GEOMETRY AND STABILITY OF SURFACES WITH CONSTANT ANISOTROPIC MEAN CURVATURE

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Abstract

We study the geometry of surfaces which are in equilibrium for a (constant coefficient) parametric elliptic functional with a volume constraint. We consider the first and second variations and the exceptional set of the Gauss map for such surfaces. The equilibrium surfaces of revolution (anisotropic Delaunay surfaces) are also discussed as is an anisotropic version of the Willmore functional.

1 Introduction

The interface between two non mixing media is often represented as a surface. Determining the shape of the interface may lead to an isoperimetric problem in which a minimum value of the free surface energy, subject to a volume constraint is sought. The free surface energy may be a homogeneous surface tension or, for more highly structured materials, an anisotropic surface energy (i.e. one that depends on the direction of the surface) may be more suitable. This paper treats the geometry of equilibrium surfaces for the resulting type of isoperimetric problem.

Let $F : U \subset S^n \to \mathbf{R}^+$ be a positive, smooth function. For a smooth, oriented immersed hypersurface (we will simply write hypersurface) $X : \Sigma \to \mathbf{R}^{n+1}$ whose Gauss map $\nu : \Sigma \to S^n$ is assumed to lie in U, we define the functional

$$\mathcal{F}_{\Lambda_0}(X) := \mathcal{F}(X) + \Lambda_0 V(X) := \int_\Sigma F(\nu) d\Sigma + \Lambda_0 V(X), \quad \Lambda_0 \in \mathbf{R}, \tag{1}$$

where $d\Sigma$ is the volume form of the induced metric and $V(X)$ denotes the algebraic $(n+1)$-dimensional volume enclosed by X,

$$V(X) := \frac{1}{n+1} \int_\Sigma \langle X, \nu \rangle \, d\Sigma.$$

Such functionals are used to model anisotropic surface energies. Applications can be found in many branches of the physical sciences including metallurgy and crystallography [14, 15]. We will impose a *convexity condition* on the functional: Denote by DF and D^2F the gradient and Hessian

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of F on S^n. Then we require that at each point in U the matrix $D^2F + F1$ is positive definite. We will refer to the case when $U = S^n$ as the *uniform case*. The functional appearing in (1) will be referred to as a (constant coefficient) *parametric elliptic functional* (PEF). They have been extensively studied from the viewpoint of geometric measure theory and convex analysis but do a lesser degree using differential geometry. One notable geometric investigation is [11].

By parallel translation in $\mathbf{R}^{n+1}$, DF may be considered as a smooth tangent vector field along X. The Euler-Lagrange equation for the functional $\mathcal{F}_{\Lambda_0}$ is

$$0 = \mathrm{div}_\Sigma DF - nHF + \Lambda_0 =: -\Lambda + \Lambda_0, \tag{2}$$

where H is the mean curvature of the immersion. The quantity Λ will be called the *anisotropic mean curvature*.

The convexity condition implies that the Euler-Lagrange equation is absolutely elliptic in the sense of [7]. This implies that a maximum principle analogous to that for constant mean curvature surfaces holds. Another interpretation of the convexity condition is the following. Consider the embedding $G : S^n \to \mathbf{R}^{n+1}$ defined by $G(\nu) = DF(\nu) + F(\nu)\nu$. Then the convexity condition implies that G defines a smooth, convex hypersurface in $\mathbf{R}^{n+1}$. The hypersurface defined by G is called the *Wulff shape* of F. An important result known as Wulff's theorem, though actually proved by Jean Taylor ([13]), is that the Wulff shape is the absolute minimizer of the functional $\mathcal{F}_0$ among all closed 'hypersurfaces' in $\mathbf{R}^{n+1}$ enclosing the same $(n+1)$-volume. Here the term 'hypersurface' can be taken in the sense of the boundary of a set of positive Lebesgue measure. Thus the Wulff shape solves the isoperimetric problem for the functional $\mathcal{F}_0$. In [10] we showed that any closed critical, stable hypersurface of a PEF $\mathcal{F}_{\Lambda_0}$ with nonzero Λ_0 is, up to translations and homotheties, the Wulff shape, where a critical point X of $\mathcal{F}_{\Lambda_0}$ is said to be stable if the second variation of $\mathcal{F}_0$ is nonnegative for all $(n+1)$-volume-preserving variations of X with compact support.

In this paper we begin by obtaining variational formulas for parametric elliptic functionals with a volume constraint (Section 2). The variational formulas lead to a Ruh-Vilms type theorem that the Gauss map of a hypersurface with constant anisotropic mean curvature satisfies an associated variational problem (Section 3). In fact, the Gauss map is the absolute minimizer of this associated functional if the surface is strongly stable (Theorem 3.1), i.e. the first Dirichlet eigenvalue of the Jacobi operator is nonnegative.

Throughout the rest of the paper, we will restrict ourselves to surfaces in $\mathbf{R}^3$. First we will show that for functionals sufficiently close to the area functional, the only complete, stable critical point, up to homothety and translation, is the Wulff shape (Corollary 4.1).

Probably the most important examples of constant mean curvature surfaces are the Delaunay surfaces, i.e. the surfaces of revolution. They are not only important examples of constant mean curvature surfaces in their own right, but also appear in a fundamental way in the study of more general constant mean curvature surfaces. We investigate the analogous class of surfaces with constant anisotropic mean curvature (Section 5). The composition of this class of surfaces is strikingly similar to the classical case. For each functional there are families of anisotropic catenoids, unduloids and nodoids. The anisotropic unduloids are embedded periodic surfaces while the anisotropic nodoids are only immersed and periodic.

In the subsequent section we study the exceptional set of the Gauss map of a surfaces with constant anisotropic mean curvature. We produce various L^1 bounds on the curvature under the assumption that the Gauss map is contained in a suitable region of the 2-sphere which in particular is contained in the compliment of neighborhoods of both poles. These bounds depend on the conformal structure of the surface but do not explicitly depend on the functional, only on upper and lower bounds on the principal curvatures of the Wulff shape.

Finally, in Section 7, we study an anisotropic version of the Willmore functional which assigns to a surface the L^2 norm of the anisotropic mean curvature. It is shown that the Wulff shape is the absolute minimizer of this functional among all closed genus zero surfaces in $\mathbf{R}^3$ (Proposition 7.2).

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We have been informed by the referee that Theorem 4.1 was obtained, but not published, by Clarenz. In addition, Clarenz obtained Proposition 7.2 in [2]. With respect to these results, we wish to state that there were significant delays in the publication of our paper.

2 Variational formulas

In this section, we consider a smooth, oriented immersion $X : \Sigma := \Sigma^n \to \mathbf{R}^{n+1}$ of an n-dimensional, oriented, connected smooth, compact manifold Σ^n, with or without boundary. We do not assume yet that the anisotropic mean curvature is constant.

Denote by ν the Gauss map of X. We will consider a variation $X_t = X + tu\nu + \mathcal{O}(t^2)$, where u is a smooth function. The variation is assumed to have compact support, i.e. $X_t \equiv X$ outside some compact set which is independent of t. Derivatives at $t = 0$ will be often denoted by "dot". The first variation of the algebraic $(n+1)$-dimensional volume V is

$$\dot{V} = \int_\Sigma u d\Sigma. \tag{3}$$

The *anisotropic mean curvature* Λ is defined by the first variation formula

$$\partial_t \mathcal{F}|_{t=0} = -\int_\Sigma \Lambda u \, d\Sigma,$$

where $\mathcal{F}$ is the functional defined in (1). It then follows that the first variation of $\mathcal{F}$ vanishes for all compactly supported variations which preserve the enclosed $(n+1)$-dimensional volume exactly when $\Lambda \equiv \Lambda_0$ holds for some constant Λ_0. The expression for Λ in (2) was derived in [10].

