exists a subspace  $M \subseteq H_1$  such that  $\|Vx\| = \|x\|$ , if  $x \in M$  and  $\|Vx\| = 0$ , if  $x \in M^\perp$ . For more details (see [2]).

Two operators  $A \in B(H)$  and  $B \in B(K)$  are said to be *similar* if there exists an invertible operator  $N \in B(H, K)$  such that  $NA = BN$  or equivalently  $A = N^{-1}BN$ , and are *unitarily equivalent* if there exists a unitary operator  $U \in B_+(H, K)$  (Banach algebra of all invertible operators in  $B(H)$ ) such that  $UA = BU$  (i.e.,  $A = U^*BU$  equivalently,  $A = U^{-1}BU$ ). An operator  $X \in B(H, K)$  is a *quasiaffinity* or a *quasi-invertible* if it is injective and has dense range. Two operators  $A \in B(H)$  and  $B \in B(K)$  are said to be quasiaffine transforms of each other if there exists a quasiaffinity  $X \in B(H, K)$  such that  $XA = BX$ .

Two operators  $A \in B(H)$  and  $B \in B(K)$  are *quasisimilar* if there exist quasiaffinities  $X \in B(H, K)$  and  $Y \in B(K, H)$  such that  $XA = BX$  and  $AY = YB$ . Two operators  $A, B \in B(H)$  are said to be *metrically equivalent* if  $A^*A = B^*B$ ,

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or equivalently,  $\|Ax\| = \|Bx\|$  for all  $x \in H$ . For more exposition on metric equivalence (see [10]). Two operators  $A, B \in B(H)$  are said to be *almost similar* if there is an invertible operator  $X \in B(H)$  such that  $A^*A = X^{-1}(B^*B)X$  and  $A^* + A = X^{-1}(B^* + B)X$ . The concept of almost similarity was introduced in [3] and studied in [8]. Clearly unitary equivalence, similarity, almost similarity and metric equivalence are equivalence relations on  $B(H)$ . Suppose that  $A$  is a positive and invertible operator. The  $A$ -adjoint of  $T \in B(H)$  is an operator  $S \in B(H)$  such that  $AS = T^*A$  (see [9]). An operator  $T \in B(H)$  is said to be  $A$ -self-adjoint if  $T^* = ATA^{-1}$ . For any operators  $A, B \in B(H)$ , we define  $[A, B] = AB - BA$ .

The *commutant* of  $T \in B(H)$  denoted by  $\{T\}'$  is the set of all operators that commute with  $T$ . That is,  $\{T\}' = \{S \in B(H) : ST = TS\}$ . Let  $A, B \in B(H)$ . We say that an operator  $T \in B(H)$  *intertwines* the pair  $(A, B)$  if  $TA = BT$ . If  $T$  intertwines both  $(A, B)$  and  $(B, A)$ , then we say that  $T$  *doubly intertwines*  $A$  and  $B$ .

Two operators  $A, B \in B(H)$  are said to be *unitarily quasi-equivalent* (denoted by  $A \stackrel{uqe}{\approx} B$ ) if there exists a unitary operator  $U \in B(H)$  such that  $A^*A = UB^*BU^*$  and  $AA^* = UBB^*U^*$ . Two operators  $A, B \in B(H)$  are said to be *absolutely equivalent* (denoted by  $A \stackrel{abe}{\approx} B$ ) if both the absolute values of the operators are unitarily equivalent. That is, if  $|A| = U|B|U^*$ . Two operators  $A, B \in B(H)$  are said to be *nearly-equivalent* (denoted by  $A \stackrel{ne}{\approx} B$ ) (see [10]) if there exists a unitary operator  $U \in B(H)$  such that  $A^*A = UB^*BU^*$ . The notion of unitary quasi-equivalence was introduced by Mahmoud [7] under the name quasi-equivalence and was investigated by Othman [11] under the near-equivalence relation.

