Bidding for money

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Received 2 March 2006; final version received 18 October 2006

Abstract

We analyze monetary exchange in a model that allows for directed search and multilateral matches. We consider environments with divisible goods and indivisible money, and compare the results with those in models that use random matching and bilateral bargaining. Two different pricing mechanisms are used: ex ante price posting, and ex post bidding (auctions). Also, we consider settings both with and without lotteries. We find that the model generates very simple and intuitive equilibrium allocations that are similar to those with random matching and bargaining, but with different comparative static and welfare properties. © 2006 Published by Elsevier Inc.

JEL classification: C78; D44; E40

Keywords: Money; Directed search; Multilateral matching

1. Introduction

Wallace [25,26] stressed the importance of continuing the development of monetary theory, while still retaining the spirit of the frictions that current models are designed to capture. As a general principle, this development involves the replacement of convenient assumptions with better alternatives wherever possible. When applying this principle to the search-based models of money as pioneered by Kiyotaki and Wright [13,14] we believe that two convenient, and

\* The inspiration for this paper came out of a workshop at the University of Copenhagen in June 2005. An earlier version was circulated under the title “Monetary Exchange with Multilateral Matching”.

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0022-0531/$ - see front matter © 2006 Published by Elsevier Inc.
doi:10.1016/j.jet.2006.10.009

Please cite this article as: B. Julien, et al., Bidding for money, Journal of Economic Theory (2006), doi: 10.1016/j.jet.2006.10.009
related, assumptions stand out as candidates for improvement: random matching and bilateral bargaining. Random matching is problematic because it removes the search process from the realm of choice theory by making it purely a technological phenomenon that agents face. Similarly, the assumption of bilateral bargaining imposes a particular price determination mechanism that has no justification from the theory of mechanism design and, moreover, impels modelers to restrict attention to bilateral matches.

In this paper, we explore how robust the results from these models are when these assumptions are replaced with ones that are (we suggest) somewhat nicer. In particular, we introduce directed search (where buyers can select over sellers, and sellers use pricing mechanisms to induce selection) and multilateral matching into a model which is, otherwise, standard, based on Shi [21] and Trejos and Wright [24], hereafter STW, with divisible goods and indivisible money. The directed search framework we use is similar to that used in models of the labor market, as in for example Julien et al. [10] and Burdett et al. [4].

Once search is directed, guided by prices, the arrival rate that sellers face is no longer parametric but, rather, a function of the particular price associated with a seller. As a result, directed search allows for the existence of monetary equilibria with price posting—something that cannot be obtained in this framework with random matching. In all the settings that we consider, we identify conditions under which stationary monetary equilibria exist, and characterize the properties of these equilibria. We also compare equilibrium allocations with those that maximize the surplus of matches. We find that, under both pricing mechanisms, and in common with the STW model, where the stationary monetary equilibrium exists, it is unique. With ex ante posted prices, conditional on a minimum value of the discount factor, there exists a unique value of the money supply that is consistent with match efficiency. With ex post bidding, by way of contrast, match efficiency can never be obtained. This last result is a consequence of the price dispersion inherent with ex post bidding: efficiency requires a unique price (and, thus, production level), and ex post bidding delivers two distinct prices.

When the model is generalized to allow for lotteries we find that, under ex ante price posting, in the region of the parameter space where lotteries are not degenerate, the unique monetary equilibrium always attains match efficiency. A similar result is found under ex post bidding: in a certain region of the parameter space, non-degenerate lotteries are offered both when one and when many buyers approach sellers. In this region, the purchase probabilities differ under the two different conditions (one or many buyers per seller) but the quantities produced are identical. Moreover, under these conditions, the unique quantity produced coincides exactly with the quantity required for efficiency. Outside these regions, with degenerate lotteries, the equilibria in these models are identical to those in the model with no lotteries.

1 See Coles [5] for an analysis of a model with two-sided indivisibility and directed search. Rocheteau and Wright [18] also analyze monetary models with directed search. However, directed search in their paper takes a different form, based on the competitive search framework developed by Moen [15]. In that setting the aggregate market is divided into submarkets, each of which is frictional, but where movement across submarkets is directed by market-makers who perform a role similar to Walrasian auctioneers. See King [12] or Rogerson et al. [19] for survey discussions of directed search.

2 With purely random matching, as in Curtis and Wright [6], sellers are aware that the arrival rate of buyers is independent of the price they post, and so post a price equal to the entire surplus of the match—eliminating the monetary equilibrium. This is a variant of the Diamond paradox [7].
The remainder of the paper is structured as follows. Section 2 introduces the physical environment, and characterizes efficient allocations. Section 3 then considers equilibria in the model without lotteries, under ex ante posting and ex post bidding. Section 4 covers the model with lotteries, using both mechanisms. Section 5 offers some concluding comments and suggestions for further research. Proofs of some of the propositions are provided in the appendix.

2. The physical environment

The economy contains a continuum of infinitely lived agents with measure normalized to one. Time is discrete and the agents have a rate of time preference \( r > 0 \) or, equivalently, a discount factor, \( \beta = 1/(1 + r) \). There is also a continuum of perfectly divisible non-storable goods, where these goods are produced only after an agreement to exchange. Each agent produces and consumes these goods, but they are specialized as follows. Each agent wishes to consume only a certain fraction \( x \) of the goods and, for any agent producing a good, only a fraction \( x \) of the population consumes that good. We assume that no agents consume the good they produce themselves—hence, trade is essential. We also assume that, when any two agents meet, direct barter is impossible—they cannot reciprocally produce a good for each other. To keep the notation to a minimum, we set \( x = 1 \). Hence, when any two agents encounter each other, there is always a single coincidence of wants, and never a double coincidence of wants.

For any agent, in any period, let \( u(q) \) be the utility of consuming \( q \) units of a good that she wishes to consume and \( c(q) \) the disutility of producing \( q \) units of her own production good. As is standard, we assume \( u(0) = c(0) = 0, u'(0) > c'(0) = 0, u'(q) > 0, c'(q) > 0, u''(q) \leq 0, \) and \( c''(q) \geq 0 \), for \( q > 0 \), with at least one of the weak inequalities strict. For future reference, we define \( q^* \) by \( u'(q^*) = c'(q^*) \). Also, there is a \( \hat{q} > 0 \) such that \( u(\hat{q}) = c(\hat{q}) \). It is important to understand, though, what is meant by quantity in this setting. In STW, \( q \) is interpreted as either the quality of a service or the quantity of a perishable good. In this model, the service interpretation is valid, but the goods interpretation requires another assumption: due to the possibility of multilateral matching, to keep things simple, we restrict sellers so that they can serve only one buyer at a time.

A fraction \( M \) of agents is initially endowed with one unit each of indivisible fiat money: an intrinsically useless object that is storable. We assume that agents holding money cannot produce. Thus no one can ever acquire more than one unit of money, and all agents hold either 0 or 1 unit of money. Since money is indivisible, at each date we can partition the agents into two different groups: “buyers”, who currently hold money, and “sellers”, who do not. The proportion of buyers is therefore given by \( M \) and the proportion of sellers by \( (1 - M) \). Let \( \Phi \) denote the ratio of buyers to sellers (market tightness), thus: \( \Phi = M/(1 - M) \). In any time period, whenever a buyer and seller meet and agree to exchange, the seller produces a quantity \( q \) of the good, and exchanges this amount in return for the unit of money that the buyer holds. The buyer then consumes the good, within the period. Moving into the subsequent period, the agent who was the buyer in the previous period becomes a seller in the new period, and the agent who was a seller now holds a unit of money and thereby becomes a buyer.

2.1. Constrained efficient allocations

In this subsection, we restrict our attention to allocations where, at the beginning of each period, each buyer is assigned randomly, arriving at each seller with equal probability. In subsequent sections, we show that this behavior is an equilibrium outcome. Given the randomness of the
assignment, and the fact that the value of $M$ influences the ratio of buyers to sellers, what values of $M$ and $q$ maximize surplus in a stationary allocation? To answer this question, we consider two different cases. In the first case, a planner can choose $M$ and $q$ independently. Here, choosing $q$ affects the intensive margin or “match efficiency”: for any given match, a choice of $q$ will influence the surplus of the match: $[u(q) - c(q)]$. Choosing $M$, on the other hand, affects the extensive margin: the choice of $M$ determines the number of matches induced by buyers randomizing over sellers is given by $X(M) = (1 - M) \left(1 - e^{-(M/(1-M))}\right)$.\(^3\) Within any period, aggregate surplus is given by the product of these two expressions.

The planner’s problem is

$$\max_{[M,q]} Y = (1 - M) \left(1 - e^{-(M/(1-M))}\right) [u(q) - c(q)].$$

The solution to this problem is given by

$$M^* \approx 0.533, \quad u'(q^*) = c'(q^*).$$

These two equations characterize optimality along the extensive margin and the intensive margin, respectively. The second condition is identical to the condition identified in STW, but the first is different. In the STW model, the number of matches is proportional to $M(1 - M)$, which has its maximum at $M = \frac{1}{2}$. Exponential matching of the type considered here induces an asymmetry, not found with Poisson arrival rates, due to the different matching roles assigned to buyers and sellers. Buyers distribute themselves over sellers, and maximizing the number of matches requires more buyers than sellers.