In order to study the second variation of the functional $\mathcal{F}_{\Lambda_0}$, we will derive a formula for the pointwise variation of the function Λ. Let $\Omega \subset\subset \Sigma$ and let $u, v \in C_c^\infty(\Omega)$. Consider the two parameter variation

$$X(s,t) := X + (tu + sv)\nu \tag{4}$$

(no higher order terms). We wish to show:

$$\partial^2_{st}\mathcal{F}|_{s=t=0} = \int_\Sigma \langle A\nabla u, \nabla v\rangle - \langle Ad\nu, d\nu\rangle uv + nH\Lambda uv \, d\Sigma, \tag{5}$$

where $A = (D^2F + F1)$. We have

$$\partial^2_{st}\mathcal{F}|_{s=t=0} = \int_\Sigma (\partial^2_{st}F) d\Sigma + \int_\Sigma (\partial_s F)(\partial_t d\Sigma) + (\partial_t F)(\partial_s d\Sigma) + \int_\Sigma F(\partial^2_{st} d\Sigma) =: I + II + III,$$

where all derivatives which appear in the integrands are evaluated at $s = t = 0$. We now use that the first two variations of the normal ν are given by

$$\partial_t \nu = -\nabla u, \qquad \partial_s \nu = -\nabla v, \qquad \partial_{st}\nu = d\nu(\nabla uv) - \langle \nabla u, \nabla v\rangle \nu.$$

It then follows that

$$I = \int_\Sigma \langle D^2F(\nabla u), \nabla v\rangle + \langle DF, d\nu(\nabla(uv))\rangle \, d\Sigma.$$

We now note that the gradient of $F(\nu)$ on Σ is given by $\nabla F = d\nu DF$. We also recall a well known formula for the transformation of the Laplacian

$$-\Delta F \circ \nu = \mathrm{trace}_\Sigma \langle (D^2F)d\nu(\cdot), d\nu(\cdot)\rangle + \langle \tau(\nu), DF\rangle,$$

where $\tau(\nu)$ is the tension field of the Gauss map. Using that the tension field of ν is given by $\tau(\nu) = -n\nabla H$, we arrive at

$$\begin{aligned}
I &= \int_\Sigma \langle D^2F(\nabla u), \nabla v\rangle + \langle DF, d\nu(\nabla(uv))\rangle \, d\Sigma \\
&= \int_\Sigma \langle D^2F(\nabla u), \nabla v\rangle - \langle d\nu(DF), \nabla(uv)\rangle \, d\Sigma \\
&= \int_\Sigma \langle D^2F(\nabla u), \nabla v\rangle - \langle \nabla F, \nabla(uv)\rangle \, d\Sigma \\
&= \int_\Sigma \langle D^2F(\nabla u), \nabla v\rangle + uv\Delta F \, d\Sigma \\
&= \int_\Sigma \langle D^2F(\nabla u), \nabla v\rangle - uv(\langle (D^2F)d\nu, d\nu\rangle - n\langle \nabla H, DF\rangle) \, d\Sigma.
\end{aligned}$$

Using $(\partial_t d\Sigma) = -nuH \, d\Sigma$, $(\partial_s d\Sigma) = -nvH \, d\Sigma$, we obtain

$$II = \int_\Sigma nH(\langle DF, \nabla(uv)\rangle \, d\Sigma.$$

Since

$$\Lambda := -\mathrm{trace}_\Sigma A d\nu = -(\mathrm{div} DF - nHF), \tag{6}$$

we can integrate the right hand side by parts to obtain

$$II = -\int_\Sigma nuv \cdot \mathrm{div}(H \cdot DF) \, d\Sigma = -\int_\Sigma nuv \cdot \langle \nabla H, DF\rangle + nuvH(nHF - \Lambda) \, d\Sigma.$$

Therefore, we arrive at

$$I + II = \int_\Sigma (\langle D^2F(\nabla u), \nabla v\rangle - uv\langle (D^2F)d\nu, d\nu\rangle - nuvH(nHF - \Lambda)) \, d\Sigma. \tag{7}$$

We next use a formula for the pointwise second variation of area to obtain

$$III = \int_\Sigma F \cdot (\langle \nabla u, \nabla v\rangle + (n^2H^2 - |d\nu|^2)uv) \, d\Sigma. \tag{8}$$

Combining (7) and (8), we arrive at

$$\begin{aligned} I+II+III &= \int_\Sigma (\langle (D^2F+F1)(\nabla u), \nabla v\rangle - \langle (D^2F+F1)d\nu, d\nu\rangle uv + nH\Lambda uv)\, d\Sigma \\ &= -\int_\Sigma v(\mathrm{div}[(D^2F+F1)(\nabla u)] + \langle (D^2F+F1)d\nu, d\nu\rangle u) - nH\Lambda uv\, d\Sigma, \end{aligned} \quad (9)$$

which is the same as (5).

We will now consider applications of the formula (9). The first will be to compute the pointwise variation of the anisotropic mean curvature Λ which will be used in later sections.

Again, we consider variations of $\mathcal{F}$ with respect to the variation given by (4). We have at any values of (s,t),

$$\partial_s \mathcal{F} = -\int_\Sigma \Lambda \langle \partial_s X, \nu\rangle\, d\Sigma.$$

Differentiating again and setting $s=t=0$ gives

$$\partial^2_{ts}\mathcal{F} = -\int_\Sigma (\partial_t \Lambda) v - nH\Lambda uv\, d\Sigma. \quad (10)$$

Comparison of (9) and (10) shows:

Proposition 2.1 *The pointwise variation of Λ is given by*

$$\partial_t \Lambda = L[u] := \mathrm{div}[(D^2F+FI)|_\nu(\nabla u)] + \langle (D^2F+F1)|_\nu d\nu, d\nu\rangle u. \quad (11)$$

The second application of (9) will the second variation formula for the functional $\mathcal{F}$ with a volume constraint. We refer the reader to [1] for details.

Assume that $X:\Sigma \to \mathbf{R}^{n+1}$ is a critical point for $\mathcal{F}_{\Lambda_0}$. We choose smooth compactly supported functions u and v with

$$\int_\Sigma u d\Sigma \quad \text{and} \quad \int_\Sigma v d\Sigma \neq 0,$$

and we consider a variation $\underline{X}_t = X + (tu + s(t)v)\nu$ where $s(t)$ is implicitly defined so that the volume $V(t, s(t)) \equiv$ constant. For this variation we have at any value of t,

$$\partial_t \mathcal{F} = -\int_\Sigma \Lambda \langle \partial_t \underline{X}, \nu\rangle\, d\Sigma.$$

Also, when $t=0$, we see $\dot{s}=0$ and so, using (11), we have

$$\begin{aligned} \partial^2_{tt}\mathcal{F} &= -\int_\Sigma (\partial_t\Lambda)u + \Lambda_0\langle \partial^2_{tt}\underline{X}, \nu\rangle + \Lambda_0 \langle \partial_t \underline{X}, \partial_t \nu\rangle - \Lambda_0 nHu^2\, d\Sigma \\ &= -\int_\Sigma uL[u]\, d\Sigma - \ddot{s}\Lambda_0 \int_\Sigma v\, d\Sigma + \Lambda_0 \int_\Sigma nHu^2\, d\Sigma. \end{aligned} \quad (12)$$

As in [1], we have that

$$\ddot{s} = -\partial^2_{tt}V/\partial_s V = (\int_\Sigma nHu^2 d\Sigma) \quad / \quad (\int_\Sigma v d\Sigma).$$

We therefore obtain

$$\partial_{tt}^2 \mathcal{F}_{\Lambda_0}|_{t=0} = \partial_{tt}^2 \mathcal{F}|_{t=0} = -\int_\Sigma uL[u]d\Sigma.$$

