A non-iterative method for solving non-linear equations

Beong In Yun
School of Mathematics, Informatics and Statistics, Kunsan National University, 573-701, South Korea

Abstract

In this paper, using a hyperbolic tangent function \( \tanh(\beta x) \), \( \beta > 0 \), we develop a non-iterative method to estimate a root of an equation \( f(x) = 0 \). The problem of finding root is transformed to evaluating an integral, and thus we need not take account of choosing initial guess. The larger the value of \( \beta \), the better the approximation to the root. Alternatively we employ the signum function \( \text{sgn}(x) \) instead of the hyperbolic tangent function, which results in an exact formula for the root. Availability of the present method is shown by some numerical examples for which the traditional Newton’s method is not appropriate.

© 2007 Elsevier Inc. All rights reserved.

Keywords: Newton’s method; Hyperbolic tangent function; Signum function; Non-linear equation

1. Introduction

Newton’s method is known to be the best procedure for finding a root of an equation \( f(x) = 0 \) due to its formal simplicity and its fast convergence rate. It has been generalized in many ways for the solution of non-linear problems, for example, systems of non-linear equations and non-linear differential equations [1,2]. However, though the Newton’s method converges very rapidly once an iteration is fairly close to the root, one cannot expect good convergence when an initial guess is not properly chosen or when the slope of \( f(x) \) is extremely flat near the root. Another drawback of the Newton’s method is the necessity of knowing the first derivative \( f'(x) \) explicitly. Recently, a lot of iterative methods [3–11] based on the Newton’s method for solving non-linear equations were developed. Noticeable comments, with comparison, on these methods are given in [12].

In this paper we present a non-iterative procedure to approximate a root \( p^* \) of an equation \( f(x) = 0 \) using two kinds of transforms based on a hyperbolic tangent function, \( \tanh(\beta x) \) and a signum function, \( \text{sgn}(x) \), respectively. In the result, the problem of solving an equation is reduced to evaluating an integral of a transformed function, say, \( \tilde{f}(x) \) of \( f(x) \). The integral of \( \tilde{f}(x) \) implicitly contains an information for the location of the root \( p^* \) which is invariant under the transform.

This work was supported by the Korea Research Foundation Grant funded by the Korean Government (MOEHRD, Basic Research Promotion Fund) (KRF-2007-521-C00034).

E-mail address: biyun@kunsan.ac.kr

0096-3003/S - see front matter © 2007 Elsevier Inc. All rights reserved.
doi:10.1016/j.amc.2007.09.006
Basically the present method requires neither any knowledge of the derivative $f'(x)$ nor any iteration process. In addition, if the resultant estimate $q$ of the root $p^*$ is not satisfactory one may apply the Newton’s method for the equation $f(x) = 0$ taking $q$ as an initial guess. We can find that the present method is useful even for a function having a bad oscillation near the root or an extreme slope on the interval.

2. Approximate solution based on a sigmoid transform

For any $\beta > 0$ we introduce a hyperbolic tangent function such as

$$
\tanh(\beta x) = \frac{e^{2\beta x} - 1}{e^{2\beta x} + 1}, \quad -\infty < x < \infty,
$$

which is strictly increasing from $-1$ to $1$ producing a sigmoid curve. As $\beta$ increases, $\tanh(\beta x)$ becomes steeper near its zero and it approaches a step function as shown in Fig. 1. Note that $\tanh(\beta x)$ is a sigmoid function [13,14], a kind of the so-called logistic function which is used for mathematical modelling in sociology or biology, for instance a model of non-sinusoidal circadian rhythms [15].

In this paper, employing the sigmoid function $\tanh(\beta x)$, we aim to develop an efficient method to approximate a solution of an equation $f(x) = 0$ under the assumption that $f(x)$ is a continuous function having a unique zero $p^*$ on an interval $[a, b]$ with $f(a) \cdot f(b) < 0$.