In the second case, we consider a problem where the planner can directly choose only $M$, and must accept a functional relationship: $q = q(M)$, where $q'(M) < 0$. Thus, in this case, the choice of $M$ determines both the intensive and extensive margins.\(^4\)

The planner’s problem is

$$\max_{[M]} Y = (1 - M) \left(1 - e^{-(M/(1-M))}\right) [u(q(M)) - c(q(M))].$$

There are two points to consider here. First, it is quite clear that, in general, we should not expect that the $M$ that maximizes this expression will equal $M^*$ above, or the $M$ that induces $q^*$ above. That is, the optimal $M$ must now trade off the intensive and extensive margins. This is a feature shared with the standard STW model. Secondly, there is no reason to expect that this expression is concave in $M$. Thus, examination of the first-order condition

$$X'(M) (u(q(M)) - c(q(M))) + X(M) (u'(q(M)) - c'(q(M))) q'(M) = 0$$

may not tell us anything about the optimum.

For this reason, throughout the remainder of this paper, we will continue to follow the literature by using $q^*$ as the key reference for efficiency.

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\(^3\)To see this, note that $(1 - M)$ is the number of sellers, $\Phi = M/(1 - M)$ is the buyer/seller ratio, and $(1 - e^{-\Phi})$ is the probability, for any seller, that she will be matched with at least one buyer.

\(^4\)A functional relationship of the type $q = q(M)$ exists in the equilibria we explore in subsequent sections.
3. Equilibrium allocations without lotteries

In this section, we consider equilibrium allocations under two alternative pricing mechanisms. Under the first mechanism, producers announce the terms of trade prior to matching. In this setting, prices are determined ex ante through posting. Under the second, the terms of trade are determined by local market conditions, which are revealed only after matching. In that setting, prices are determined ex post through a bidding game or auction.

3.1. Ex ante pricing (price posting)

In this subsection, we consider a directed search model where, in each time period, buyers’ selection of a trading partner is directed by an ex ante quantity announcement, and commitment, by sellers. Each seller announces, in advance of matching, the quantity level, \( q \), implying a nominal price \( p = 1/q \), at which they will produce upon a match with a buyer. Buyers observe the array of quantities posted and choose one and only one seller to approach. A lack of coordination among buyers implies that they use mixed strategies in selecting over sellers. There is then a possibility for several buyers to select the same seller. We assume that each seller can only serve one buyer per period and, when facing several buyers, one is picked at random.\(^5\) In effect, here, sellers bid for money.

When solving a directed search problem such as this, there are two approaches that are commonly used. One is to set the problem up in a way similar to Peters [16], Julien et al. [10] and Burdett et al. [4], and solve for a subgame perfect Nash equilibrium of a finite game and then let take the limit as the number of agents goes to infinity while keeping the buyer–seller ratio constant. The equilibrium mixed strategies, used by buyers in the finite game, generate a binomial distribution over sellers which, in large markets, is approximated by an exponential distribution.

The second approach is based on the market utility property: there exists a maximum equilibrium expected utility buyers can receive from participating in the market. This approach is used in Acemoglu and Shimer [1] and Peters [17]. The two approaches are known to give identical results in large markets, but the second is easier to present and solve, so we use it here.\(^6\) We solve first for the equilibrium posted \( q \) in a particular period, and then solve for steady state. Let \( \bar{V}_1 \) be the value for a buyer entering the market with one unit of money and \( \bar{V}_0 \) the value for a seller entering the market with no money. These values embed the equilibrium quantity, \( Q \), simultaneously posted by sellers in each period.

Let \( \phi(q) \) (not necessarily equal to \( \Phi \)) be the expected queue length for a seller posting \( q \). Let \( \xi(q) \) be the probability of attracting at least one buyer, and hence, the probability of trade when posting \( q \). The value, to a seller, of posting \( q \), is given by

\[
V_0(q) = \xi(q)[\beta \bar{V}_1 - c(q)] + [1 - \xi(q)]\beta \bar{V}_0.
\]

Each seller posts a quantity that he is willing to trade for one unit of money, and if many buyers show up, one buyer is chosen at random for the trade. Let \( \psi(q) \) be the probability that a buyer

\(^{5}\) This differs from existing directed search models of the labor market in the following way: in those models, sellers compete in prices and generate a queue for their indivisible good (labor). Here, the indivisibility is on the other side of the market (buyers), and sellers adjust their quantity to attempt to acquire indivisible fiat money.

\(^{6}\) In a previous version of the paper we used the first approach. See Julien et al. [11].
gets served when selecting a seller posting \( q \). The value, to a buyer, from selecting a seller posting \( q \), is given by

\[
V_1(q) = \psi(q) [u(q) + \beta \bar{V}_0] + [1 - \psi(q)]\beta \bar{V}_1. 
\]  
(2)

The arrival rates for sellers and buyers are, respectively,

\[
\xi(q) = 1 - e^{-\phi(q)} 
\]  
(3)

and

\[
\psi(q) = \frac{1 - e^{-\phi(q)}}{\phi(q)}. 
\]  
(4)

Let \( \Delta = \bar{V}_1 - \bar{V}_0 \). The incentive constraint, within a match, for a seller to be willing to trade, at a given posted \( q \), is \( \beta \bar{V}_1 - c(q) \geq \beta \bar{V}_0 \), or:

\[
\beta \Delta - c(q) \geq 0. 
\]  
(5)

Similarly, the incentive constraint, within a match, for a buyer to be willing to trade is \( u(q) + \beta \bar{V}_0 \geq \beta \bar{V}_1 \), or

\[
u(q) - \beta \Delta \geq 0. 
\]  
(6)

These constraints imply a maximum \( \bar{q} \), defined by \( c(\bar{q}) = \beta \Delta \), that a seller is willing offer, and a minimum \( \underline{q} \), defined by \( u(\underline{q}) = \beta \Delta \) that a buyer would be willing to accept.

A seller solves the following problem:

\[
\max_{q \in [\underline{q}, \bar{q}]} \left\{ V_0(q) = \xi(q)[\beta \bar{V}_1 - c(q)] + [1 - \xi(q)]\beta \bar{V}_0 \right\} 
\]  
(7)

s.t.

\[
V_1(q) = \psi(q) [u(q) - \beta \Delta] + \beta \bar{V}_1 = \bar{V}_1, 
\]  
(8)

where the constraint determines how the expected queue length \( \phi(q) \) changes with the posted \( q \). The expected queue length is determined by buyers being indifferent between selecting the seller posting \( q \) and any other sellers in the market. The value of selecting the seller posting \( q \) must be equal to the buyer’s equilibrium market utility value \( \bar{V}_1 \). It follows that, for any \( q \leq \bar{q} \), we have \( V_1(q) < \bar{V}_1 \) or \( \phi(q) = 0 \): no buyer would select a seller posting \( q \leq \bar{q} \) even if he were to be served with probability 1.

Taking the derivative of (7) yields

\[
V_0'(q) = e^{-\phi(q)} \phi'(q) \beta \Delta - \xi(q) c'(q). 
\]  
(9)

Using the implicit function theorem on (8) yields

\[
\phi'(q) = \frac{u'(q) \phi(q) (1 - e^{-\phi(q)})}{[1 - e^{-\phi(q)} - \phi(q) e^{-\phi(q)}][u(q) - \beta \Delta]}. 
\]  
(10)

It is easily shown that \( V_0(q) \) is concave in \( q \) and \( \phi'(q) > 0 \). Using (10) in (9), and setting (9) equal to zero yields a necessary condition for the equilibrium posted \( q \), as summarized in the following lemma.

\footnote{A derivation of these arrival rates is provided in the appendix.}
Lemma 1. The Nash equilibrium quantity announcement by a seller in any period is characterized by

\[
\frac{c'(q)}{u'(q)} = \frac{\phi(q)e^{-\phi(q)}}{1 - e^{-\phi(q)} - \phi(q)e^{-\phi(q)}} \left[ \beta \Delta - c(q) \right].
\]

Let the solution to (11) be \( q = q(\Delta) \). A seller takes \( \Delta \) as given when choosing his quantity announcement in any period—the solution represents the best response of a particular seller to the aggregate variable \( \Delta \). In order to solve for equilibrium, we write the value functions \( \bar{V}_0 \) and \( \bar{V}_1 \), letting \( Q > 0 \) and \( \bar{V}_1 \) be the quantity posted by sellers in any period. We have

\[
\bar{V}_0 = \xi[\beta\Delta - c(Q)] + \beta \bar{V}_0
\]

(12) and

\[
\bar{V}_1 = \psi[u(Q) - \beta\Delta] + \beta \bar{V}_1.
\]

(13)

Notice that \( \bar{V}_0 \) and \( \bar{V}_1 \) are functions \( Q \) and \( \Delta \). Furthermore, the arrival rates are now expressed as \( \xi = 1 - e^{-\Phi} \) and \( \psi = \frac{1-e^{-\Phi}}{\Phi} \) to reflect the fact that buyers select all sellers with the same probability. (That is: \( \phi(q) = \Phi = \frac{M}{1-\beta} \).)

Using (12) and (13) and the definition of \( \Delta \), we solve for

\[
\Delta(Q) = \bar{V}_1(Q) - \bar{V}_0(Q) = \frac{\psi u(Q) + \xi c(Q)}{(1 - \beta(1 - \psi - \xi))} > 0
\]

(14)

as long as \( Q > 0 \). The relationship \( \Delta = \Delta(Q) \) is an aggregate consistency requirement. It determines the value of \( \Delta \) when all sellers are committing to trade \( Q \) for a unit of fiat money (see [23]).