Moreover, from (3) and (12), for any compactly supported variation $X_t = X + tu\nu + \mathcal{O}(t^2)$ of X, we have

$$\partial_{tt}^2 \mathcal{F}_{\Lambda_0}|_{t=0} = -\int_\Sigma uL[u]d\Sigma.$$

Proposition 2.2 *Let $X : \Sigma \to \mathbf{R}^{n+1}$ be a smooth, oriented, immersed hypersurface with $\Lambda \equiv \Lambda_0$. Let $X_t = X + tu\nu + \mathcal{O}(t^2)$ be a compactly supported variation of X. Then the second variation of $\mathcal{F}_{\Lambda_0}$ is given by*

$$\partial_{tt}^2 \mathcal{F}_{\Lambda_0}|_{t=0} = -\int_\Sigma uL[u]d\Sigma, \tag{13}$$

where L is the self-adjoint operator

$$L[u] = \operatorname{div}((D^2F + F1)|_\nu \nabla u) + \langle (D^2F + F1)|_\nu d\nu, d\nu\rangle u.$$

If X_t satisfies a further assumption that it is volume-preserving, then

$$\partial_{tt}^2 \mathcal{F}_{\Lambda_0}|_{t=0} = \partial_{tt}^2 \mathcal{F}|_{t=0} = -\int_\Sigma uL[u]d\Sigma. \tag{14}$$

From now on, we will write the right hand side of (13) (which is the same as (14)) as $\delta_u^2 \mathcal{F}_{\Lambda_0}$, that is

$$\delta_u^2 \mathcal{F}_{\Lambda_0} := -\int_\Sigma uL[u]d\Sigma.$$

3 Applications of the second variation

If we consider a smooth variation $X_\epsilon = X + \epsilon\psi\nu + \cdots$ of an immersion $X : \Sigma^n \to \mathbf{R}^{n+1}$, then, from Proposition 2.1, the first order change in Λ is given by

$$\partial_\epsilon(\Lambda) = L[\psi] := \operatorname{div}(A\nabla\psi) + \langle A d\nu, d\nu\rangle\psi,$$

where $A = D^2F + F1$. Since a one parameter family of translations $X_\epsilon = X + \epsilon C$, where C is a constant vector, leaves the functional invariant, its normal component $\nu_C := \langle \nu, C\rangle$ solves $L[\nu_C] = 0$, i.e.

$$\operatorname{div}(A\nabla\nu_j) + \langle A \cdot d\nu, d\nu\rangle\nu_j = 0, \quad j = 1, \cdots, n+1, \tag{15}$$

here we write $\nu = (\nu_1, \nu_2, \cdots, \nu_{n+1}) : \Sigma^n \to S^n \subset \mathbf{R}^{n+1}$.

Proposition 3.1 *The equation $\Lambda =$ constant is absolutely elliptic in the sense that its linearization at any surface is elliptic.*

Consequently, as explained in [7] the maximum principle holds for the equation $\Lambda =$ constant. If two hypersurfaces satisfying this condition are in oriented contact at a point p and one hypersurface lies on one side of the other near to p, then the two hypersurfaces must agree on a neighborhood of p.

Assume that $X : \Sigma \to \mathbf{R}^{n+1}$ is a critical point of a PEF $\mathcal{F}_\Lambda$. We will say that a relatively compact subdomain $\Omega \subset \Sigma$ is *stable* if $\delta^2_\psi \mathcal{F}_\Lambda \geq 0$ holds for all smooth ψ with compact support in Ω satisfying

$$\Lambda \int_\Omega \psi d\Sigma = 0.$$

A domain Ω is called *strongly stable* if $\delta^2_\psi \mathcal{F}_\Lambda \geq 0$ holds for all smooth ψ with compact support in Ω. Note that Ω is strongly stable if and only if the first eigenvalue $\lambda_1(L, \Omega)$ of the eigenvalue problem

$$L[\psi] = -\lambda\psi, \qquad \psi \in C^\infty_c(\Omega)$$

is nonnegative.

Assume that $X : \Sigma \to \mathbf{R}^{n+1}$ is a critical point of a PEF $\mathcal{F}_\Lambda$.

Remark 3.1 Define

$$\sigma := \min_{\nu \in S^n} \min_{\xi \in T_\nu S^n,\, |\xi|=1} \langle (D^2F + F1)|_\nu \xi, \xi\rangle, \qquad \tau := \max_{\nu \in S^n} \max_{\xi \in T_\nu S^n,\, |\xi|=1} \langle (D^2F + F1)|_\nu \xi, \xi\rangle. \quad (16)$$

In the uniform case, we have $0 < \sigma \leq \tau < \infty$. Note the inequalities

$$\begin{aligned} \sigma \int |\nabla\psi|^2 - (\tau/\sigma)|d\nu|^2\psi^2 d\Sigma &\leq \int \langle (D^2F + F1)\nabla\psi, \nabla\psi\rangle - \langle (D^2F + F1)|_\nu d\nu, d\nu\rangle \psi^2 d\Sigma \\ &\leq \tau \int |\nabla\psi|^2 - (\sigma/\tau)|d\nu|^2\psi^2 d\Sigma. \end{aligned}$$

Proposition 3.2 *Let $X : \Sigma \to \mathbf{R}^{n+1}$ be a smooth, oriented, immersed hypersurface which is a critical point of a uniform parametric elliptic functional $\mathcal{F}_\Lambda$, $\Lambda \neq 0$. If Ω is a relatively compact, strongly stable subdomain in Σ, then the first Dirichlet eigenvalue λ_1 of the Laplacian for Ω satisfies*

$$\lambda_1 \geq n\Lambda^2/(2\tau^2).$$

Proof. Note that

$$\tau\langle (D^2F + F1)|_\nu d\nu, d\nu\rangle \geq \sum h^2_{ij}/\mu^2_i \geq \frac{1}{n}(\sum h_{ij}/\mu_i)^2 = \Lambda^2/n$$

holds so that on any relatively compact, strongly stable subdomain, we have

$$0 \leq \int_\Omega \langle (D^2F + F1)|_\nu \nabla\psi, \nabla\psi\rangle - \langle (D^2F + F1)|_\nu d\nu, d\nu\rangle \psi^2 d\Sigma \leq \int_\Omega \tau|\nabla\psi|^2 - \frac{n\Lambda^2}{2\tau}\psi^2 d\Sigma$$

for all $\psi \in C^\infty_c(\Omega)$ from which the result follows. **q.e.d.**

Theorem 3.1 *Let $(\Sigma^n, \partial\Sigma)$ be a smooth manifold with smooth boundary and let*

$$X : \Sigma \to \mathbf{R}^{n+1}$$

be a critical point of a PEF $\mathcal{F}$ (with or without a volume constraint). Assume that the immersion is strongly stable, i.e. $\lambda_1(L,\Sigma)\geq 0$. Then the Gauss map ν has the following minimizing property: for every smooth map $f:\Sigma\to S^n$ with $f\equiv\nu$ on $\partial\Sigma$, there holds

$$\int_\Sigma \langle (D^2F+F1)|_\nu \nabla\nu,\nabla\nu\rangle\, d\Sigma \leq \int_\Sigma \langle (D^2F+F1)|_\nu \nabla f,\nabla f\rangle\, d\Sigma. \tag{17}$$

If the domain is stable for the functional with a volume constraint, then (17) holds for all maps f such that $f\equiv\nu$ on $\partial\Sigma$ and

$$\int_\Sigma f-\nu\, d\Sigma = 0,$$

i.e. f and ν have the same center of mass.