First, we transform $f(x)$ via $\tanh(\beta x)$ such as

$$
f^{[\beta]}(x) := \tanh(\beta \cdot f(x)), \quad a \leq x \leq b.
$$

We call $f^{[\beta]}(x)$ a sigmoid transform of $f(x)$. The present work is motivated by the following observations:

(i) The zero of the original function $f(x)$ is invariant under the sigmoid transform $f^{[\beta]}(x)$.
(ii) The graph of $f^{[\beta]}(x)$ converges to that of $\text{sgn}(f(x))$ as $\beta$ goes to the infinity (see Fig. 2), where $\text{sgn}(t)$ is the signum function defined by

$$
\text{sgn}(t) := \begin{cases} 
1, & t > 0 \\
0, & t = 0 \\
-1, & t < 0.
\end{cases}
$$

That is, for $\beta$ large enough the transformed function $f^{[\beta]}(x)$ moves within a limited interval $[-1, 1]$ clustering almost every points toward the limits $\pm 1$ and its slope near the root is very steep.

![Fig. 1. Graphs of tanh(\beta x) with respect to \beta.](image-url)
Referring to the remarks mentioned above, we may consider an approximation formula

$$Z_a^b f\left(\frac{b}{C_1}x\right) \text{d}x = C_2 \text{sgn}(f(a)) \left\{ (p^* - a) - (b - p^*) \right\}$$

(4)

for \( \beta \) large enough. The graphs in Fig. 2 are helpful in understanding the construction of this formula. Setting \( I(f^{[\beta]}) := \int_a^b f^{[\beta]}(x) \text{d}x \) in (4), we have

$$p^* \approx \frac{1}{2} \{ a + b + \text{sgn}(f(a)) \cdot I(f^{[\beta]}) \}. \quad (5)$$

The integral \( I(f^{[\beta]}) \) in (5) implicitly contains an information of the root \( p^* \) which is invariant under the sigmoid transform. Replacing \( I(f^{[\beta]}) \) by its numerical evaluation \( \tilde{I}(f^{[\beta]}) \) and denoting by \( q(\beta) \) the related approximation to \( p^* \) in (5), we have a formula

$$q(\beta) = \frac{1}{2} \{ a + b + \text{sgn}(f(a)) \cdot \tilde{I}(f^{[\beta]}) \}. \quad (6)$$

Since the transformed function \( f^{[\beta]}(x) \) with large \( \beta \) has a simple shape, one may obtain a good approximation \( \tilde{I}(f^{[\beta]}) \) by any standard numerical integration rule. Thus, we can expect an excellent approximation \( q(\beta) \) to \( q^* \) for any function having an improper behavior such as bad oscillation, flat or blowup. Furthermore, the evaluation of \( q(\beta) \) is useful even for a function which is not differentiable.

When the approximation \( q(\beta) \) obtained by (6) is not sufficient one may additionally apply the Newton’s method, with an initial guess \( q_0 = q(\beta) \), such as

$$q_{n+1} = q_n - \frac{f^{[\beta]}(q_n)}{f'(q_n)}, \quad n = 0, 1, 2, \ldots \quad (7)$$

as long as the first derivative of \( f(x) \) in the vicinity of the root \( p^* \) is available. If \( \beta \cdot f(q_0) \) is small enough for some \( \beta \), the iteration (7) can be replaced by

$$q_{n+1} = q_n - \frac{f(q_n)}{f'(q_n)}, \quad n = 0, 1, 2, \ldots \quad (8)$$

because

$$\frac{f^{[\beta]}(q_0)}{f'(q_0)} \approx \frac{f'(q_0)}{f'(q_0)} = 1.$$

3. Solving an equation using a signum function

Noting that the sigmoid transform \( f^{[\beta]}(x) \) converges to \( \text{sgn}(f(x)) \) as \( \beta \to \infty \) in (4), we have an exact equation

$$\int_a^b \text{sgn}(f(x)) \text{d}x = \text{sgn}(f(a)) \left\{ (p^* - a) - (b - p^*) \right\}. \quad (9)$$
which implies, for \( I(\text{sgn}(f)) := \int_a^b \text{sgn}(f(x)) \, dx \),

\[
p^* = \frac{1}{2} \{a + b + \text{sgn}(f(a)) \cdot I(\text{sgn}(f))\}. \tag{10}
\]