Observe that there is a link between the incentive constraints (5) and (6) and the steady-state values. In equilibrium, \( q = Q \), and substitution of \( \bar{V}_0(q) \) and \( \bar{V}_1(q) \) into the constraints reveals that (5) is satisfied whenever \( \bar{V}_0(q) > 0 \) and (6) holds whenever \( \bar{V}_1(q) > 0 \). Furthermore, since \( u(q) > c(q) \) for all \( q \in (0, \hat{q}) \) and \( (1 - \beta(1 - \xi)) > \beta \xi \) for all \( \beta \) and \( \Phi \), it follows that \( \bar{V}_1(q) > 0 \) for all parameter values and \( q \in (0, \hat{q}) \). In the steady state, a buyer would always want to participate in the market because money has value as a medium of exchange. Therefore, to show existence of a steady-state monetary equilibrium, we only need to focus on the conditions under which \( \bar{V}_0(q) > 0 \).

Definition 1. A symmetric steady-state equilibrium is a triple \((q, \bar{V}_0, \bar{V}_1)\) satisfying the following conditions:

(i) \( q = q(\Delta) = Q \) solves the seller’s problem (7) of the ex ante game characterized by Eq. (11) taking \( \bar{V}_0 \) and \( \bar{V}_1 \) as given;

(ii) the value functions \( \bar{V}_0(Q) \) and \( \bar{V}_1(Q) \) satisfying (14), taking \( Q \) as given;

(iii) the incentive constraints (5) and (6) evaluated at \( q \) are satisfied.

The steady-state equilibrium is a joint solution to \( q = q(\Delta) \) and \( \Delta = \Delta(q) \). The following proposition characterizes the solution.

Proposition 1. For any \( \beta \in (0, 1) \) and \( M \in (0, 1) \) (hence \( \Phi \)), there exists a unique symmetric steady state monetary equilibrium with \( q \in (\hat{q}, \bar{q}) \) and \( \bar{q} < \hat{q} \).
Proof. If we set \( q = Q \) and use \( \Delta = \Delta(q) \) into (11) and write

\[
T(q) = [\beta \Delta(q) - c(q)]u'(q) - [u(q) - \beta \Delta(q)]c'(q)/g(q),
\]
where \( g(q) = \frac{\phi(q)e^{-\phi(q)}}{1-e^{-\phi(q)\Delta(q)}e^{-\phi(q)}} \). This implies that \( T(q) = 0 \) represents the necessary and sufficient condition for a steady-state equilibrium. Observe that \( T(q) > 0 \) since \( u(q) = \beta \Delta(q) \) and, using L’Hospital’s rule at \( \phi(q) = 0, 1/g(q) = 0 \). Next, \( T(q) = -[u(q) - \beta \Delta(q)]c'(q)/g(q) < 0 \). Since \( T(q) \) is continuous, there exists \( q \in (\bar{q}, \bar{q}) \) such that \( T(q) = 0 \). This establishes the existence of an unconstrained equilibrium.

To show uniqueness of \( q \in (\bar{q}, \bar{q}) \), rewrite \( T(q) = 0 \) as

\[
\frac{c'(q)}{u'(q)} = g(q) \frac{[\beta \Delta(q) - c(q)]}{[u(q) - \beta \Delta(q)]}.
\]

The left-hand side is positive at \( q \) and strictly increasing. The right-hand side increases without bound as \( q \to \bar{q} \), is strictly decreasing for all \( q \in (\bar{q}, \bar{q}) \), and equals zero at \( \bar{q} \). By continuity, there exists a unique solution such that \( T(q) = 0 \) in \( (\bar{q}, \bar{q}) \).

We must also show that \( \bar{q} < \bar{q} \). Suppose that \( \bar{q} = \bar{q} \), we have that \( V_0(\bar{q}) = \frac{1}{1-\beta(1-\psi)} c(\bar{q}) > 0 \) holds if \( (\beta\psi u(\bar{q}) - (1 - \beta(1 - \psi)) c(\bar{q})) > 0 \) or \( u(\bar{q})/c(\bar{q}) > (1 - \beta(1 - \psi))/\beta\psi > 1 = u(\bar{q})/c(\bar{q}) \), which implies that \( \bar{q} < \bar{q} \). The above demonstration is valid for all \( \beta \in (0, 1) \) and \( M \in (0, 1) \). Finally, notice that \( q \) is defined by \( u(q) = \beta \Delta \), and it is easily shown that as \( M \) goes to 1, or \( \Phi \) goes to infinity, \( q \to \bar{q} \) to zero.

3.1.1. Properties of the equilibrium

Proposition 2. For any given \( \beta \in (0, 1) \), the equilibrium \( q = q(\Phi, \beta) \) is strictly decreasing in \( M \) (and hence in \( \Phi \)) for all \( M \in (0, 1) \). For any given \( \Phi \), the equilibrium \( q \) is strictly decreasing in \( \beta \).

Proof. See the Appendix.

The first result in this proposition contrasts with the result, in STW, that \( q \) can be increasing in \( M \) when \( M \) is small and \( \beta \) is large. Trejos and Wright [24], for example, identify a critical value of \( M(\beta) \), beyond which \( q \) is strictly decreasing in \( M \). The above lemma also suggests that there may exist a value of \( M \) for which the equilibrium is efficient along the intensive margin. The following proposition establishes this fact.

Proposition 3. There exists a \( \beta > 0 \) such that for all \( \beta \geq \beta \), there exists a \( \bar{M} \) (hence a \( \bar{\Phi} \)) for which the monetary equilibrium is match efficient: \( q(\bar{\Phi}, \beta) = q^* \), where \( u'(q^*) = c'(q^*) \). Otherwise the monetary equilibrium is match inefficient as follows: if \( \Phi \geq \Phi \) then \( q(\Phi, \beta) \leq q^* \).

Proof. See the Appendix.

This result has an interesting analogy with the one derived by STW. It is known that if one uses generalized (as opposed to symmetric) Nash bargaining in STW, one can show that there exists a value of the bargaining power parameter for buyers, \( \theta \), such that \( q = q^* \), and otherwise if \( \theta \geq \bar{\theta} \) then \( q(\theta) \leq q^* \). The analogy comes from the fact that the buyer–seller ratio \( \Phi \) determines the
agents bargaining power in the ex ante pricing environment. Implicitly, the bargaining power of buyers in our model is equal to \(e^{-\phi M_{e^{-\phi}}}/(1-e^{-\phi})\), which is determined endogenously by \(M\). In fact, \(e^{-\phi M_{e^{-\phi}}}/(1-e^{-\phi})\) represents the probability that a buyer will be alone when selecting a seller, conditional on getting a match. Hence, this is the probability that when a buyer selects a seller, he will be served with probability one. The higher is this probability, the higher is the bargaining power for buyers, ex ante. Since \(e^{-\phi M_{e^{-\phi}}}/(1-e^{-\phi})\) is strictly decreasing in \(\phi\), an increase in \(\phi\) implies that there are now more buyers in the market, and hence, decreases their bargaining power. For relatively high buyer–seller ratios, sellers post a quantity lower than the efficient level. They can do so in equilibrium because they have high relative ex ante bargaining power in this case.

3.2. Ex post pricing (auctions)

In the previous subsection the terms of trade were determined prior to the matching allocation. This mechanism assumes a price commitment on the part of sellers: they charge the announced price, no matter how many buyers approach them. Here, we investigate an alternative where the terms of trade are determined once the number of buyers assigned to a seller is realized. Since we allow for the possibility of multilateral matches to form, auctions are the natural ex post mechanism to consider.

It is common knowledge that each buyer has only one unit of fiat money. Bidding, therefore takes the following structure. Because the margin of adjustment is on the sellers’ side (the quantity choice of the divisible good) a bid from a buyer is a proposition of a quantity required in exchange for his unit of fiat money, hence, bidding for money. It follows that, within a multilateral match, where one seller is matched with several buyers, competition between the buyers reduces the minimal quantity of output acceptable for the unit of money. (In effect, this is a procurement auction, where the unit of money corresponds to the fixed budget of the buyer, and the quantity level is analogous to the cost of input.) In all other respects, the model remains the same.

The matching process determines the number, \(n\), of buyers matched with a particular seller. Here, the outcome depends on the number of buyers bidding. The bidding outcomes are reduced to two possible cases: when \(n = 1\) (a pairwise match) and when \(n > 1\) (a multilateral match). Let the outcome of the auction be \(q_p\) when \(n = 1\) and \(q_m\) when \(n > 1\). Let \(q = (q_p, q_m) \in \mathbb{R}_+^2\) be the vector of possible outcomes from the auction in any period. The value function for a seller is given by

\[
V_0(q) = \xi_p[-c(q_p) + \beta V_1] + \xi_m[-c(q_m) + \beta V_1] + (1 - \xi_m - \xi_p)\beta V_0, \tag{15}
\]

where, for a seller, \(\xi_p = e^{-\phi}\) is the probability of a pairwise match, \(\xi_m = 1 - \phi e^{-\phi} - e^{-\phi}\) is the probability of a multilateral match, \(1 - \xi_m - \xi_p\) is the probability of no match. The values \(V_0\) and \(V_1\) are the values embedding the ongoing equilibrium bidding quantities.