Proof. We write $\nu=(\nu_1,\cdots,\nu_{n+1})$, $f=(f_1,\cdots,f_{n+1})$. Then from the usual characterization of λ_1, we have for $j=1,\cdots,(n+1)$,

$$\lambda_1\int_\Sigma (\nu_j-f_j)^2\, d\Sigma \leq \int_\Sigma \langle A\nabla(\nu_j-f_j),\nabla(\nu_j-f_j)\rangle - \langle A\nabla\nu,\nabla\nu\rangle(\nu_j-f_j)^2\, d\Sigma,$$

where $A:=(D^2F+F1)|_\nu$. Expanding this out and using that the mappings are into a sphere and using the self-adjointness of A, gives

$$\lambda_1\int_\Sigma 2-2\langle f,\nu\rangle\, d\Sigma \leq \int_\Sigma \langle A\nabla\nu,\nabla\nu\rangle + \langle A\nabla f,\nabla f\rangle - 2\langle A\nabla\nu,\nabla f\rangle - \langle A\nabla\nu,\nabla\nu\rangle(2-2\langle f,\nu\rangle)\, d\Sigma. \tag{18}$$

Note that

$$\sum_j \oint_{\partial\Sigma} f_j\langle A\nabla\nu_j,n\rangle\, d\hat{s} = \sum_j\int_\Sigma f_j \mathrm{div} A(\nabla\nu_j) + \langle\nabla f_j, A\nabla\nu_j\rangle\, d\Sigma, \tag{19}$$

where $d\hat{s}$ is the volume form of $\partial\Sigma$. Note that

$$\begin{aligned}\sum_j \oint_{\partial\Sigma} f_j\langle A\nabla\nu_j,n\rangle\, d\hat{s} &= \sum_j \oint_{\partial\Sigma} \nu_j\langle A\nabla\nu_j,n\rangle\, d\hat{s} = \frac{1}{2}\sum_j\oint_{\partial\Sigma}\langle A\nabla\nu_j^2,n\rangle\, d\hat{s}\\ &= \frac{1}{2}\oint_{\partial\Sigma}\langle A\nabla(\sum_j\nu_j^2),n\rangle\, d\hat{s} = 0.\end{aligned}$$

This gives from (19) and (15),

$$\sum_j\int_\Sigma\langle\nabla f_j, A\nabla\nu_j\rangle\, d\Sigma = -\sum_j\int_\Sigma f_j\mathrm{div}A(\nabla\nu_j)\, d\Sigma = \int_\Sigma\langle A\nabla\nu,\nabla\nu\rangle\langle f,\nu\rangle\, d\Sigma.$$

Inserting this in (18), gives

$$\lambda_1\int_\Sigma 2-2\langle f,\nu\rangle\, d\Sigma \leq \int_\Sigma\langle A\nabla f,\nabla f\rangle - \langle A\nabla\nu,\nabla\nu\rangle\, d\Sigma.$$

This proves the result. **q.e.d.**

We make note of the following simple corollary.

Corollary 3.1 *Let $X:\Sigma\to\mathbf{R}^{n+1}$ be an immersed compact CMC (possibly minimal) hypersurface with boundary with $\lambda_1(L,\Sigma)\geq 0$. Then the Gauss map ν is the absolute minimizer of the Dirichlet energy among all sufficiently smooth maps into S^n with the same boundary values as the map ν.*

4 Two dimensional results

The main result of this section is the following.

Theorem 4.1 *Let $\mathcal{F}$ be a uniform PEF. Let $X : \Sigma^2 \to \mathbf{R}^3$ be an oriented, critical immersion of $\mathcal{F}_\Lambda$ for some $\Lambda \neq 0$, which is complete with respect to the induced metric. Assume that the functional is close to the area functional in the sense that*

$$\frac{2\sigma}{\tau} \geq 1 \tag{20}$$

holds, where σ and τ are those which were defined by (16). Then the Morse index of the immersion is finite if and only if Σ is compact.

In the case of constant mean curvature $F \equiv 1$, this result was obtained independently by da Silveira [3] and by the author [9] and later by Lopez-Ros [8]. The approach we will use below is similar to that used in [3] and [8], which is heavily based on work of Fischer-Colbrie [4].

Since any closed critical, stable hypersurface of a PEF $\mathcal{F}_\Lambda$ with nonzero Λ is, up to homotheties and translations, the Wulff shape ([10]), we obtain the following

Corollary 4.1 *Let $\mathcal{F}$ be a uniform PEF satisfying (20). Let $X : \Sigma^2 \to \mathbf{R}^3$ be an oriented, critical immersion of $\mathcal{F}_\Lambda$ for some $\Lambda \neq 0$, which is complete with respect to the induced metric. Then the immersion is stable if and only if $X(\Sigma)$ is up to a homothety and translation, equal to the Wulff shape.*

Proof of Theorem 4.1. Note that the operator $D^2F + F1$ is symmetric on S^2. Let $\{e_1, e_2\}$ be a locally defined frame on S^2 such that $(D^2F + F1)e_i = (1/\mu_i)e_i$. If ν is the Gauss map of Σ, note that the basis $\{e_1, e_2\}$ at ν also serves as an orthonormal basis for $dX(T_p\Sigma)$. Let $(-h_{ij})$ be the matrix representing $d\nu$ with respect to this basis. Then

$$(D^2F + F1)d\nu = \begin{pmatrix} -h_{11}/\mu_1 & -h_{12}/\mu_1 \\ -h_{12}/\mu_2 & -h_{22}/\mu_2 \end{pmatrix}.$$

This with (6) gives $\Lambda = h_{11}/\mu_1 + h_{22}/\mu_2$. Note that $D^2F + F1$ is the inverse of the differential of the Gauss map of the Wulff shape W and so its eigenvalues $1/\mu_j$ are the negatives of the reciprocals of the principal curvatures of W with respect to the outward unit normal. Letting $K_W = \mu_1\mu_2$, denote the Gauss curvature of W, we have $K_\Sigma/K_W = (h_{11}h_{22} - h_{12}^2)/(\mu_1\mu_2)$ from which we have

$$\begin{aligned} \Lambda^2/2 - 2K_\Sigma/K_W &= (1/2)(h_{11}^2/\mu_1^2 + h_{22}^2/\mu_2^2 + 2h_{11}h_{22}/\mu_1\mu_2) - 2(h_{11}h_{22} - h_{12}^2)/\mu_1\mu_2 \\ &= (1/2)(h_{11}/\mu_1 - h_{22}/\mu_2)^2 + 2h_{12}^2/\mu_1\mu_2 \geq 0. \end{aligned}$$

Note that $\tau^{-1} \leq \mu_i \leq \sigma^{-1}$ $(i = 1, 2)$ holds. Therefore,

$$\begin{aligned} \Lambda^2\mu_1\mu_2/2 &\leq \Lambda^2\mu_1\mu_2/2 + (\Lambda^2/2 - 2K_\Sigma/K_W)\mu_1\mu_2 \\ &= (\frac{h_{11}^2}{\mu_1^2} + \frac{h_{22}^2}{\mu_2^2} + \frac{2h_{12}^2}{\mu_1\mu_2})\mu_1\mu_2 \end{aligned}$$

$$
\begin{aligned}
&= \frac{h_{11}^2}{\mu_1}\mu_2 + \frac{h_{22}^2}{\mu_2}\mu_1 + h_{12}^2(\frac{\mu_1}{\mu_1} + \frac{\mu_2}{\mu_2}) \\
&\leq \sigma^{-1}[\frac{h_{11}^2}{\mu_1} + \frac{h_{22}^2}{\mu_2} + h_{12}^2(\frac{1}{\mu_1} + \frac{1}{\mu_2})] \\
&= \sigma^{-1}\langle (D^2F + F1)|_\nu d\nu, d\nu\rangle.
\end{aligned}
$$