Replacing the integral \( I(\text{sgn}(f)) \) by its numerical evaluation \( \tilde{I}(\text{sgn}(f)) \) and denoting by \( \tilde{p} \) the related approximation to \( p^* \), we have

\[
\tilde{p} = \frac{1}{2} \{a + b + \text{sgn}(f(a)) \cdot \tilde{I}(\text{sgn}(f))\}. \tag{11}
\]

In practical computation there are problems that the integrand \( \text{sgn}(f) \) in (11) is a logical function and that it is a discontinuous function, and thus basic numerical quadrature rules do not result in excellent estimation in general. Once, nevertheless, a good estimation \( \tilde{I}(\text{sgn}(f)) \) of \( I(\text{sgn}(f)) \) is possible in (11), we may obtain an accurate approximation to the root \( p^* \).

On the other hand, suppose that \( g(x) \) is a continuous function having two simple zeros \( p_1 \) and \( p_2 \) in \((a, b)\) with \( p_1 < p_2 \). For a midpoint \( c = (a + b)/2 \) and any even integer \( m \geq 0 \), set

\[
I_m(g(x)) := \int_a^b (x - c)^m \text{sgn}(g(x)) \, dx.
\]

Taking \( m = 0 \) and \( m = 2 \) to determine the roots \( p_1 \) and \( p_2 \), we have

\[
I_0(g(x)) = \text{sgn}(g(a)) \{2(p_1 - p_2) - (a - b)\},
\]

\[
I_2(g(x)) = \frac{\text{sgn}(g(a))}{3} \left\{ 2[(p_1 - c)^3 - (p_2 - c)^3] - \frac{1}{4}(a - b)^3 \right\}.
\]

Then it follows that

\[
p_1 = c + A + \frac{\sqrt{-A^4 + 4AB}}{2\sqrt{3}A}, \quad p_2 = p_1 - A, \tag{14}
\]

where

\[
A = \frac{1}{2} \{a - b + \text{sgn}(g(a)) \cdot I_0(\text{sgn}(g))\},
\]

\[
B = \frac{1}{2} \left\{ \frac{1}{4}(a - b)^3 + 3\text{sgn}(g(a)) \cdot I_2(\text{sgn}(g)) \right\}. \tag{15}
\]

Substituting \( I_0(\text{sgn}(g)) \) and \( I_2(\text{sgn}(g)) \) in (14) by numerical estimations \( \tilde{I}_0(\text{sgn}(g)) \) and \( \tilde{I}_2(\text{sgn}(g)) \), respectively, we have approximate roots, say, \( \tilde{p}_1 \) and \( \tilde{p}_2 \). Consequently, we only have to evaluate two integrals \( I_0(\text{sgn}(g)) \) and \( I_2(\text{sgn}(g)) \) in (15) to find the roots \( p_1 \) and \( p_2 \) simultaneously.

4. Error analysis

The following theorem provides an upper bound of an error of the present approximation \( q(\beta) \) given in the formula (6).

**Theorem 1.** Suppose a continuous function \( f(x) \) has a unique zero \( p^* \) on an interval \((a, b)\) with \( f(a)f(b) < 0 \) and its first derivative exists on \((a, b)\).

Then for some \( \delta > 0 \), \( q(\beta) \) defined in (6) satisfies

\[
|p^* - q(\beta)| \leq \frac{\delta^2}{2} \{\tau_{\max}(\beta) - \tau_{\min}(\beta)\} + Ce^{-2\epsilon} + |I(f(\beta)) - \tilde{I}(f(\beta))|,
\]

where \( C > 0 \) is a constant, \( \epsilon = \inf_{|\beta - p^*| > \delta} |f(x)| \) and

\[
\tau_{\max}(\beta) = \max_{0 < |\beta - p^*| < \delta} F(\beta, \eta), \quad \tau_{\min}(\beta) = \min_{0 < |\beta - p^*| < \delta} F(\beta, \eta)
\]

for a function \( F(\beta, \eta) := f^\beta(\eta) = \beta f'(\eta) \sech^2(\beta f(\eta)) \).
Proof. From (6) and (10) we have
\[ |p^* - q(\beta)| = |I(\text{sgn}(f)) - \tilde{I}(f^{|\beta|})| \leq |I(\text{sgn}(f) - f^{|\beta|})| + |I(f^{|\beta|}) - \tilde{I}(f^{|\beta|})|. \]