## 2. Main Results

It has been shown in [11] that absolute equivalence implies near equivalence of operators. We observe that near-equivalence of operators is weaker than unitary quasi-equivalence. We note that any two unitary operators and in general any two isometries on the same Hilbert space are absolutely equivalent, metrically equivalent and nearly-equivalent.

**Theorem 2.1.** *Unitary quasi-equivalence is an equivalence relation on  $B(H)$ .*

*Proof.* Let  $A, B, C \in B(H)$ . Clearly  $A \stackrel{uqe}{\approx} A$  since  $A^*A = IA^*AI^*$  and  $AA^* = IAA^*I^*$ , by taking  $U = I$ . If  $A \stackrel{uqe}{\approx} B$  then  $A^*A = UB^*BU^*$  and  $AA^* = UBB^*U^*$ . Pre-multiplying and post-multiplying each of these equations by  $U^*$  and  $U$ , respectively, we have  $B^*B = UA^*AU^*$  and  $BB^* = UAA^*U^*$ . This proves that  $B \stackrel{uqe}{\approx} A$ . Now suppose that  $A \stackrel{uqe}{\approx} B$  and  $B \stackrel{uqe}{\approx} C$ . Then  $A^*A = UB^*BU^*$  and  $AA^* = UBB^*U^*$  and  $B^*B = WC^*CW^*$  and  $BB^* = WCC^*W^*$ , for some unitary operators  $U$  and  $W$ . A simple computation shows that  $A^*A = UB^*BU^* = UWC^*CW^*U^* = ZC^*CZ^*$ , where  $Z = UW$  is unitary. Also,  $AA^* = UBB^*U^* = UWCC^*W^*U^* = ZCC^*Z^*$ . This proves that  $A \stackrel{uqe}{\approx} C$ . This proves that  $\stackrel{uqe}{\approx}$  is an equivalence relation. □

The following result shows that unitary quasi-equivalence is weaker than unitary equivalence.

**Theorem 2.2.** *If  $A, B \in B(H)$  are unitarily equivalent then they are unitarily quasi-equivalent.*

*Proof.* Suppose  $A = UBU^*$  for some unitary operator  $U \in B(H)$ . Then  $A^*A = UB^*U^*UBU^* = UB^*BU^*$  and  $AA^* = UBU^*UB^*U^* = UBB^*U^*$ . This proves the claim. □

The converse of Theorem 2.2 is not true in general. The operators  $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$  and  $B = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$  are unitarily quasi-equivalent but not unitarily equivalent.

The next result gives a condition under which the converse of Theorem 2.2 is true.

**Theorem 2.3.** *Let  $A, B \in B(H)$  be unitarily quasi-equivalent. If  $A$  and  $B$  are projections, then they are unitarily equivalent.*

**Remark 2.4.** We note that by ([9], Theorem 3.17) that unitary equivalence preserves self-adjointness and even normality of operators but it does not preserve  $A$ -self-adjointness of operators.

The following result shows that unitary quasi-equivalence preserves normality.

**Theorem 2.5.** Let  $A, B \in B(H)$  be unitarily quasi-equivalent. Then  $A$  is normal if and only if  $B$  is normal.

*Proof.* Suppose that  $A \overset{uqe}{\approx} B$  and suppose  $A$  is normal. Then  $B^*B = UA^*AU^*$  and  $BB^* = UAA^*U^*$ . Thus  $B^*B = UA^*AU^* = UAA^*U^* = BB^*$ . The converse is proved similarly.  $\square$

**Lemma 2.6.** Two operators  $A, B \in B(H)$  are unitarily quasi-equivalent if and only if  $A^*A - AA^* = U(B^*B - BB^*)U^*$ .

**Theorem 2.7.** Let  $A, B \in B(H)$  be unitarily quasi-equivalent. Then  $A$  is hyponormal if and only if  $B$  is hyponormal.