Hence, on any relatively compact subdomain $\Omega \subset\subset \Sigma$, for all ψ with compact support, we have

$$
\begin{aligned}
&\tau^{-1}\int \langle (D^2F + F1)\nabla\psi, \nabla\psi\rangle - \langle (D^2F + F1)d\nu, d\nu\rangle\psi^2 \, d\Sigma \\
&\leq \tau^{-1}\int \tau|\nabla\psi|^2 - \sigma\mu_1\mu_2\{\Lambda^2/2 + (\Lambda^2/2 - 2K_\Sigma/(\mu_1\mu_2))\}\psi^2 \, d\Sigma \\
&= \int |\nabla\psi|^2 - \sigma\tau^{-1}\{\Lambda^2\mu_1\mu_2/2 + (\Lambda^2\mu_1\mu_2/2 - 2K_\Sigma)\}\psi^2 \, d\Sigma \\
&= -\frac{\sigma\Lambda^2}{2\tau}\int \mu_1\mu_2\psi^2 \, d\Sigma + \int |\nabla\psi|^2 - \frac{2\sigma}{\tau}(\frac{\Lambda^2\mu_1\mu_2}{4} - K_\Sigma)\psi^2 \, d\Sigma \\
&\leq -\frac{\sigma\Lambda^2}{2\tau^3}\int \psi^2 \, d\Sigma + \int |\nabla\psi|^2 - \frac{2\sigma}{\tau}(\frac{\Lambda^2\mu_1\mu_2}{4} - K_\Sigma)\psi^2 \, d\Sigma.
\end{aligned}
$$

If

$$
\frac{2\sigma}{\tau} \geq 1
$$

is satisfied, then, using the fact that the quantity $\Lambda^2/2 - 2K_\Sigma/K_W$ is non-negative,

$$
\begin{aligned}
&\tau^{-1}\int \langle (D^2F + F1)\nabla\psi, \nabla\psi\rangle - \langle (D^2F + F1)d\nu, d\nu\rangle\psi^2 \, d\Sigma \\
&\leq -\frac{\sigma\Lambda^2}{2\tau^3}\int \psi^2 \, d\Sigma + \int |\nabla\psi|^2 - (\frac{\Lambda^2\mu_1\mu_2}{4} - K_\Sigma)\psi^2 \, d\Sigma \\
&\leq -\frac{\sigma\Lambda^2}{2\tau^3}\int \psi^2 \, d\Sigma + \int |\nabla\psi|^2 + K_\Sigma\psi^2 \, d\Sigma \\
&= \int |\nabla\psi|^2 - (\frac{\sigma\Lambda^2}{2\tau^3} - K_\Sigma)\psi^2 \, d\Sigma \\
&= -\int \Big\{\Delta\psi + (\frac{\sigma\Lambda^2}{2\tau^3} - K_\Sigma)\psi\Big\}\psi \, d\Sigma.
\end{aligned}
$$

Therefore, if the index of L is finite, then the index of

$$
J := \Delta + \frac{\sigma\Lambda^2}{2\tau^3} - K_\Sigma
$$

is finite. We can now apply a method developed in Fischer-Colbrie [4]. There exists a compact subset C of Σ such that there exists a positive solution u of the equation

$$
\Delta u + (\frac{\sigma\Lambda^2}{2\tau^3} - K_\Sigma)u = 0
$$

on $\Sigma \setminus C$. By arguing as in Lopez-Ros [8], one sees that Σ must be compact. **q.e.d.**

Example 4.1 Without the uniformity assumption things may change radically. We consider surfaces which are given as graphs $x_3 = u(x_1, x_2)$ over some domain $\Omega \subset\subset \mathbf{R}^2$. Here u is a smooth function and we choose the orientation on the graph Γ so that the normal is given by $\nu := (1 + |Du|^2)^{-1/2}(-Du, 1)$. For such surfaces one can introduce the PEF's

$$\mathcal{F}_\Lambda = \int_\Gamma \nu_3^{-1} - \nu_3 \, d\Sigma + \Lambda V = \int_\Omega |Du|^2 d^2x + \Lambda V.$$

This functional appears in [11]. It is easy to check that the functional is indeed elliptic, in fact its Wulff shape is the elliptic paraboloid $x_3 = (1/2)(x_1^2 + x_2^2)$. For a PEF such that $F = F(\nu_3)$, the eigenvalues $1/\mu_j$ defined above are given by

$$1/\mu_2 = F - \nu_3 F', \qquad 1/\mu_1 = (1 - \nu_3^2)F'' + 1/\mu_2,$$

so that for $F = \nu_3^{-1} - \nu_3$ we have $1/\mu_2 = 2/\nu_3$ which is unbounded on the open hemisphere.

Note that when $\Lambda = 0$ the critical points of $\mathcal{F}$ are exactly the graphs of harmonic functions since $\mathcal{F}$ corresponds to the Dirichlet energy. There are therefore many 'entire' (and hence complete) examples, e.g. the graph of $\Re(z^n)$ where n is any positive integer. Note also that since the second variation of the Dirichlet energy is equal to the Dirichlet energy, these examples are globally, strongly stable, i.e. they have Morse index zero. When $\Lambda \neq 0$ holds, it is again easy to produce complete, examples by taking graphs of functions of the form

$$u = f + \Lambda(x_1^2 + x_2^2),$$

where f is an entire harmonic function. These examples are also globally strongly stable. To see this, note that vertical translation in $\mathbf{R}^3$ is a symmetry of the variational problem. If an immersion is subjected to a one parameter family of vertical translations, then the normal component of the resulting variation field is the function ν_3 which is positive for a graph. It follows that the first eigenvalue of the stability operator is positive on every relatively compact subdomain in Σ.

Example 4.2 Using the Legendre transformation (cf. Reilly [11]), one finds that the PEF whose Wulff shape is the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is given by

$$\mathcal{F} = \int \sqrt{a^2\nu_1^2 + b^2\nu_2^2 + c^2\nu_3^2} \, d\Sigma,$$

or in non-parametric form $u = u(x, y)$,

$$\mathcal{F} = c \int \sqrt{1 + (au_x/c)^2 + (bu_y/c)^2} \, dxdy.$$

The transformation $x' = x/a$, $y' = y/b$, $z' = z/c$ converts $\mathcal{F}$ into the area functional multiplied by abc. It follows that an immersion $X = (x, y, z)$ is a critical point of some $\mathcal{F}_{\Lambda_0}$ if and only if (x', y', z') is a constant mean curvature immersion.

5 Anisotropic Delaunay surfaces

Let

$$\mathcal{F}(X) = \int_\Sigma F(\nu_3)\, d\Sigma$$

be a rotationally invariant parametric elliptic functional. We will look for surfaces of revolution (with vertical axes) which are critical points of $\mathcal{F}$, possibly with a volume constraint. We will do this by expressing the corresponding one-dimensional Euler-Lagrange equation for the generating curve as a conservation law using a "flux formula".