And we may assume that there exists some \( \delta > 0 \) such that the first derivative satisfies \( f'(x) \geq 0 \) (or \( f(x) \leq 0 \) for all \( x \) on the interval \( |x - p^*| < \delta \)).

Therefore, \( f^{|\beta|}(p^*) = 0 \), we have
\[ f^{|\beta|}(x) = F(\beta, \eta)(x - p^*) \]
for each \( x \) such that \( |x - p^*| < \delta \) and for some \( \eta = \eta(x) \) between \( x \) and \( p^* \). In addition, from the assumption above we have \( F(\beta, \eta) = \beta f'(\eta) \text{sech}^2(\beta f(\eta)) \geq 0 \) (or \( \leq 0 \)) on the interval \( |x - p^*| < \delta \). Thus it follows that
\[
\left| \int_{|x-p^*|<\delta} (\text{sgn}(f(x)) - f^{|\beta|}(x)) \, dx \right| = \left| - \int_{p^* - \delta}^{p^* + \delta} f^{|\beta|}(x) \, dx \right|
\leq \frac{\tau_{\min}(\beta)}{p^* - \delta} \int_{p^* - \delta}^{p^* + \delta} (x - p^*) \, dx + \frac{\tau_{\max}(\beta)}{p^* - \delta} \int_{p^* - \delta}^{p^* + \delta} (x - p^*) \, dx
= \frac{\delta^2}{2} \{ \tau_{\max}(\beta) - \tau_{\min}(\beta) \}.
\]

On the other hand, for \( x > p^* + \delta, f(x) > \epsilon \), and thus we have
\[ |\text{sgn}(f(x)) - f^{|\beta|}(x)| = |1 - \text{tanh}(\beta f(x))| = \frac{2}{e^{2\beta f(x)}} + 1 \leq 2e^{-2|\beta|}. \]
Similarly, for \( x < p^* - \delta, f(x) < -\epsilon \) so that
\[ |\text{sgn}(f(x)) - f^{|\beta|}(x)| = |1 + \text{tanh}(\beta f(x))| = \frac{2e^{2\beta f(x)}}{e^{2\beta f(x)} + 1} \leq 2e^{-2|\beta|}. \]
Therefore,
\[ \left| \int_{|x-p^*|>\delta} (\text{sgn}(f(x)) - f^{|\beta|}(x)) \, dx \right| \leq Ce^{-2|\beta|} \]
for some constant \( C > 0 \), and the proof is completed. \( \square \)

Theorem 1 implies that for sufficiently large \( \beta \) the error of \( q(\beta) \) depends on an error of the numerical integration, \( \tilde{I}(f^{|\beta|}) \) of the sigmoid transform \( f^{|\beta|} \). That is, we may obtain an excellent approximation \( q(\beta) \) to the root \( p^* \) as long as sufficiently accurate evaluation of \( \tilde{I}(f^{|\beta|}) \) is possible for \( \beta \) large enough. For \( \beta \) not large enough, however, the upper bound of the error in (16) depends mainly on a deviation of the function \( F(\beta, x) = f^{|\beta|}(x) \) over the interval \( 0 < |x - p^*| < \delta \).

5. Numerical examples

In this section we take three examples, \( f_i(x) = 0, i = 1, 2, 3 \), for which the traditional Newton’s method is not available due to the existence of the extreme slope or bad oscillation. The graphs of each function \( f_i \) and its sigmoid transform \( f_i^{|\beta|} \) needed in the present method are illustrated in Fig. 3.

We have used the Mathematica V.5 with 25 digits of precision in fulfillment of the required numerical integrations.