The value function for a buyer is given by

\[
V_1(q) = \psi_p[u(q_p) + \beta V_0] + \psi_m[u(q_m) + \beta V_0] + (1 - \psi_m - \psi_p)\beta V_1. \tag{16}
\]
where, for a buyer, $\psi_p = e^{-\Phi}$ is the probability of a pairwise match for a buyer, $\psi_m = \frac{1-e^{-\Phi}}{\Phi}$ is the joint probability of a multilateral match and winning the bidding game, and $1 - \psi_m - \psi_p$ is the probability of not winning the bidding game.

We are now ready to consider the determination of quantities in equilibrium. From the structure of the auction, it should be clear that the quantity resulting from a pairwise match will be higher than from a multilateral match (see Lemma 2 below). In any pairwise matching event, the buyer is bidding alone and thus proposes a (high) quantity level solving

$$\bar{q} = \arg\max_q \left\{ u(q) + \beta V_0 \right\}$$
$s.t. \quad -c(q) + \beta V_1 \geq \beta V_0,$

(17)

where the constraint is required for the seller to participate in trade. $^9$ This has the obvious solution:

$$\bar{q} = c^{-1}(\beta \Delta),$$

(18)

where $\Delta = V_1 - V_0$ as before. Thus, $q_p = \bar{q}$.

In a multilateral matching event, the seller has multiple buyers and thus the buyers propose a quantity level that cannot be beat by other buyers. This is a simple Bertrand pricing game—the outcome of a simple auction. Therefore, the equilibrium bid is one where each buyer is indifferent between trading and not trading (i.e., remaining a buyer in the next period). Since all bids are the same in this scenario in equilibrium, whether or not a particular buyer wins the auction, he receives the same expected payoff. The equilibrium quantity can therefore be found from the following problem:

$$\underline{q} = \arg\max_q \left\{ -c(q) + \beta V_1 \right\}$$
$s.t. \quad u(q) + \beta V_0 \geq \beta V_1.$

(19)

This has the solution:

$$\underline{q} = u^{-1}(\beta \Delta).$$

(20)

Thus, $q_m = \underline{q}$. The following lemma verifies the existence of price dispersion in any candidate equilibrium and, as argued above, prices with multilateral matching are higher than with pairwise matching. $^{10}$

**Lemma 2.** For all $\Delta > 0$, $(q, \bar{q}) \in (0, \hat{q})^2$ and $\underline{q} < \bar{q}$, where $\hat{q}$ is defined by $u(\hat{q}) = c(\hat{q})$. Otherwise, $\Delta = 0$ if and only if $\underline{q} = \bar{q} = 0$, and $\Delta = \hat{\Delta} > 0$, if and only if $\underline{q} = \bar{q} = \hat{q}$.

**Proof.** In the appendix.

$^9$ In a pairwise match such as this, one could think of alternative ways of determining the price (in particular: bargaining). Here, to be consistent with auctions, in the event of pairwise matches, the buyer has the power to push the seller to her outside option with the minimum acceptable quantity demanded or required. Halko et al. [9] show that this mechanism is evolutionary stable, while the hybrid mechanism with bargaining is not. In Section 4, below, we also consider the introduction of lotteries with pairwise matches.

$^{10}$ Notice that price dispersion occurs in this auction setting where the buyer has the bargaining power in pairwise matches and the seller has the power with multilateral matches. Hence, here, the bargaining power within a match is determined by the matching process.
In steady state, using (18) and (20) in (15) and (16), and solving for $V_0(q) = \tilde{V}_0$ and $\tilde{V}_1(q) = \tilde{V}_1$ we find

$$\Delta = \tilde{V}_1 - \tilde{V}_0 = \frac{\psi_p u(\tilde{q}) + \xi_m c(\tilde{q})}{1 - \beta(1 - \psi_p - \xi_m)} > 0$$  \hspace{1cm} (21)

as long as $q > 0$. Using Lemma 2, it follows that $\tilde{V}_1 > 0$ for all parameter values. Once again, this implies that, for existence of a steady-state equilibrium we need only establish conditions under which $\tilde{V}_0 > 0$.

**Definition 2.** A steady-state equilibrium is a tuple $(q, \tilde{q}, \tilde{V}_0, \tilde{V}_1)$ satisfying the following conditions:

1. $q$ and $\tilde{q}$ satisfy (18) and (20), respectively, taking $\tilde{V}_0$ and $\tilde{V}_1$ as given;
2. $\tilde{V}_0$ and $\tilde{V}_1$ solve (15) and (16) for given $(q, \tilde{q})$.

**Proposition 4.** For any $\beta \in (0, 1)$ and $M \in (0, 1)$, there exists a unique symmetric steady state monetary equilibrium, with $0 < q < \tilde{q} < \hat{q}$.

**Proof.** First note that $q$ and $\tilde{q}$ are maximized values such that, respectively, a seller and a buyer is indifferent between trading or not. Therefore, we do not have to worry about participation constraints. For any $\beta \in (0, 1)$ and $M \in (0, 1)$ to show the existence of a monetary equilibrium with $0 < q < \tilde{q}$, insert the steady-state value functions into (18) and (20) to get

$$c(\tilde{q}) = \beta \frac{\psi_p u(\tilde{q}) + \xi_m c(\tilde{q})}{1 - \beta(1 - \psi_p - \xi_m)}$$  \hspace{1cm} (22)

and

$$u(q) = c(\tilde{q}).$$  \hspace{1cm} (23)

Therefore, quantities are linked by a strictly convex function $q = u^{-1}(c(\tilde{q}))$. Using this into (22) gives

$$c(\tilde{q}) = \beta \frac{\psi_p u(\tilde{q}) + \xi_m c(u^{-1}(c(\tilde{q})))}{1 - \beta(1 - \psi_p - \xi_m)} = H(\tilde{q}).$$

Define $T(\tilde{q}) = H(\tilde{q}) - c(\tilde{q})$. Clearly $T(\tilde{q})$ is continuous, $T(0) = 0$, and $T'(0) = \beta \frac{\psi_p u'(0)}{1 - \beta(1 - \psi_p - \xi_m)} > 0$ for all parameter values satisfying the assumptions of the proposition. Observe that $T(\tilde{q}) = -u(\tilde{q}) \left( \frac{1}{1 - \beta(1 - \psi_p - \xi_m)} \right) < 0$ meaning that $\tilde{q} = \hat{q}$ cannot be an equilibrium. By continuity and the Weierstrass Intermediate Value Theorem, there exists a $\bar{q} > 0$, such that $T(\bar{q}) = 0$ and $\bar{q} = u^{-1}(c(\tilde{q})) > 0$. This implies $\Delta > 0$ and from Lemma 2 we have $q < \bar{q}$. Therefore, $0 < q < \bar{q} < \hat{q}$.

To show uniqueness, observe that $H(\tilde{q})$ is strictly increasing and a linear combination of a concave and a convex function. This implies that there is a unique inflexion point below which $H(\tilde{q})$ is concave and above which $H(\tilde{q})$ is convex. Therefore, the function $T(\tilde{q})$ has also a unique concave and a unique convex portion. Since $T(0) = 0$, $T'(0) > 0$ and $T(\hat{q}) < 0$, it implies that there must be a unique value $\bar{q} \in (0, \hat{q})$ such that $T(\bar{q}) = 0$. Since $u(q) = c(\tilde{q})$ then $q > 0$ is also unique. \(\square\)
3.2.1. Properties of the equilibrium

Proposition 5. For any given $\beta \in (0, 1)$, the equilibrium values $(q(\Phi, \beta), \tilde{q}(\Phi, \beta)) \in (0, \hat{q})^2$ are decreasing in $M$ (and hence $\Phi$) for all $M \in (0, 1)$. For any given $\tilde{\Phi}$, the equilibrium values $(q, \tilde{q})$ are strictly increasing in $\beta$.

Proof. See Appendix.

Proposition 6. The steady-state equilibrium of the ex post pricing game is inefficient along the intensive margin.

Proof. From the assumptions on $u(q)$ and $c(q)$, we know that $q^*$ is unique. From Lemma 2, we know that, in any monetary equilibrium $q < \tilde{q}$. It follows immediately that we cannot have $\underline{q} = \tilde{q} = q^*$. □

It is clear that we can have either $\underline{q} = q^*$ or $\tilde{q} = q^*$, but never both. The inefficiency may entail either underproduction in all matches (with high $\phi$ or low $\beta$), overproduction in all matches (with low $\phi$ or high $\beta$), or underproduction in some and overproduction in others. Interestingly, even if the average production level is equal to the efficient level ($q^*$), the price dispersion inherent in auctions guarantees that the equilibrium will always be inefficient.\(^\text{11}\)

4. Lotteries

In this section we introduce lotteries as in Berentsen et al. [2] and Berentsen and Rocheteau [3]. Now, whenever a match is formed, the buyer may propose a lottery. In particular, the buyer proposes a contract asking the seller to produce a quantity $q$ and a lottery in which the seller receives the buyer’s money with probability $\tau \in [0, 1]$.

4.1. Ex ante pricing with lotteries

Under ex ante pricing, each seller can commit to a quantity $q \in \mathbb{R}_+$ and a probability of monetary transfer $\tau \in [0, 1]$ to induce buyer to select them as a potential trading partner. Using the market utility property once again, we solve for the equilibrium pair $(q, \tau)$. All else in this environment is identical to Section 3.1. Again, let $(\hat{V}_0, \hat{V}_1)$ be the values of entering the market with zero and one unit of money.