Let X be a surface of revolution which is a critical point of the functional $\mathcal{F}_\Lambda$. Then, the endomorphism field $D^2F + F1$ has eigendirections corresponding to

$$E_2 = \nu \times E_1, \quad E_1 = (0,0,1) - \nu_3\nu$$

as long as the normal is not vertical. E_1, E_2 define an orthonormal basis $\{e_1, e_2\}$ on TS^2 as long as X does not intersect with the vertical axis. The eigenvalues μ_1^{-1}, μ_2^{-1} corresponding to these directions are given by

$$1/\mu_2 = F - \nu_3 F', \qquad 1/\mu_1 = (1-\nu_3^2)F'' + 1/\mu_2.$$

Since $F = F(\nu_3)$, the Euler-Lagrange equation for $\mathcal{F}_\Lambda$ becomes

$$\begin{aligned} 0 = \mathrm{div}_\Sigma(DF) - 2HF + \Lambda &= \mathrm{div}_\Sigma(F'(\nu_3)\nabla x_3) - 2HF + \Lambda \\ &= F''(\nu_3)\langle \nabla\nu_3, \nabla x_3\rangle - 2H(F - \nu_3 F') + \Lambda. \end{aligned}$$

From this it is easy to derive the equation

$$\Lambda\nu_3 = \mathrm{div}_\Sigma(F - \nu_3 F')\nabla x_3 = \mathrm{div}_\Sigma(1/\mu_2)\nabla x_3.$$

We can obtain a flux formula by integration. For example, if X is a surface with boundary, then the following flux formula holds.

$$\Lambda \int_\Sigma \nu_3\, d\Sigma = \oint_{\partial\Sigma} (F - \nu_3 F')\langle E_3, n\rangle\, d\hat{s} = \oint_{\partial\Sigma} (1/\mu_2)\langle E_3, n\rangle\, d\hat{s}, \tag{21}$$

where n is the outward pointing unit normal along $\partial\Sigma$ and $E_3 = (0,0,1)$.

We consider the part of such a surface which is bounded by two circles which lie in parallel horizontal planes. Then the surface is parameterized as

$$X(s,\theta) = (x(s)e^{i\theta}, z(s)), \quad x \geq 0, \qquad s \in [0, L],$$

$$z(0) < z(L),$$

where s is arc length of the generating curve. Note that the area element $d\Sigma$ of X is given by

$$d\Sigma = x ds d\theta$$

at any point where x does not vanish. We choose the "outward pointing" Gauss map $\nu = (\nu_1, \nu_2, \nu_3)$ of X as

$$\nu(s,\theta) = (z'(s)\cos\theta, z'(s)\sin\theta, -x'(s)).$$

Then we apply the Divergence theorem to the constant vector E_3 to obtain

$$\int_\Sigma \nu_3 \, d\Sigma + \pi[(x(L)^2 - x(0)^2] = 0. \tag{22}$$

Next, we apply the flux formula (21) for X. In the case of a surface of revolution, $\partial_n z$ are constant on each boundary circle and we obtain

$$2\pi\Big([(1/\mu_2)(\partial_n z)x](L) + [(1/\mu_2)(\partial_n z)x](0) \Big) = \Lambda \int_\Sigma \nu_3 \, d\Sigma. \tag{23}$$

By combining (22) and (23), we arrive at

$$2[(1/\mu_2)(\partial_n z)x](L) + \Lambda(x(L))^2 = -2[(1/\mu_2)(\partial_n z)x](0) + \Lambda(x(0))^2.$$

It follows easily, that

$$2(1/\mu_2)z'(s)x(s) + \Lambda(x(s))^2 \equiv \text{constant} =: c. \tag{24}$$

This is a first order conservation law for the generating curve $(x(s), z(s))$ of a surface of constant anisotropic mean curvature Λ. The Euler-Lagrange equation is the second order equation obtained by differentiating (24). The resulting surfaces will be called *anisotropic Delaunay surfaces*. Note that for a CMC surface, we can take $\mu_2 \equiv 1$ and we have $\Lambda = 2H$.

By taking a new parameter $\tilde{s} := L - s$, we have

$$2(1/\mu_2)z'(\tilde{s})x(\tilde{s}) - \Lambda(x(\tilde{s}))^2 = -c.$$

So below we will assume without loss of generality that $\Lambda \leq 0$ holds . With this normalization, we have the following cases:

- (I-1) $\Lambda = 0$ and $c = 0$: *horizontal plane.*
- (I-2) $\Lambda = 0$ and $c \neq 0$: *anisotropic catenoid.*
- (II-1) $\Lambda < 0$ and $c = 0$: *Wulff shape (up to homothety and translation).*
- (II-2) $\Lambda < 0$ and $c = ((\mu_2|_{\nu_3=0})^2|\Lambda|)^{-1}$: *cylinder.*
- (II-3) $\Lambda < 0$ and $((\mu_2|_{\nu_3=0})^2|\Lambda|)^{-1} > c > 0$: *anisotropic unduloid.*
- (II-4) $\Lambda < 0$ and $c < 0$: *anisotropic nodoid.*

In cases (I-2), (II-3) and (II-4), we will refer to the respective generating curve as an *anisotropic catenary*, *anisotropic undulary*, and *anisotropic nodary*. In general, we will call generating curves of anisotropic Delaunay surfaces as *anisotropic Delaunay curves*.

We will first justify the statement that case (II-1) does indeed give the Wulff shape and then consider the surfaces in cases (I-2) and (II-2) $\sim$ (II-4).

The orthonormal basis on $T\Sigma$ corresponding to $\{E_1, E_2\}$ above is $\{\partial/\partial s, x^{-1}\partial/\partial\theta\}$. The matrix $(-h_{ij})$ representing $d\nu$ with respect to this basis is given by

$$h_{11} = x''z' - x'z'', \quad h_{12} = h_{21} = 0, \quad h_{22} = -x^{-1}z'. \tag{25}$$

From (24) and (25), we have

$$(-2h_{22}/\mu_2 + \Lambda)x^2 = c.$$

On the other hand, $\Lambda = -\text{trace}(A \cdot d\nu) = h_{11}/\mu_1 + h_{22}/\mu_2$. It then follows that

$$(h_{11}/\mu_1 - h_{22}/\mu_2)x^2 = c, \tag{26}$$

$$(2h_{11}/\mu_1 - \Lambda)x^2 = c. \tag{27}$$

Suppose $\Lambda \neq 0$ and $c = 0$. Then, at any point where $x > 0$ holds, from (26), it holds that $\Lambda = 2h_{11}/\mu_1 = 2h_{22}/\mu_2$ and so the surface is, up to homothety and translation, part of the Wulff shape. In particular, if a surface of revolution with constant non zero anisotropic mean curvature has the topology of the disc, then we have $x = 0$ at some point and so, from (24), $c = 0$ holds. Therefore, it is, up to homothety and translation, part of the Wulff shape.

The Wulff shape W is parameterized as

$$(\sigma, \alpha) \mapsto (u(\sigma)e^{i\alpha}, v(\sigma)),$$

where σ is arc length of the generating curve. Let $X(s, \theta) = (x(s)e^{i\theta}, z(s))$ be another anisotropic Delaunay surface. At points where the Gauss maps of these two surfaces agree, we have

$$x_s = u_\sigma \qquad z_s = v_\sigma.$$

The quantity μ_2 is the negative of the principal curvature k_2 of W. Thus

$$\mu_2 = -k_2 = v_\sigma/u = z_s/u\,.$$

If we use this expression in the conservation law (24), then we obtain

$$2ux + \Lambda x^2 = c\,.$$

We can consider u as the independent variable and solve

$$x = \begin{cases} c/(2u), & \Lambda = 0, \\ \frac{-u \pm \sqrt{u^2 + \Lambda c}}{\Lambda}, & \Lambda \neq 0. \end{cases}$$

By considering the direction of the Gauss map for the case of anisotropic nodoids, we have the following:

Proposition 5.1 *Let W be the Wulff shape of a rotationally invariant parametric elliptic functional $\mathcal{F}(X) = \int_\Sigma F(\nu_3)\, d\Sigma$. Let*

$$\sigma \mapsto (u(\sigma), v(\sigma)), \quad \sigma \in (-\infty, \infty),$$

be the profile curve of W, where σ is the arc length. Then

$$\mu_2^{-1} v_\sigma - u = 0$$

holds. Let $X(s, \theta) = (x(s)e^{i\theta}, z(s))$ be a critical point of $\mathcal{F}_\Lambda$ with a constant $\Lambda \leq 0$. Then the immersion X is given as follows.