*Proof.* From Lemma 2.6,  $A^*A - AA^*$  is unitarily equivalent to  $B^*B - BB^*$ . If  $A^*A - AA^* \geq 0$ , then  $B^*B - BB^* = U(A^*A - AA^*)U^* \geq 0$ . This shows that unitary quasi-equivalence preserves the hyponormality property.  $\square$

Trivially, unitary quasi-equivalence preserves quasinormality and binormality of operators. This follows from Theorem 2.7 and the fact that these classes of operators are contained in the class of hyponormal operators.

### 3. Unitary Quasi-equivalence, Related Relations and Some Classes of Operators

Unitary quasi-equivalence preserves invertibility of operators. This may not be true for absolute equivalence and near-equivalence of operators.

**Lemma 3.1.** Let  $A, B \in B(H)$  be unitarily quasi-equivalent. If  $A^*A$  is invertible, then  $B^*B$  is invertible.

We note that unitary quasi-equivalence in Theorem 3.1 cannot be replaced by near-equivalence or absolute equivalence of operators. There exist operators  $A$  such that  $A^*A$  is invertible but  $A$  is not invertible. An example is the unilateral shift operator on  $\ell^2(\mathbb{N})$ . Invertibility of  $A^*A$  implies invertibility of  $A$  if and only if  $A$  and  $A^*$  are injective.

**Lemma 3.2.** Let  $A, B \in B(H)$  be unitarily quasi-equivalent. If  $A$  is invertible, then  $B^*B$  and  $BB^*$  are invertible.

We note that in Lemma 3.2 unitary quasi-equivalence of  $A$  and  $B$  and the invertibility of  $A$  is sufficient for the invertibility of  $B$ .

**Theorem 3.3.** Let  $A, B \in B(H)$  be unitarily quasi-equivalent. If  $A$  is invertible, then  $B$  is also invertible.

*Proof.* From Lemma 3.1 and Lemma 3.2, we conclude that  $B^*B$  and  $BB^*$  are invertible and hence  $0 \notin \sigma(B^*B)$  and  $0 \notin \sigma(BB^*)$ . Using the fact that  $\sigma(B^*B) \subseteq \sigma(B^*)\sigma(B) = \sigma(B)\sigma(B^*)$ , provided  $0 \notin \sigma(B^*B)$  and  $0 \notin \sigma(BB^*)$ . Thus and  $0 \notin \sigma(B)$ , and hence  $B$  is invertible.  $\square$

We note that Theorem 3.3 says that invertibility is invariant under unitary quasi-equivalence of operators. For two algebras  $a$  and  $b$ , a Jordan isomorphism  $\varphi : a \rightarrow b$  is a bijective linear map satisfying  $\varphi(T^2) = (\varphi(T))^2$ , for every  $T \in a$ . Every Jordan isomorphism between unital algebras preserves invertibility. The map  $\varphi : a \rightarrow b$  defined by  $\varphi(A^*A) = UB^*BU^*$  and  $\varphi(AA^*) = UBB^*U^*$  is a Jordan isomorphism on  $B(H)$ . It was proved in [10] that the class of nearly-equivalent operators contains the class of metrically equivalent operators.

**Example 3.4.** The unilateral shift operator and the identity operator on  $\ell^2(\mathbb{N})$  are metrically equivalent, absolutely equivalent, near-equivalent, while their adjoints are not. This shows that these operators are not unitarily quasi-equivalent.

**Lemma 3.5.**  $T \in B(H)$  is unitarily equivalent to a unitary operator if and only if it is a unitary operator.

*Proof.* Suppose that  $T = VUV^*$ , where  $U, V \in B(H)$  are unitary operators. Then

$$T^*T = VU^*V^*VUV^* = I$$

and

$$TT^* = VUV^*VU^*V^* = I.$$

□

Lemma 3.5 can be extended to the class of unitarily quasi-equivalent operators.

**Theorem 3.6.**  $T \in B(H)$  is unitarily quasi-equivalent to a unitary operator if and only if it is a unitary operator.