Example 1
\[ f_1(x) = (1 - x) \log x + 2 = 0, \quad 0 \leq x \leq 1 \]
whose root is \( p^* \approx 0.1066027644557068 \). Since \( f_1'(x) = O(1/x) \) near \( x = 0 \) and \( f_1'(1) = 0 \), any endpoint of the interval \( (0,1) \) is not suitable for an initial guess of the Newton’s method. However, this situation is not a troublesome in applying the present method which does not require any information of the derivative. The second column in Table 1 shows absolute errors of the present approximation \( q(\beta) \) given in (6) for various values of \( \beta \).
It is suspected that the slow convergence rate of $q(\beta)$, with respect to $\beta$, is due to a relatively large deviation of a function $F_1(\beta, \eta) = \beta f_1'(\eta) \sech^2(\beta f_1(\eta))$, in Theorem 1, in the neighborhood of the root $p^*$.

To improve the error of $q(\beta)$, the Newton’s method can be applied with an initial guess $q_0 = q(\beta)$ as given in (7) or (8). Numerical results are included in Table 2 compared with those of the traditional Newton’s method.

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$f_1$</th>
<th>$f_2$</th>
<th>$f_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>$8.4 \times 10^{-5}$</td>
<td>$4.3 \times 10^{-4}$</td>
<td>$4.4 \times 10^{-6}$</td>
</tr>
<tr>
<td>40</td>
<td>$2.1 \times 10^{-5}$</td>
<td>$3.0 \times 10^{-6}$</td>
<td>$7.1 \times 10^{-9}$</td>
</tr>
<tr>
<td>80</td>
<td>$5.2 \times 10^{-6}$</td>
<td>$2.7 \times 10^{-10}$</td>
<td>$8.6 \times 10^{-10}$</td>
</tr>
<tr>
<td>120</td>
<td>$2.3 \times 10^{-6}$</td>
<td>$3.2 \times 10^{-14}$</td>
<td>$1.7 \times 10^{-9}$</td>
</tr>
<tr>
<td>400</td>
<td>$2.1 \times 10^{-7}$</td>
<td>$1.1 \times 10^{-16}$</td>
<td>$1.1 \times 10^{-9}$</td>
</tr>
</tbody>
</table>

It is suspected that the slow convergence rate of $q(\beta)$, with respect to $\beta$, is due to a relatively large deviation of a function $F_1(\beta, \eta) = \beta f_1'(\eta) \sech^2(\beta f_1(\eta))$, in Theorem 1, in the neighborhood of the root $p^*$.

To improve the error of $q(\beta)$, the Newton’s method can be applied with an initial guess $q_0 = q(\beta)$ as given in (7) or (8). Numerical results are included in Table 2 compared with those of the traditional Newton’s method.

Table 2

| $k$ | $|p^* - p_k|$ with $p_0 = 0.001$ | $|p^* - q_k|$ with $q_0 = q(10)$ | $|p^* - q_k|$ with $q_0 = q(80)$ |
|-----|----------------|----------------|----------------|
| 0   | $1.1 \times 10^{-1}$ | $3.4 \times 10^{-4}$ | $5.2 \times 10^{-6}$ |
| 1   | $1.0 \times 10^{-1}$ | $8.0 \times 10^{-7}$ | $1.9 \times 10^{-10}$ |
| 2   | $8.3 \times 10^{-2}$ | $2.9 \times 10^{-12}$ | 0 |
| 3   | $4.6 \times 10^{-2}$ | 0 | 0 |
| 4   | $1.1 \times 10^{-2}$ | 0 | 0 |
| 5   | $6.2 \times 10^{-4}$ | 0 | 0 |
with an initial guess $p_0 = 0.001$. Therein all the real numbers with magnitude less than $1 \times 10^{-18}$ in the computation were replaced by 0.

**Example 2**

$$f_2(x) = -(x - \frac{\pi}{6})^3 = 0, \quad -1 \leq x \leq 1,$$

which has a multiple root $p^* = \pi/6$ and it has an extremely small slope near the root.