(i) When X is an anisotropic catenoid,

$$x = c/(2u)$$

for some nonzero constant c.

(ii) When X is an anisotropic unduloid,

$$x = \frac{u \pm \sqrt{u^2 + \Lambda c}}{-\Lambda}$$

for some constants $c > 0$ and $\Lambda < 0$, where $x = x(u(\sigma))$ is defined in $\{\sigma | u \geq \sqrt{-\Lambda c}\}$.

(iii) When X is an anisotropic nodoid,

$$x = \frac{u + \sqrt{u^2 + \Lambda c}}{-\Lambda}$$

for some constants $c < 0$ and $\Lambda < 0$, where $x = x(u(\sigma))$ is defined in $\{-\infty < \sigma < \infty\}$.

In all cases above, z is given by

$$z = \int^u v_u x_u \, du. \tag{28}$$

Conversely, for a Wulff shape W defined as above, define x and z as in (i) $\sim$ (iii) and (28). Then $X(s, \theta) = (x(s)e^{i\theta}, z(s))$ is an anisotropic Delaunay surface which satisfies

$$2\mu_2^{-1} z_s x + \Lambda x^2 = c,$$

where s is the arc length of (x, z), and Λ is supposed to be zero for Case (i). Moreover, X has the same regularity as W.

Proof. If we think of z as also depending on u, then

$$z_u = (z_x)(x_u) = (z_s/x_s)x_u = (v_\sigma/u_\sigma)x_u = v_u x_u.$$

Then $(x, \int^u z_u \, du)$ gives the profile curve of an anisotropic Delaunay surface.

We need to prove the regularity of X at $u = \sqrt{-\Lambda c}$ for the case (ii). We see

$$s = \pm \int^u \sqrt{x_u^2 + z_u^2} \, du = \pm \int^u \sqrt{1 + v_u^2} \, |x_u| \, du.$$

We can choose the direction of s so that

$$s = \begin{cases} -\int^u \sqrt{1 + v_u^2} \, |x_u| \, du, & x = \frac{u + \sqrt{u^2 + \Lambda c}}{-\Lambda}, \\ \int^u \sqrt{1 + v_u^2} \, |x_u| \, du, & x = \frac{u - \sqrt{u^2 + \Lambda c}}{-\Lambda} \end{cases}$$

holds. And so

$$ds/du = \begin{cases} -\sqrt{1 + v_u^2} \, |x_u|, & x = \frac{u + \sqrt{u^2 + \Lambda c}}{-\Lambda}, \\ \sqrt{1 + v_u^2} \, |x_u|, & x = \frac{u - \sqrt{u^2 + \Lambda c}}{-\Lambda}, \end{cases}$$

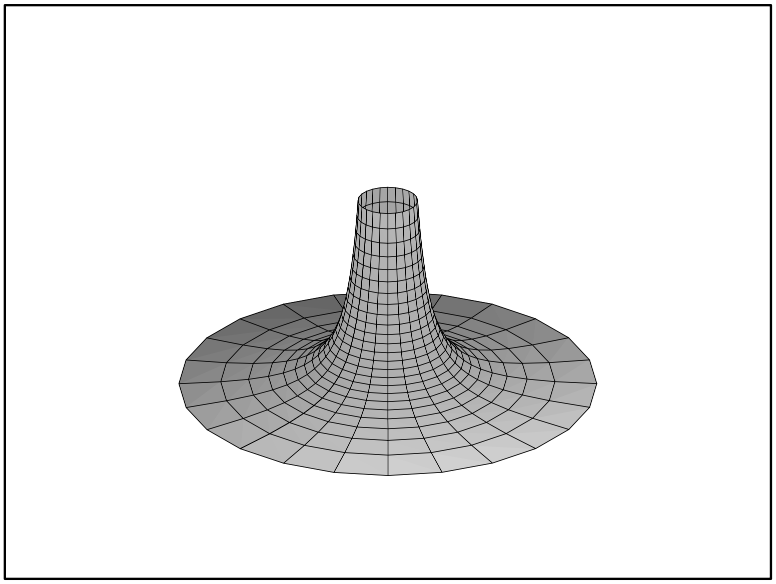

Figure 1: An end of an anisotropic catenoid

and

$$x_s = x_u u_s = -(1+v_u^2)^{-1/2}.$$

Therefore, X is of class C^1. It is shown that v_u is of class C^1 with respect to s as follows. We see

$$(v_u)_s = v_{uu} u_s = \begin{cases} -v_{uu}(1+v_u^2)^{-1/2} \frac{-\Lambda\sqrt{u^2+\Lambda c}}{\sqrt{u^2+\Lambda c}+u}, & x = \frac{u+\sqrt{u^2+\Lambda c}}{-\Lambda}, \\ v_{uu}(1+v_u^2)^{-1/2} \frac{-\Lambda\sqrt{u^2+\Lambda c}}{\sqrt{u^2+\Lambda c}-u}, & x = \frac{u-\sqrt{u^2+\Lambda c}}{-\Lambda}. \end{cases}$$

Hence, at $u = \sqrt{-\Lambda c}$, $(v_u)_s = 0$ holds and v_u is of C^1 with respect to s. Similarly we see that v_u is of C^r with respect to s if W is of C^r, and therefore X has the same regularity as W. **q.e.d.**

Example 5.1 We will carry this out to find the anisotropic catenoid in the case where the Wulff shape W_p is given by $(x_1^2+x_2^2)^p + x_3^{2p} = 1$, with $p > 1$. The profile curve is $u^{2p} + v^{2p} = 1$. We normalize so that $c = 2$ and consider the part of the anisotropic catenoid C_p which corresponds to the branch $v(u) = +(1-u^{2p})^{1/2p}$, $0 < u < 1$. We obtain

$$x = 1/u\,, \quad z = -\int^u w^{2p-3}(1-w^{2p})^{\frac{1-2p}{2p}}\, dw, \quad 0 < u < 1.$$

The "pole" of W_p, $(u,v) = (0,1)$ corresponds to an end of C_p. Because the singularity in z_u is integrable at $u = 0$ for $p > 1$, this end is planar. One end of the surface when $p = 2$ is shown above.

From now on, we will assume that

$$\Lambda < 0$$

holds.

Let $(x(s), z(s))$ be an anisotropic Delaunay curve which is defined by (24) with $\Lambda < 0$ and $c \neq 0$. Let B (bulge) and N (neck) be, respectively, the maximum and minimum of x. For the values of s for which these extrema occur, we have $z' = \pm 1$ and $\nu_3 = 0$. If an extremum of $x(s)$ occurs at $s = s_0$, we have

$$\pm 2(\mu_2|_{\nu_3=0})^{-1} x(s_0) + \Lambda(x(s_0))^2 = c.$$

It is easy to see that if $c > 0$ holds, then $z' = +1$ must hold at any positive local extremum of $x(s)$ while for $c < 0$, $z' = +1$ must hold at a bulge and $z' = -1$ must hold at a neck. Observe that for the anisotropic unduloid, $z' > 0$ must hold and so the generating curve is embedded.

For the anisotropic unduloid, by substituting $z' = 1$ for (24), we obtain

$$B = (-\Lambda)^{-1}\big((\mu_2^{-1}|_{\nu_3=0}) + \sqrt{(\mu_2^{-1}|_{\nu_3=0})^2 + \Lambda c}\big), \quad N = (-\Lambda)^{-1}\big((\mu_2^{-1}|_{\nu_3=0}) - \sqrt{(\mu_2^{-1}|_{\nu_3=0})^2 + \Lambda c}\big), \tag{29}$$

where

$$0 < c < ((\mu_2|_{\nu_3=0})^2 |\Lambda|)^{-1}$$

holds.