*Proof.* Suppose that  $T$  is unitarily quasi-equivalent to a unitary operator  $V \in B(H)$ . Then there exists a unitary operator  $U \in B(H)$  such that  $T^*T = U(V^*V)U^* = I$  and  $TT^* = U(VV^*)U^* = I$ . This shows that  $T^*T = TT^* = I$ . □

The converse follows from Lemma 3.5 and the definition,  $T$  unitary means it is unitarily equivalent to a unitary and hence by Theorem 2.2, it is unitarily quasi-equivalent to a unitary operator.

From Theorem 3.6, we note that “unitary equivalence to a unitary operator” is the same as “unitary quasi-equivalence to a unitary operator”. This means that a necessary and sufficient condition that an operator is unitarily equivalent to a unitary is that it be unitarily quasi-equivalent to a unitary.

Two operators  $A, B \in B(H)$  are said to be almost unitarily equivalent if  $A^*A = U(B^*B)U^*$  and  $A^* + A = U(B^* + B)U^*$ . Clearly almost unitary equivalence implies almost similarity of operators.

**Theorem 3.7.** If  $A, B \in B(H)$  are self-adjoint and unitarily quasi-equivalent then  $A^2$  and  $B^2$  are unitarily equivalent.

*Proof.* By definition  $A^*A = UB^*BU^*$  and  $AA^* = UBB^*U^*$ . Using the self-adjointness of  $A$  and  $B$ , we have that  $A^2 = UB^2U^*$ , which establishes the claim. □

**Theorem 3.8.** If  $A, B \in B(H)$  are unitarily quasi-equivalent projections then  $A$  and  $B$  are unitarily equivalent.

*Proof.* The proof follows from Theorem 3.6 and the fact that projections are idempotent operators. □

Note that two unitarily quasi-equivalent self-adjoint operators are unitarily equivalent and hence nearly equivalent. Thus unitary quasi-equivalence coincides with near-equivalence in the class of self-adjoint operators.

**Remark 3.9.** If  $T$  has polar decomposition  $T = U|T|$ , where  $U$  is a partial isometry and  $|T| = \sqrt{T^*T}$ , then  $\hat{T} = |T|U$  is the Duggal transform of  $T$ .

**Theorem 3.10.** Let  $A, B \in B(H)$  be invertible unitarily quasi-equivalent. Then  $|A|$  and  $|B|$  are nearly-equivalent.

**Theorem 3.11.** Let  $A, B \in B(H)$  be metrically normal operators. Then their Duggal transforms are nearly-equivalent.

*Proof.* Suppose  $A = U|A|$  and  $B = V|B|$ , where  $U, V$  are partial isometries. Since  $A$  and  $B$  are normal  $U, V$  are unitary operators. Using ([10], Theorem 2.18) we have

$$\begin{aligned} \hat{A}^* \hat{A} &= U^* |A| |A| U = U^* |A|^2 U = U^* |B|^2 U \\ &= U^* [V (\hat{B}^* \hat{B}) V^*] U \\ &= W (\hat{B}^* \hat{B}) W^*, \end{aligned}$$

where  $W = V^*U$  is a unitary operator. □

**Corollary 3.12.** *If the Duggal transforms of  $A, B \in B(H)$  are unitarily quasi-equivalent, then  $A$  and  $B$  are nearly-equivalent.*

*Proof.* Suppose  $\hat{A}^*\hat{A} = W(\hat{B}^*\hat{B})W^*$  and  $\hat{A}\hat{A}^* = W(\hat{B}\hat{B}^*)W^*$ , where  $W$  is unitary. A simple computation shows that  $A^*A = XB^*BX^*$ , where  $X = UW$  is unitary. □

**Theorem 3.13.** *For orthogonal projection operators  $P, Q \in B(H)$  the following assertions are equivalent.*

- (1).  $P$  and  $Q$  are almost unitarily equivalent.
- (2).  $P$  and  $Q$  are unitarily equivalent.
- (3).  $P$  and  $Q$  are unitarily quasi-equivalent.