Since $f_2'(\eta) = -3(\eta - p^*)^2$, a deviation of $F_2(\beta, \eta) = \beta f_2'(\eta) \text{sech}^2(\beta f_2(\eta))$ in Theorem 1 is small near the root $p^*$. Therefore we can expect better approximations $q(\beta)$ than those in the case of Example 1. Numerical results of the present approximation $q(\beta)$ are given in the third column of Table 1.

**Example 3**

$$f_3(x) = -(x - \sqrt{2}) \sin[1/(x - \sqrt{2})] - 2x + 2\sqrt{2} = 0, \quad 1 \leq x \leq 2$$

whose exact root is $p^* = \sqrt{2}$. We note that $f_3(x)$ is constantly oscillating in the neighborhood of the root $p^*$ and the first derivative satisfies $\lim_{x \to p^*} f_3'(x) = \infty$. Thus any iterative method like the Newton’s method is not available in this case, however close to $p^*$ an initial guess is.

Based on Theorem 1, unfortunately, we are not confident of a good approximation to $p^*$ by the present method because the derivative $f_3'(x)$ oscillates between 0 and $\infty$. Nevertheless the present method produces satisfactory results as given the fourth column in Table 1. And a halt of convergence for $\beta > 80$ seems to be due to an error of the numerical integration $\tilde{I}(\beta_{\beta/3})$ for $f_3(\beta)$ which has lasting wiggles near the root.

In addition, we consider an example with two simple roots:

**Example 4**

$$g(x) = \left(x - \frac{1}{2}\right) \{e^{\sin[10(x - \pi)]} + 4(x - \pi) - 1\} = 0, \quad 0 \leq x \leq 4$$

which has two roots, $p_1^* = \frac{1}{2}$ and $p_2^* = \pi$. Fig. 4 shows the graphs of $g(x)$ and its sigmoid transforms $g^{[\beta]}(x)$ for $\beta = 1, 40$.

Replacing $\text{sgn}(g)$ by $g^{[\beta]}$ in the formulas (12)–(15), we have approximate roots $q_1(\beta)$ and $q_2(\beta)$ for $p_1^*$ and $p_2^*$, respectively, as below.

$$q_1(\beta) = c + \frac{A}{2} + \frac{\sqrt{-A^4 + 4AB}}{2\sqrt{3}A}, \quad q_2(\beta) = q_1(\beta) - A,$$  

(17)

![Fig. 4. Sigmoid transforms $g^{[\beta]}(x)$, $\beta = 1, 40$, for $g(x)$ in Example 4.](image-url)
\[ A = \frac{1}{2} \{ a - b + \text{sgn}(g(a)) \cdot I_0(g[b]) \}, \]
\[ B = \frac{1}{2} \left\{ \frac{1}{4} (a - b)^3 + 3 \text{sgn}(g(a)) \cdot I_2(g[b]) \right\} \]

and

\[ I_m(g[b]) = \int_a^b (x - c)^m g[b](x) \, dx, \quad m = 0, 2. \]

6. Conclusions

In this paper, we have proposed a non-iterative method to estimate a solution \( p^* \) of a non-linear equation \( f(x) = 0 \). Using the so-called sigmoid transform \( f^{[\beta]}(x) = \tanh(\beta f(x)) \), we have derived the formula (6) for \( q(\beta) \approx p^* \) where only a numerical integration of \( f^{[\beta]}(x) \) is required. Conclusive remarks on the present method are summarized as follows.

(i) Once an accurate numerical integration of \( f^{[\beta]}(x) \) is possible, we can expect an excellent approximation \( q(\beta) \) for large \( \beta \).

(ii) The present method is always useful regardless of the condition of the derivative \( f'(x) \).

(iii) If the present approximation \( q(\beta) \) in (6) is not sufficient, we may additionally apply an iterative method with an initial guess \( q_0 = q(\beta) \) to obtain further accurate result. That is, the present method provides a good initial guess of an iterative method without any constraint.

Additionally, we have derived another formula (11) using a signum function. As mentioned in Section 3 partially, we may extend this idea to find a finite number of solutions simultaneously.

References