To derive the equilibrium ex ante posting, assume that a seller posts a pair $(q, \tau) \in \mathbb{R}_+\times[0, 1]$. This generates an expected queue length, $\phi(q, \tau)$, faced by the seller. The value for a seller is given by

$$V_0(q, \tau) = \xi(q, \tau)[-c(q) + \beta(\tau\tilde{V}_1 + (1 - \tau)\hat{V}_0)] + (1 - \xi(q, \tau))\beta\hat{V}_0,$$

where $\xi(q, \tau)$ is the probability of attracting at least one buyer, and hence trading. The value, to a buyer, from selecting a seller posting $(q, \tau)$ is given by

$$V_1(q, \tau) = \psi(q, \tau)[u(q) + \beta(\tau\hat{V}_0 + (1 - \tau)\tilde{V}_1)] + (1 - \psi(q, \tau))\beta\tilde{V}_1,$$

\(^\text{11}\) Also, this inefficiency is not due to the usual holdup problem common in search models with pairwise matching and bargaining. In particular, the equilibrium in this model can be rationalized as a directed search equilibrium. In models of this type, allowing sellers to announce reserve prices ex ante, and buyers directing their search using mixed strategies, the equilibrium reserve price equals the sellers’ outside option and buyers perfectly randomize as they do here (see [10]).
where \( \psi(q, \tau) \) is the probability that a buyer is served by selecting a seller. As above, we have
\[
\tilde{\zeta}(q, \tau) = 1 - e^{-\phi(q, \tau)}
\]
and
\[
\psi(q, \tau) = \frac{1 - e^{-\phi(q, \tau)}}{\phi(q, \tau)}.
\]

Let \( \Delta = \bar{V}_1 - \bar{V}_0 \) as before. The incentive constraint, within a match, for a seller to be willing to trade at a given posted pair \((q, \tau)\), is
\[
\beta(\tau \bar{V}_1 + (1 - \tau) \bar{V}_0) - c(q) \geq \beta \bar{V}_0,
\]
or
\[
\beta \tau \Delta - c(q) \geq 0.
\]  
Similarly, the incentive constraint within a match for a buyer to be willing to trade, at a given posted pair \((q, \tau)\), is
\[
u(q) + \beta(\tau \bar{V}_0 + (1 - \tau) \bar{V}_1) \geq \beta \bar{V}_1,
\]
or
\[
u(q) - \beta \tau \Delta \geq 0.
\]  
These constraints imply that, for any given \( \tau \), there is a minimum \( q \), where \( u(q) = \beta \tau \Delta \), that a seller is able to offer to make it worthwhile for a buyer to trade. Similarly, there is a maximum \( \bar{q} \) where \( c(\bar{q}) = \beta \tau \Delta \), that a seller is willing offer for it to be worthwhile to trade with a buyer. In addition, for any given \( q \in [q, \bar{q}] \), there is a minimum \( \tau \) where \( u(q) = \beta \tau \Delta \), that a buyer is willing to accept to make it worthwhile to select a seller. Notice that these bounds are functions of \( \Phi \) and \( \beta \).

The seller’s problem, after some algebraic manipulation, is
\[
\max_{\tau \in [\underline{\tau}, \bar{\tau}], q \in [q, \bar{q}]} \quad \tilde{\zeta}(q, \tau)[-c(q) + \beta \tau \Delta] + \beta \bar{V}_0 \tag{28}
\]
subject to
\[
\psi(q, \tau)[u(q) - \beta \tau \Delta] + \beta \bar{V}_1 = \bar{V}_1, \tag{29}
\]
where again, \( \bar{V}_1 \) is the equilibrium maximum market value a buyer can get by selecting any other sellers. Taking partial derivatives with respect to \( q \) and \( \tau \), yields:
\[
V_{0q}(q, \tau) = e^{-\phi(q, \tau)} \phi_q(q, \tau)[\beta \tau \Delta - c(q)] - \tilde{\zeta}(q, \tau) c'(q) \tag{30}
\]
and
\[
V_{0\tau}(q, \tau) = e^{-\phi(q, \tau)} \phi_\tau(q, \tau)[\beta \tau \Delta - c(q)] + \tilde{\zeta}(q, \tau) \beta \Delta. \tag{31}
\]

Using the Implicit Function Theorem on (29) yields
\[
\phi_q(q, \tau) = \frac{u'(q) \phi(q, \tau)(1 - e^{-\phi(q, \tau)})}{[1 - e^{-\phi(q, \tau)} - \phi(q, \tau)e^{-\phi(q, \tau)}][u(q) - \beta \tau \Delta]} \tag{32}
\]
and
\[
\phi_\tau(q, \tau) = \frac{-\beta \Delta \phi(q, \tau)(1 - e^{-\phi(q, \tau)})}{[1 - e^{-\phi(q, \tau)} - \phi(q, \tau)e^{-\phi(q, \tau)}][u(q) - \beta \tau \Delta]} \tag{33}
\]
As one would expect, to \( \phi_q(q, \tau) > 0 \) and \( \phi_\tau(q, \tau) < 0 \). It is easily shown that \( V_0(q, \tau) \) is concave in both arguments. Using (32) into (30) and (33) into (31), the necessary and sufficient conditions for an equilibrium is given by setting \( V_{0q}(q, \tau) \leq 0 \) (= 0 if \( q > \bar{q} \)) and \( V_{0\tau}(q, \tau) \leq \lambda \) (= 0 if \( \tau > \bar{\tau} \)), where \( \lambda \) is the non-negative multiplier for the constraint that \( \tau \) cannot exceed 1. These conditions are summarized in the following lemma.
Lemma 3. The Nash equilibrium posting \((q, \tau)\) by a seller in any period is characterized by
\[
\frac{\phi(q, \tau)e^{-\phi(q, \tau)}}{1 - e^{-\phi(q, \tau)} - \phi(q, \tau)e^{-\phi(q, \tau)}} \frac{\beta \tau \Delta - c(q)}{[u(q) - \beta \tau \Delta] \leq c'(q) / u'(q)}.
\]
\[\tag{34}\]
and
\[
1 - \frac{\phi(q, \tau)e^{-\phi(q, \tau)}}{1 - e^{-\phi(q, \tau)} - \phi(q, \tau)e^{-\phi(q, \tau)}} \frac{\beta \tau \Delta - c(q)}{[u(q) - \beta \tau \Delta] \leq \frac{\lambda}{\beta \Delta(1 - e^{-\phi(q, \tau)})}}.
\]
\[\tag{35}\]
Let the solutions to (34) and (35) be \(q = q(\tau, \Delta) > q\) and \(\tau = \tau(q, \Delta) > 0\). This is a seller’s best response to the aggregate \(\Delta\). In order to solve for the steady-state equilibrium, we write the value functions \(\bar{V}_0\) and \(\bar{V}_1\), letting \(Q\) and \(T\) be the quantity and the probability of monetary transfer pair posted by sellers in any period. We have
\[
\bar{V}_0 = \bar{\zeta} [\beta T \Delta - c(Q)] + \beta \bar{V}_0
\]
and
\[
\bar{V}_1 = \psi [u(Q) + \beta T \Delta] + \beta \bar{V}_1.
\]
\[\tag{36} \tag{37}\]
Notice that \(\bar{V}_0\) and \(\bar{V}_1\) are functions \(Q, T\) and \(\Delta\). If every seller posts the same \((Q, T)\), the expected queue length faced by all sellers is identical and equal to the buyer–seller ratio, \(\Phi = M/(1 - M)\). The arrival rates are now expressed as \(\bar{\zeta} = 1 - e^{-\Phi}\) and \(\psi = \frac{1 - e^{-\Phi}}{\Phi}\) to reflect the fact that buyers select all sellers with the same probability.

Using (36) and (37) and the definition of \(\Delta\), we solve for
\[
\Delta(Q, T) = \frac{\psi u(Q) + \bar{\zeta} c(Q)}{1 - \beta (1 - T(\psi + \bar{\zeta}))} > 0.
\]
\[\tag{38}\]
The relationship \(\Delta = \Delta(Q, T)\) is again an aggregate consistency requirement. It determines value of \(\Delta\) when all sellers commit to trade \(Q\) for the probability \(T\) of getting a unit of fiat money.

Definition 3. A symmetric steady-state equilibrium is a tuple \((q, \tau, \bar{V}_0, \bar{V}_1)\) satisfying the following conditions:

(i) \(q = q(\tau, \Delta) = Q\) and \(\tau = \tau(Q, \Delta) = T\) solves the seller’s problem (28) of the ex ante game characterized by Eqs. (34) and (35) taking \(\bar{V}_0\) and \(\bar{V}_1\) as given;

(ii) the value functions \(\bar{V}_0(Q, T)\) and \(\bar{V}_1(Q, T)\) satisfy (36) and (37), taking \(Q\) as given;

(iii) the incentive constraints (26) and (27) evaluated at \(q\) and \(\tau\) are satisfied.

Proposition 7. For any \(\beta \in (0, 1)\) and \(M \in (0, 1)\) (hence \(\Phi\)), there exists a unique symmetric steady-state monetary equilibrium as follows:

(i) if \(0 < \Phi \leq \Phi_4\), then \(q = Q = q^*\) and \(\tau = T \in (\bar{\tau}, 1]\) is given by
\[
\bar{\tau} = \frac{(1 - \beta)}{\beta} \frac{([1 - e^{-\Phi} - \Phi e^{-\Phi}]u(q^*) + \Phi e^{-\Phi} c(q^*))}{(2e^{-\Phi} - 1 + \Phi e^{-\Phi})(1 - e^{-\Phi})(u(q^*) - c(q^*))},
\]
with \(\bar{\tau} > 0\);

(ii) if \(\Phi > \Phi_4\), then \(q < Q \leq q^*\) and \(\tau = T = 1\), where \(u'(q^*) = c'(q^*)\), \(\frac{\partial u}{\partial \Phi} < 0\) and \(\lim_{\Phi \to \Phi_4} q = q^*\).