*Proof.* (1)  $\Rightarrow$  (2): Suppose there exists a unitary operator  $U$  such that  $P^*P = U(Q^*Q)U^*$  and  $P^* + P = U(Q^* + Q)U^*$ . Since  $P$  and  $Q$  are orthogonal projections, a simple computation shows that  $P = UQU^*$ .

(2)  $\Rightarrow$  (3): Suppose that there exists a unitary operator  $U$  such that  $P = UQU^*$ . A simple calculation shows that  $P^*P = U(Q^*Q)U^*$  and  $PP^* = U(QQ^*)U^*$ .

(3)  $\Rightarrow$  (1): This is trivial. □

It has been proved in [9] that metric equivalence of orthogonal projections acting on the same space coincides with equality.

**Example 3.14.** *The orthogonal projections  $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$  and  $Q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$  are unitarily equivalent and hence unitarily quasi-equivalent (with the unitary quasi-equivalence being implemented by  $U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ ). However, these operators are not metrically equivalent, since they are not equal.*

*There exist metrically equivalent operators which are not unitarily quasi-equivalent. The operators  $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$  and  $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$  are metrically equivalent but not unitarily quasi-equivalent.*

## 4. Unitary Quasi-equivalence and Some Operator Characterizations

Unitary quasi-equivalence need not preserve the spectrum and the numerical range of operators.

**Lemma 4.1.** *Let  $A, B \in B(H)$  be unitarily quasi-equivalent. Then  $W(A^*A) = W(B^*B)$ .*

It was proved in ([10], Corollary 2.3) that an operator is normal if and only if the operator and its adjoint are metrically equivalent. It was also proved that two metrically equivalent operators have equal norm (see [10], Corollary 2.14). This is also true for unitarily quasi-equivalent operators.

**Theorem 4.2.** *Let  $A, B \in B(H)$  be unitarily quasi-equivalent. Then  $\|A\| = \|B\|$ .*

*Proof.*  $\|A\|^2 = \|A^*A\| = \|UB^*BU^*\| = \|B^*B\| = \|B\|^2$ . Taking square roots both sides gives the required result. □

Note that unitarily quasi-equivalent operators need not have equal spectrum. The operators  $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  are unitarily quasi-equivalent with the equivalence being implemented by the unitary operator  $U = I$ . However,  $\sigma(A) = \{-1, 1\} \neq \{1\} = \sigma(B)$ . Note also that  $W(A) = [-1, 1] \neq \{1\} = W(B)$ .

**Proposition 4.3.** *Let  $T \in B(H)$ . Then*

- (1).  $Ker(T^*T) = Ker(T)$ .
- (2).  $\overline{Ran(TT^*)} = \overline{Ran(T)}$ .

*Proof.*

- (1).  $Ker(T^*T) = \{x \in H : T^*Tx = 0\}$   
 $= \{x \in H : Tx = 0\}$   
 $= Ker(T)$ .
- (2).  $\overline{Ran(TT^*)} = \overline{\{y \in H : y = TT^*x, x \in H\}}$   
 $= \overline{\{y \in H : y = T(T^*x)\}}$   
 $= \overline{Ran(T)}$ .

□

**Theorem 4.4.** *If  $A, B \in B(H)$  are unitarily quasi-equivalent then  $Ker(A) = Ker(B)$  and  $\overline{Ran(|A|)} = \overline{Ran(|B|)}$ .*

*Proof.* The proof follows from Proposition 4.3 and the definition of unitary quasi-equivalence of operators. □

**Corollary 4.5.** *If  $A, B \in B(H)$  are unitarily quasi-equivalent and  $A$  is injective, then  $B$  is injective.*

We add that unitarily quasi-equivalence, unlike metric equivalence, preserves injectivity and also range of absolute value of operators. We also note that unitary quasi-equivalence preserves self-adjointness but not positivity of operators. The operators  $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$  are unitarily quasi-equivalent. But  $B$  is not positive.

Recall that an operator  $T \in B(H)$  is skew-normal if  $T^2 = T^{*2}$ . Skew-normal operators were introduced by [6] and were also studied by [4] and [5]. Note that if  $T \in B(H)$  is skew-normal, then  $T^2$  is normal.

**Theorem 4.6.** *Let  $S, T \in B(H)$  be unitarily quasi-equivalent. If  $T$  is self-adjoint and skew-normal, then  $S$  is normal.*

*Proof.* A simple computation using the self-adjointness and skew-normality of  $A$  gives

$$T^2 = US^*SU^* = USS^*U^*.$$

Thus  $U(S^*S - SS^*)U^* = 0$ . This implies that  $S^*S - SS^* = 0$ . This proves the claim. □

**Remark 4.7.** *Note that for an operator  $T \in B(H)$ , normality of  $T^2$  need not imply the normality of  $T$ . If  $T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ , then  $T^2 = 0$ , which is normal. However,  $T$  is not normal. The following result gives a condition when the normality of  $T^2$  implies that of  $T$ .*

**Theorem 4.8.** *Let  $T \in B(H)$  be such that  $T^2$  is unitarily quasi-equivalent to  $T^*$ . If  $T^2$  is normal, then  $T$  is normal.*

*Proof.* Using the hypothesis, we have  $T^{2*}T^2 = UTT^*U^*$  and  $T^2T^{2*} = UT^*TU^*$ . Since  $T$  is normal, we have  $U(T^*T - TT^*)U^* = 0$ , which implies that  $T^*T - TT^* = 0$ . This establishes the claim. □

Note that unitary quasi-equivalence does not preserve the trace of operators. Consider  $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$  and  $B = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$ . A simple computation shows that  $A$  and  $B$  are unitarily quasi-equivalent (with the unitary quasi-equivalence being implemented by  $U = I$ ). However,  $2 = \text{tr}(A) \neq \text{tr}(B) = -2$ .

**Theorem 4.9.** *Unitarily quasi-equivalent positive operators have the same point spectrum.*

*Proof.* Suppose  $A, B \in B(H)$  are unitarily quasi-equivalent. Then by definition,  $A^*A = U(B^*B)U^*$  and  $AA^* = U(BB^*)U^*$ , for some unitary operator  $U \in B(H)$ . Suppose that  $Ax = \lambda x$  and  $By = \beta y$ ,  $x, y \in H$ . By positivity of  $A$  and  $B$ , it is clear that  $\lambda, \beta \geq 0$ . Without loss of generality, we may assume that  $x, y$  are unit vectors. Then  $\langle x, x \rangle = \langle y, y \rangle = 1$ . From the first equation, we have that

$$\begin{aligned} A^*A = U(B^*B)U^* &\Leftrightarrow \langle Ax, Ax \rangle = U \langle Bx, Bx \rangle U^* \\ &\Leftrightarrow \lambda \bar{\lambda} = \beta \bar{\beta} \\ &\Leftrightarrow \lambda^2 = \beta^2 \\ &\Leftrightarrow \lambda = \beta \\ &\Leftrightarrow \sigma_p(A) = \sigma_p(B). \end{aligned}$$

We note that the positivity condition in Theorem 4.8 cannot be dropped. Self-adjointness alone is not enough in the conclusion of Theorem 4.8. In general, unitarily quasi-equivalent operators may have unequal or even disjoint spectra.  $\square$

**Theorem 4.10.** *Let  $T \in B(H)$  be unitarily quasi-equivalent to  $T^*$ . Then  $T^*T - TT^*$  is unitarily equivalent to  $TT^* - T^*T$ .*

*Proof.* From the hypothesis we have

$$T^*T = U(TT^*)U^*$$

and

$$TT^* = U(T^*T)U^*.$$

Therefore  $T^*T - TT^* = U(TT^* - T^*T)U^*$ .  $\square$

**Corollary 4.11.** *Let  $T \in B(H)$  be unitarily quasi-equivalent to  $T^*$ . Then  $T^*T - TT^*$  is quasi-nilpotent.*