Please cite this article as: B. Julien, et al., Bidding for money, Journal of Economic Theory (2006), doi: 10.1016/j.jet.2006.10.009
**Proof.** First consider the case where $\tau \in (\xi, 1]$, which implies that $\lambda = 0$. In a symmetric equilibrium $q = Q$ and $\tau = T$ and $\phi(q, \tau) = \Phi$. Both conditions (34) and (35) hold and they imply $u'(q) = c'(q)$ or $q = q^*$, with a solution $\tau = \tau(q^*, \Delta) = \tau(\Delta)$. Using $q^*$ into (34) implies that the condition can hold iff $c(q^*) < \beta \tau \Delta$ and $u(q^*) > \beta \tau \Delta$, meaning that $q^* \in (\bar{q}, \tilde{q})$.

Substituting $q^*$ into the steady-state values yields

$$\tilde{V}_0 = \frac{\tilde{\zeta}(\beta \tau \Delta - c(q^*))}{(1 - \beta)}$$

and

$$\tilde{V}_1 = \frac{\psi(u(q^*) - \beta \tau \Delta)}{(1 - \beta)},$$

with

$$\Delta(\tau) = \frac{\psi u(q^*) + \tilde{\zeta} c(q^*)}{1 - \beta(1 - \tau(\psi + \tilde{\zeta}))} > 0.$$ (41)

Solving $\tau = \tau(\Delta)$ and $\Delta = \Delta(\tau)$ simultaneously yields:

$$\tilde{\tau} = \frac{(1 - \beta)}{\beta} \left( \frac{1 - e^{-\Phi} - \Phi e^{-\Phi} u(q^*) + \Phi e^{-\Phi} c(q^*)}{2e^{-\Phi} - 1 + \Phi e^{-\Phi}(1 - e^{-\Phi})(u(q^*) - c(q^*))} \right).$$

It easily shown that $\tilde{\tau}$ is strictly decreasing in $\beta$, strictly increasing (and convex) in $\Phi$, and there exists a $\Phi > 0$ such that $\tilde{\tau}(\Phi) = 1$. To check that the equilibrium $\tilde{\tau}$ satisfies the incentive constraint, use $\tilde{\tau}$ into the incentive constraint for a seller $c(q^*) = \beta \tau \Delta(\tilde{\tau})$ to get

$$\tilde{\tau} = \frac{(1 - \beta)}{\beta} \frac{\Phi c(q^*)}{1 - e^{-\Phi}(u(q^*) - c(q^*))} > 0.$$ (41)

For all $\beta \in (0, 1)$, we find $\tilde{\tau} > \tau$ for all $0 < \Phi \leq \Phi$. Given that $\tilde{\tau}$ is strictly increasing in $\Phi$ over the range $(0, \Phi]$ implies uniqueness. Hence, we conclude that there exists a unique equilibrium with $\tau = \tilde{\tau}$ and $q = q^*$ iff $0 < \Phi \leq \Phi$.

Next, consider the case where $\tau = 1$, which implies $\lambda \geq 0$. Since $\Delta > 0$ is required for an equilibrium to exist, using the two conditions (34) and (35), we get $u'(q) \geq c'(q)$, implying that $q \leq q^*$ in any equilibrium with $\tau = 1$, and with strict inequality as long as $\lambda > 0$. Using $q = Q$ and $\tau = T = 1$, from (38) we get $\Delta = \Delta(q)$. Substitute it into $q = q(\Delta)$ from (34), we can apply the proof of Proposition 1 to show that there exists a unique equilibrium $q(\Phi) \in (q_0, q^*)$. Using the Implicit Function Theorem, it is easy to show that $\frac{\partial q}{\partial \Phi} < 0$ and that $q(\Phi) = q^*$. Since, for an equilibrium with $\tau = 1$, we need $q(\Phi) \leq q^*$, an equilibrium of this type cannot exist if $\Phi < \Phi$. If $\Phi < \Phi$ then $q(\Phi) < q^*$, but we need to show that the condition for $\tau$ given by (35) is satisfied at $\tau = 1$. Use $q(\Phi) < q^*$ into (34) and substituting into (35) to get

$$1 - \frac{c'(q)}{u'(q)} = \frac{\lambda}{\beta \Delta(q)(1 - e^{-\Phi(q, \tau)})}.$$ (39)

The condition for $\tau$ is satisfied for $\tau = 1$ since there is a value $\lambda > 0$ such that the above equality holds. Finally, when $q(\Phi) = q^*$, the condition also holds since it is consistent with $\lambda = 0$. We conclude that $\tau = 1$ and $q = q(\Phi) \in (q_0, q^*)$ satisfy conditions (34) and (35) and the incentive constraints, and hence, all the conditions for an equilibrium. □
Fig. 1 illustrates the equilibrium values of \( q \) and \( \tau \) as functions of \( \Phi \). For all values of \( M \) where \( \Phi < \Phi_1 \), \( \tau \) increases in \( M \), but \( q = q^* \) independently of the value of \( M \). Beyond \( \Phi_1 \), however, \( \tau \) has reached its maximal value (unity) and \( q \) is a decreasing function of \( M \).

Fig. 1. Monetary equilibrium with ex ante posting and lotteries.

This equilibrium shares similar properties with the equilibrium derived by Berentsen et al. [2] under pairwise matching and bargaining. They find a unique monetary equilibrium and a critical value of the bargaining power parameter for the buyer which they called \( \tilde{\Phi}_1 \). If \( \Phi \leq \tilde{\Phi}_1 \) the equilibrium entails \( q = 1 \) and \( q^* \); and if \( \Phi > \tilde{\Phi}_1 \) the equilibrium entails \( q \in (0, 1) \) and \( q = q^* \).

Intuitively, when buyers have less bargaining power, sellers are able to ask for the unit of money with certainty, and to adjust the intensive margin \( q < q^* \). When buyers bargaining power is high, sellers offer \( q^* \), and they adjust the probability of monetary transfer \( q \in (0, 1) \) accordingly. Our result is analogous to theirs in the sense that the buyer–seller ratio \( \Phi \) determines the agents bargaining power as discussed in the previous section, that is, the ex ante bargaining power for buyers is given by \( \Phi e^{-\Phi} \), decreasing in \( \Phi \).

4.2. Ex post pricing with lotteries

The model in this subsection is identical to the one studied in Section 3.2, except now a bidding strategy for a buyer is a pair \((q_i, \tau_i) \in \mathbb{R}_+ \times [0, 1] \) with \( i \in \{p, m\} \), and where \( p \) and \( m \), respectively, refer to the strategy in pairwise and multilateral matches. Let \((q, \tau) = ((q_p, q_m), (\tau_p, \tau_m)) \in \mathbb{R}_+^2 \times [0, 1]^2 \) be the vector of possible bidding strategies in different matches.

The value for a seller entering the market under buyers bidding strategy \((q, \tau)\) is given by

\[
V_0(q, \tau) = \xi_p [\tau_p + \beta (\tau_p V_1 + (1 - \tau_p) V_0)] \\
+ \xi_m [\tau_m + \beta (\tau_m V_1 + (1 - \tau_m) V_0)] \\
+ (1 - \xi_p - \xi_m) \beta V_0
\]

whereas in Section 3.2, \( \xi_p \) and \( \xi_m \), respectively, refer to the probabilities of pairwise and multilateral matches. The value for a buyer is given by

\[
V_1(q, \tau) = \psi_p [u(q_p) + \beta (\tau_p V_0 + (1 - \tau_p) V_1)] \\
+ \psi_m [u(q_m) + \beta (\tau_m V_0 + (1 - \tau_m) V_1)] \\
+ (1 - \psi_p - \psi_m) \beta V_1
\]
where $\psi_p$ and $\psi_m$, respectively, are the probabilities that a buyer gets served in pairwise or multilateral matches, and defined as above.