*Proof.* From Theorem 4.9, and the fact that unitarily equivalent operators have equal spectra, we have

$$\sigma(T^*T - TT^*) = \sigma(TT^* - T^*T).$$

But a closer look shows that

$$TT^* - T^*T = -(T^*T - TT^*).$$

Therefore  $\lambda \in \sigma(T^*T - TT^*) \Rightarrow -\lambda \in \sigma(T^*T - TT^*)$ . This implies that  $\lambda = -\lambda$ , which implies that  $\lambda = 0$ . Thus  $\sigma(T^*T - TT^*) = \{0\}$ . This proves the claim.  $\square$

**Theorem 4.12.** *Let  $T \in B(H)$  be an invertible contraction. Then  $T$  is unitarily quasi-equivalent to its inverse if and only if  $T$  is unitary.*

*Proof.* By unitary quasi-equivalence of  $T$  and  $T^{-1}$  and by using Theorem 4.2, we have  $\|T^{-1}\| = \|T\| \leq 1$  and  $\|T^{-1*}\| = \|T^*\| = \|T\| \leq 1$ . Thus, for all  $x \in H, \|x\| = \|T^{-1}Tx\| \leq \|Tx\| \leq \|x\|$ , which implies that  $\|Tx\| = \|x\|$  and hence  $T^*T = I$ . By symmetry, since  $\|T^{*-1}\| = \|T^{-1*}\| = \|T^{-1}\| \leq 1$ , we have that  $TT^* = I$ . Thus  $T^*T = TT^* = I$ , which means that  $T$  is unitary. The proof of the converse follows easily from the fact that if  $T$  is unitary then  $T^{-1}$  is unitary and the application of Theorem 2.2.  $\square$

Recall that an operator  $V \in B(H)$  is a partial isometry if and only if  $V^*V$  is an orthogonal projection. Partial isometries are generalizations of isometries. Clearly every orthogonal projection is a partial isometry.

**Theorem 4.13.** *An operator  $V \in B(H)$  is a partial isometry if and only if it is metrically equivalent to a projection.*

*Proof.* Suppose  $V \in B(H)$  is a partial isometry with initial space  $M$  and let  $P$  be the orthogonal projection of  $H$  onto  $M$ . If  $x \in M$ , then

$$\langle V^*Vx, x \rangle = \|Vx\|^2 = \|x\|^2 = \langle Px, x \rangle = \langle P^2x, x \rangle = \langle P^*Px, x \rangle.$$

Thus  $V^*V = P^*P$ . This proves metric equivalence between  $V$  and  $P$ . Conversely, suppose that  $V^*V = P^*P = P^2 = P$ , where  $P$  be the orthogonal projection onto  $M$ . For every  $x \in M$ , we have  $\|Vx\|^2 = \langle V^*Vx, x \rangle = \langle Px, x \rangle = \|x\|^2$  and for  $x \in M^\perp$ ,  $\|Vx\|^2 = \langle V^*Vx, x \rangle = \langle Px, x \rangle = 0$ . This proves that  $V$  is a partial isometry with  $M = \text{Ran}(P)$  as initial space.  $\square$

We note that a partial isometry need not be a normal operator. The following result gives a condition under which a partial isometry is normal.

**Theorem 4.14.** *If a partial isometry  $V \in B(H)$  is unitarily quasi-equivalent to a projection then  $V$  is normal.*

*Proof.* Suppose that  $V$  is unitarily quasi-equivalent to a projection  $P \in B(H)$ . Then there exists some unitary operator such that

$$V^*V = U(P^*P)U^* = U(PP^*)U^* = VV^*.$$

$\square$

## 5. Discussion

Metric equivalence and unitary quasi-equivalence of operators come in handy in the solution of the operator interpolation problem with norm constant:  $AX = B$  and  $\|X\| \leq 1$  associated with the Douglas Factorization Theorem (see [1]).

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