The incentive compatibility constraints are always binding in this case due to the assumed ex post pricing mechanism. Hence, with $\Delta = \tilde{V}_1 - \tilde{V}_0$,

$$\beta \tau_p \Delta = c(q_p)$$

and

$$\beta \tau_m \Delta = u(q_m),$$

which means that $\tau_m = u(q_m)/c(q_p)$ in any equilibrium. In any pairwise match, a buyer makes a take-it-or-leave-it offer to the seller by solving

$$\max_{\tau_p, q_p} u(q_p) - \beta (\tau_p \Delta - \tilde{V}_1)$$

s.t. \( c(q_p) = \beta \tau_p \Delta, \quad \tau_p \in [0, 1], \)

where $\lambda_p$ and $\lambda_\tau$ are the multipliers for the two constraints. Necessary and sufficient conditions for a solution are

$$u'(q_p) - \lambda_q c'(q_p) \geq 0 \quad (= 0 \text{ if } q_p > 0),$$

$$-\beta (1 - \lambda_q) - \lambda_\tau \leq 0 \quad (= 0 \text{ if } \tau_p > 0).$$

In multilateral matches, the bidding behavior of buyers implies that the problem is equivalent to a seller making a take-it-or-leave-it offer to a buyer selected at random. A seller solves

$$\max_{\tau_m, q_m} -c(q_m) + \beta (\tau_m \Delta - \tilde{V}_0)$$

s.t. \( u(q_m) = \beta \tau_m \Delta, \quad \tau_m \in [0, 1], \)

where $\varsigma_p$ and $\varsigma_\tau$ are the multiplier for the two constraints. Necessary and sufficient conditions for a solution are

$$-c'(q_m) + \varsigma_q u'(q_m) \geq 0 \quad (= 0 \text{ if } q_m > 0),$$

$$\beta (1 - \varsigma_q) - \varsigma_\tau \leq 0 \quad (= 0 \text{ if } \tau_m > 0).$$

**Proposition 8.** For $\beta < 1$, there exist critical values $(\Phi_p, \Phi_m)$ (constructed in the proof), where $0 < \Phi_m < \Phi_p < \infty$ for all parameter values such that:

(i) for all $\Phi \in [\Phi_m, \Phi_p)$, there exists a unique monetary equilibrium with $\tau_p = \tau_m = 1$ and $q_m < q_p < q^*$ with $\lim_{\Phi \to \Phi_p} q_p = q^*$;

(ii) for all $\Phi \in (\Phi_m, \Phi_p)$, there exists a unique monetary equilibrium with $\tau_p \in (0, 1)$, $\tau_m = 1$ and $q_m < q_p = q^*$ with $\lim_{\Phi \to \Phi_p} \tau_p = 1$;

(iii) for all $\Phi \in (0, \Phi_m]$, there exists a unique monetary equilibrium with $\tau_p \in (0, 1)$, $\tau_m \in (0, 1)$ and $q_p = q_m = q^*$ with $\lim_{\Phi \to \Phi_m} \tau_m = 1$; where $\partial q_i/\partial \Phi < 0$ whenever $q_i < q^*$, and $\partial \tau_i/\partial \Phi > 0$ for equilibrium values of $\tau_i \in (0, 1)$.

**Proof.** Consider the case where $\tau_i \in (0, 1)$ for $i \in \{p, m\}$. It implies that $\lambda_p = \varsigma_p = 0$ and $\lambda_q = \varsigma_q = 1$. Combining the first-order conditions yields $u'(q_i) = c'(q_i)$, and hence, $q_i = q^*$.
for \( i \in \{p, m\} \). Solving the steady-state values using \( V_0(q, \tau) = \bar{V}_0 \) and \( V_1(q, \tau) = \bar{V}_1 \), and inserting \( q_i = q^* \) yields

\[
\Delta = \frac{(\psi_p + \psi_m)u(q_p) + (\xi_p + \xi_m)c(q^*)}{1 - \beta(1 - (\psi_p + \xi_p)\tau_p - (\psi_m + \xi_m)\tau_m)} > 0.
\]

Using the incentive constraints and the value of \( \Delta \) we get

\[
\tilde{\tau}_p = \frac{(1 - \beta)c(q^*)}{\beta[(\psi_p - \xi_m)(u(q^*) - c(q^*))]} < \tilde{\tau}_m = \frac{u(q^*)}{c(q^*)}\tilde{\tau}_p,
\]

where \( \psi_p \) and \( \xi_m \) are functions of \( \Phi \). Note that \( \tilde{\tau}_p \in (0, 1) \) iff \( \phi < \tilde{\Phi}_p \), where

\[
(\psi_p(\tilde{\Phi}_p) - \xi_m(\tilde{\Phi}_p)) = \frac{(1 - \beta)c(q^*)}{\beta[(u(q^*) - c(q^*))]},
\]

Similarly, \( \tilde{\tau}_m \in (0, 1) \) iff \( \Phi < \tilde{\Phi}_m \) where

\[
(\psi_p(\tilde{\Phi}_m) - \xi_m(\tilde{\Phi}_m)) = \frac{(1 - \beta)u(q^*)}{\beta[(u(q^*) - c(q^*))]}.
\]

Since \( \frac{\partial \psi_p(\Phi)}{\partial \tilde{\tau}_p} < 0 \) and \( \frac{\partial \xi_m(\Phi)}{\partial \tilde{\tau}_m} > 0 \) with \( u(q^*) > c(q^*) \), it follows that \( \tilde{\Phi}_m < \tilde{\Phi}_p \). The uniqueness of \( \tilde{\tau}_i \) follows directly from the properties of \( u \) and \( c \).

Consider the case \( \tau_p \in (0, 1) \) and \( \tau_m = 1 \). It implies that \( \lambda_\tau = 0, \zeta_\tau > 0 \) and \( \lambda_q = 1 \geq \zeta_q \). Therefore, \( q_p = q^* \) and \( u'(q_m) > c'(q_m) \), so \( q_m < q^* \) with strict inequality as long as \( \zeta_q > 0 \). Using the incentive constraints (44) and (45), \( \tau_p = c(q^*)/\beta \Delta \) and since \( \tau_m = 1, u(q_m) = \beta \Delta \), and we have \( \tau_p = c(q^*)/u(q_m) \). Using these in the steady-state value

\[
\Delta = \frac{\psi_p u(q_p) + \psi_m u(q_m) + \xi_p c(q_p) + \xi_m c(q_m)}{(1 - \beta(1 - (\psi_p + \xi_p)\tau_p - (\psi_m + \xi_m)\tau_m)} > 0,
\]

yields the steady-state condition for an equilibrium \( q_m \):

\[
\beta \psi_p[(u(q^*) - c(q^*))] = (1 - \beta)u(q_m) + \beta \xi_m[(u(q_m) - c(q_m))].
\]

Note that \( q_m > q^* \) cannot be an equilibrium since it implies that \( \zeta_q > 1 \) which occurs when \( \xi_q < 1 \). For all \( q_m \leq q^* \), the right-hand side of (48) is strictly increasing in \( q_m \), meaning that there is a unique \( q_m < q^* \) satisfying (48), yielding a unique \( \Delta \) and a unique \( \tau_p \). Let the unique solution be \( q_m = q_m(\Phi) \in [0, q^*) \). It is easy to check that \( q_m'(\Phi) < 0 \) by totally differentiating (48) and that \( q_m(\tilde{\Phi}_m) = q^* \).

Therefore, an equilibrium with \( \tau_m = 1 \) and \( q_m(\Phi) \leq q^* \) cannot exist for \( \Phi < \tilde{\Phi}_m \). Finally, for \( \Phi \in [\tilde{\Phi}_m, \tilde{\Phi}_p) \) we have \( \tau_p \in (0, 1) \) and \( q_p = q^* \), and \( \tau_m = 1 \) with \( q_m < q^* \) and \( \lim_{\Phi \to \Phi_m} q_m = q^* \).

Consider the case \( \tau_p = \tau_m = 1 \). It implies that \( \lambda_\tau > 0 \) and \( \zeta_\tau > 0 \) and to satisfy the first-order conditions it must be that \( \lambda_q > 1 \) and \( \zeta_q < 1 \). It implies \( u'(q_i) > c'(q_i) \) and \( q_i < q^* \) for \( i \in \{p, m\} \). Using the steady-state value of \( \Delta \) into the incentive constraints along with \( \tau_p = \tau_m = 1 \) yields \( c(q_p) = u(q_m) \), which implies \( q_m < q_p < q^* \) as a solution. The rest of the proof is similar to the one for the model without lotteries. Therefore, there exist unique \( q_m = q_m(\Phi) \) and \( q_p = q_p(\Phi) \), with \( q_p'(\Phi) < 0 \), for \( i \in \{p, m\} \), and it is easy to show that \( q_p(\tilde{\Phi}_p) = q^* \). Therefore, we have for \( \Phi > \tilde{\Phi}_p, \tau_p = \tau_m = 1 \) and \( q_m < q_p < q^* \), with \( \lim_{\Phi \to \Phi_p} q_p(\Phi) = q^* \).
Notice that the case \( \tau_m \in (0, 1) \) and \( \tau_p = 1 \) cannot be an equilibrium because \( \tau_p \) implies \( \tau_m \). □

Fig. 2 illustrates the equilibrium \((\tau_i, q_i)\) pair as a function of \( \Phi \). For small values of \( M \), where \( \Phi < \Phi_m \), \( \tau_p \) and \( \tau_m \) are both increasing in \( M \), \( \tau_p < \tau_m \), but \( q_m = q_p = q^* \), independently of \( M \). Once \( \Phi \) hits the range \([\Phi_m, \Phi_p]\), \( \tau_m \) has reached its maximal value (unity) but \( \tau_p < 1 \). Along this range, \( q_m \) is decreasing and \( \tau_p \) is increasing in \( M \), but, still, \( q_p = q^* \), independently of \( M \). Once \( \Phi \geq \Phi_p \), \( \tau_p \) hits its maximal value and \( \tau_p = \tau_m = 1 \), independently of further increments in \( M \). Here, both \( q_m \) and \( q_p \) are decreasing functions of \( M \), with \( q_m < q_p \).

5. Conclusion

Adapting the STW model, to allow for directed search and multilateral matches, is a relatively straightforward exercise that preserves many of the basic properties of the model. The central message remains the same: the existence of money as a medium of exchange depends on key parameters such as peoples’ discount factors and the quantity of money itself, in relatively intuitive ways. This framework, then, is robust to these changes; the key conclusions are not dependent on either random search or bilateral bargaining. Some of the results do change, however. First, in the absence of lotteries, the equilibrium quantity produced in this framework is a strictly decreasing function of the supply of money.

Secondly, when auctions are introduced they generate price dispersion which implies dispersion of production levels. This introduces a novel type of inefficiency within matches: at least some of the matches must involve suboptimal production levels. Introducing lotteries into this setting generates results that are quite analogous to those found in Berentsen et al. [2], and Berentsen and Rocheteau [3]. In particular, when the money supply is small, efficiency along the intensive margin is obtained in equilibrium, independent of the particular level of the money supply. Increasing the money supply beyond a critical value eliminates this efficiency. It is natural to consider, as a next step, introducing divisible money into this framework. Galenianos and Kircher [8] make a start in this direction.
Acknowledgment

We would like to thank Randall Wright for encouragement and help at a very early stage, and Richard Dutu, Merwan Engineer, Allan Head, Miguel Molico, and Guillaume Rocheteau for useful discussions and suggestions. We also thank participants at the Conference in Honor of Neil Wallace at Penn State University 2006, the Penn S&M Workshop, and the Cleveland Fed Workshops on Money, Banking and Payments 2006. The usual disclaimer applies.

Appendix A. Derivation of the probabilities $\xi$ and $\psi$

Here, we demonstrate that, in large markets, the probability for a buyer to be served when selecting a seller is given by $\psi = \frac{1 - e^{-\Phi}}{\Phi}$ and the probability for a seller of selling his good is $\xi = (1 - e^{-\Phi})$. We start with the case of finite numbers of buyers and sellers denoted by $B$ and $S$, respectively. When all buyers select a seller with probability $\theta$, the probability for a buyer to face $n$ other buyers at a particular seller is given by the binomial distribution

$$
\binom{B - 1}{n} \theta^n (1 - \theta)^{B-1-n}
$$

Since we assume that the seller chooses a buyer at random when more than one show up, the probability for a buyer to be allocated the trade is $\frac{1}{n + 1}$. Adding this to the above probability, the probability for a buyer to be served when selecting a seller is given by

$$
\sum_{n=1}^{B-1} \frac{1}{n + 1} \binom{B - 1}{n} \theta^n (1 - \theta)^{B-1-n}
$$

$$
= \sum_{n=1}^{B-1} \frac{1}{B} \binom{B}{n + 1} \theta^n (1 - \theta)^{B-1-n}
$$

$$
= \frac{1}{B\theta} \left[ \sum_{n=0}^{B} \binom{B}{n} \theta^n (1 - \theta)^{B-n} \right]
$$

$$
= \frac{1 - (1 - \theta)^B}{B\theta}.
$$

When $\theta = 1/S$, taking the limit as $B$ and $S$ go to infinity but keeping the ratio $\Phi = B/S$ yields the result $\psi = \lim_{S \to \infty} \frac{1 - (1 - \frac{1}{S})^{\Phi}}{\Phi} = \frac{1 - e^{-\Phi}}{\Phi}$. The probability that a seller is selected is derived similarly. If all buyers select a seller with probability $\theta = 1/S$, then $(1 - \theta)^B$ is the probability that no buyers select a seller. Hence, $(1 - (1 - \theta)^B)$ where again, $\xi = \lim_{S \to \infty} (1 - (1 - \theta)^B) = (1 - e^{-\Phi})$.

Proof of Proposition 2. Using the Implicit Function Theorem write the equilibrium $q = q(\Phi, \beta)$. Let $D_0 = [\beta \psi u(q) - (1 - \beta (1 - \psi)) c(q)] > 0$ and $D_1 = [(1 - \beta (1 - \xi)) u(q) - \beta \xi c(q)] > 0$, which, respectively, holds if $V_0 > 0$ and $V_1 > 0$. Using the value of $\Delta(q)$ from (14) we can rewrite (11) as

$$
\frac{c'(q)}{u'(q)} = g(\Phi) \frac{[D_0]}{[D_1]}.
$$

Please cite this article as: B. Julien, et al., Bidding for money, Journal of Economic Theory (2006), doi: 10.1016/j.jet.2006.10.009
but now taking $\Phi$ as a parameter. Take $\beta$ as given and totally differentiate this equation to get
\[
\frac{c''u - c'u''}{u'^2} dq = g(\Phi) \left[ \frac{D_0}{D_1} \right] d\Phi + g(\Phi) \left[ \frac{D_0D_1 - D_0D_1\Phi}{(D_1)^2} \right] d\Phi
\]
\[\quad + g(\Phi) \left[ \frac{D_0D_1 - D_0D_1'}{(D_1)^2} \right] dq.
\]
Collecting terms
\[
dq = \left( g(\Phi) \left[ \frac{D_0}{D_1} \right] + g(\Phi) \left[ \frac{D_0D_1 - D_0D_1\Phi}{(D_1)^2} \right] \right) dq.
\]
From the definitions we find $D_0\Phi = \beta\phi(u(q) - c(q)) < 0$ and $D_1\Phi = \beta\xi(\Phi(u(q) - c(q)) > 0$, and it is easy to show that $g(\Phi) < 0$. Therefore, the numerator of the above equation is always negative. Next, the first term in the denominator is always positive since $u'' \leq 0$. It remains to show that the term $\left[ \frac{D_0D_1 - D_0D_1'}{(D_1)^2} \right] < 0$ for all $q \in (q, \tilde{q})$. One can verify this that the case for all $\beta$ and $\phi$ if and only if $\frac{\psi}{c} < \frac{\bar{u}}{c}$ which holds given the properties of the functions $u$ and $c$.

To show the impact of $\beta$ on $q$ consider the following derivatives $D_0\beta = \psi(u - c) > 0$ and $D_1\beta = \xi(u - c) > 0$. Totally differentiate the first-order condition to get
\[
\frac{c''u - c'u''}{u'^2} dq = g(\Phi) \left[ \frac{D_0\betaD_1 - D_0D_1\beta}{(D_1)^2} \right] d\beta
\]
\[\quad = g(\Phi)(u - c) \frac{(\psi D_1 - \xi D_0)}{(D_1)^2} d\beta,
\]
where the sign of $(\psi D_1 - \xi D_0)$ as the same sign as $\Delta = \tilde{V}_1 - \tilde{V}_0 > 0$. \(\Box\)

**Proof of Proposition 3.** The proof follows from Lemma 2 and define $\bar{\beta} = q(\Phi, \beta) = q^*$. That
is for any $\Phi$, one can find a value of $\bar{\beta}$ such that the maximum sellers can post in equilibrium is efficient. Since equilibrium $q$ is increasing in $\beta$, any $\beta < \bar{\beta}$ implies $q(\Phi, \beta) < q^*$, and no value of $\Phi$ can generate an efficient $q$. \(\Box\)

**Proof of Lemma 2.** Assume $\Delta' > 0$, then possible outcomes within a match implies $c(\tilde{q}) = u(q) > 0$, and hence, $q > 0$. Under the assumptions about $u(\bullet)$ and $c(\bullet)$, $u^{-1}(c(\tilde{q})) = q$ is a strictly convex (one to one) equilibrium relationship, and note that $u(\tilde{q}) > c(\tilde{q}) = u(q)$ for $\tilde{q} \in (0, \hat{q})$. If $\Delta' = 0$, clearly $\hat{q} = \tilde{q} = 0$. Since $c(\tilde{q}) = \beta\Delta'$ implies $\tilde{q}$ is increasing in $\Delta'$, there exist a $\Delta'$ such that $\tilde{q} = \hat{q}$. \(\Box\)

**Proof of Proposition 5.** First we must show that $\hat{q}$ is decreasing in $\Phi$. Using the definition of $T(\tilde{q})$ from the proof of Proposition 4
\[
T(\tilde{q}) = \beta\psi p u(\tilde{q}) + \xi m c(u^{-1}(c(\tilde{q}))) \quad \frac{1}{1 - \beta(1 - \psi p - \xi m)} - c(\tilde{q})
\]
and the fact that we have uniqueness, it must be that $T'(\tilde{q}) < 0$ at the equilibrium $\tilde{q} > 0$. Totally differentiating $T(\tilde{q})$ with respect to $\tilde{q}$ and $\phi$, we need to show that
\[
\frac{d\tilde{q}}{d\Phi} = -\frac{T_\Phi}{T'(\tilde{q})} < 0.
\]
Therefore, we are left to demonstrate that $T_{\phi} < 0$. Taking the derivative and using $q = u^{-1}(c(\bar{q}))$ into the equilibrium value functions we find

$$T_{\phi} = \frac{\beta}{1 - \beta(1 - \psi_p - \xi_m)} \psi_m \Phi \frac{(1 - \beta) V_1(\bar{q})}{\psi_m} - \frac{(1 - \beta) V_0(\bar{q})}{\xi_p} < 0$$

since $\psi_m \Phi < 0$ and in equilibrium it must be that $V_1(\bar{q}) > 0$ and $V_0(\bar{q}) > 0$.

Next we must show $\bar{q}$ is increasing in $\beta$. Totally differentiating $T(\bar{q})$ again with respect to $\bar{q}$ and $\beta$, we need to show that

$$\frac{d \bar{q}}{d \beta} = - \frac{T_{\beta}}{T'(\bar{q})} > 0,$$

which holds since $T_{\beta} = (1 - \beta) \psi_p u(\bar{q}) + \xi_m c(u^{-1}(c(\bar{q}))) \left(1 - \beta(1 - \psi_p - \xi_m)^2\right) > 0$. □

References