For the anisotropic nodoid, $z' = +1$ must hold at a bulge and $z' = -1$ must hold at a neck. And so we have, from (24),

$$B = (-\Lambda)^{-1}\big((\mu_2^{-1}|_{\nu_3=0}) + \sqrt{(\mu_2^{-1}|_{\nu_3=0})^2 + \Lambda c}\big), \quad N = (-\Lambda)^{-1}\big(-(\mu_2^{-1}|_{\nu_3=0}) + \sqrt{(\mu_2^{-1}|_{\nu_3=0})^2 + \Lambda c}\big), \tag{30}$$

where

$$c < 0$$

holds.

Lemma 5.1 *For each anisotropic Delaunay curve $(x(s), z(s))$, there are a unique local maximum B and a unique local minimum N of x, which we will call a bulge and a neck respectively. B and N are given as (29) and (30). An anisotropic nodary has no inflection point. Each anisotropic undulary is embedded and has a unique inflection point between each B and the next N, which satisfies $x = \sqrt{c/(-\Lambda)}$.*

Proof. From (25), x is an inflection point if and only if $h_{11} = 0$ holds. Therefore, from (27), x is an inflection point if and only if $-\Lambda x^2 = c$ holds. **q.e.d.**

We will derive representation formulas of anisotropic Delaunay surfaces. Set

$$(x'(s), z'(s)) = (\cos\varphi(s), \sin\varphi(s)).$$

From (24), we have

$$-\Lambda x^2 - 2\mu_2^{-1}(\sin\varphi)x + c = 0.$$

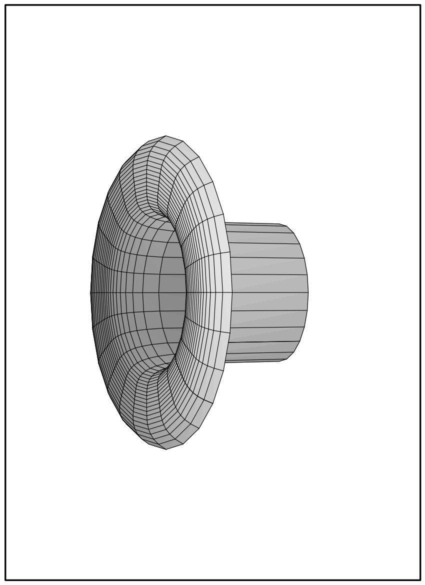

Figure 2: A part of the anisotropic nodoid for the Wulff shape W_2 defined above with $\Lambda = c = -2$.

Hence we obtain

$$x = (-\Lambda\mu_2)^{-1}\big(\sin\varphi \pm \sqrt{\sin^2\varphi + \mu_2^2\Lambda c}\big). \tag{31}$$

We differentiate x with respect to φ and obtain the following:

$$x_\varphi = (-\Lambda)^{-1}\big(\mu_2^{-1}\cos\varphi + (\mu_2^{-1})_\varphi \sin\varphi\big)\left(1 \pm \frac{\sin\varphi}{\sqrt{\sin^2\varphi + \mu_2^2\Lambda c}}\right). \tag{32}$$

Recall

$$\mu_2^{-1} = F(\nu_3) - \nu_3 F'(\nu_3), \qquad \nu_3 = -x' = -\cos\varphi,$$

from which we see that

$$(\mu_2^{-1})_\varphi = F'' \cos\varphi \sin\varphi. \tag{33}$$

Moreover, recall

$$\mu_1^{-1} = (1-\nu_3^2)F'' + \mu_2^{-1} = F''\sin^2\varphi + \mu_2^{-1}. \tag{34}$$

From (32), (33), and (34), we obtain

$$x_\varphi = (-\Lambda\mu_1)^{-1}\cos\varphi\left(1 \pm \frac{\sin\varphi}{\sqrt{\sin^2\varphi + \mu_2^2\Lambda c}}\right). \tag{35}$$

By using (35), we can compute locally

$$z = \int dz = \int \frac{dz}{dx}\frac{dx}{d\varphi}d\varphi = \int (-\Lambda\mu_1)^{-1}\sin\varphi\left(1 \pm \frac{\sin\varphi}{\sqrt{\sin^2\varphi + \mu_2^2\Lambda c}}\right) d\varphi. \tag{36}$$

(31) and (36) give global representation formulas of anisotropic Delaunay surfaces:

Proposition 5.2 (Representation formula for anisotropic nodoids) *An anisotropic nodary is parameterized as follows:*

$$\begin{cases} x(\varphi) = (-\Lambda\mu_2)^{-1}(\sin\varphi + \sqrt{\sin^2\varphi + \mu_2^2\Lambda c}), \\ z(\varphi) = \displaystyle\int_0^{\varphi} \frac{\sin\varphi}{-\Lambda\mu_1}\left(1 + \frac{\sin\varphi}{\sqrt{\sin^2\varphi + \mu_2^2\Lambda c}}\right) d\varphi + z_0, \end{cases} \qquad z_0 \in \mathbf{R},\ \Lambda < 0,\ c < 0,\ -\infty < \varphi < \infty.$$

Therefore, it is periodic along the vertical direction in the following sense:

$$(x(\varphi + 2k\pi), z(\varphi + 2k\pi)) = (x(\varphi), z(\varphi)) + k(0, z(2\pi) - z(0)), \qquad \forall k \in \mathbf{Z},\ \forall \varphi \in \mathbf{R}.$$

Remark 5.1 Let us consider an anisotropic nodoid. If $F(\nu_3) = F(-\nu_3)$, then

$$z|_{\varphi=2\pi} - z|_{\varphi=0} = \frac{2}{-\Lambda}\int_0^{\pi} \frac{\sin\varphi}{\mu_1\sqrt{\sin^2\varphi + \mu_2^2\Lambda c}}\, d\varphi > 0.$$

If there is an example which satisfies $z|_{\varphi=2\pi} - z|_{\varphi=0} = 0$, then the surface is an embedded critical surface with genus 1, therefore it is an embedded critical surface which is different from the Wulff shape.

Corollary 5.1 *An anisotropic nodoid is either non-embedding or a compact embedded surface with genus 1.*

Proof. If $z|_{\varphi=2\pi} - z|_{\varphi=0} = 0$, then the surface is a compact embedded surface with genus 1. If $z|_{\varphi=2\pi} - z|_{\varphi=0} \neq 0$, then the surface is periodic along the vertical direction, and it has both bulges and necks. On the other hand, from (24), $z' = +1$ must hold at a bulge and $z' = -1$ must hold at a neck, which leads us to the desired conclusion. **q.e.d.**

Proposition 5.3 (Representation formula for anisotropic unduloids) *One period of an anisotropic undulary C is parametrized as follows:*

$$\begin{cases} x_1(\varphi) = (-\Lambda\mu_2)^{-1}(\sin\varphi + \sqrt{\sin^2\varphi + \mu_2^2\Lambda c}), \\ z_1(\varphi) = \displaystyle\int_{\pi/2}^{\varphi} \frac{\sin\varphi}{-\Lambda\mu_1}\left(1 + \frac{\sin\varphi}{\sqrt{\sin^2\varphi + \mu_2^2\Lambda c}}\right) d\varphi + z_0, \end{cases} \qquad \pi/2 - \delta_1 < \varphi < \pi/2 + \delta_2, \tag{37}$$

$$\begin{cases} x_2(\varphi) = (-\Lambda\mu_2)^{-1}(\sin\varphi - \sqrt{\sin^2\varphi + \mu_2^2\Lambda c}), \\ z_2(\varphi) = \displaystyle\int_{\pi/2+\delta_2}^{\varphi} \frac{\sin\varphi}{-\Lambda\mu_1}\left(1 - \frac{\sin\varphi}{\sqrt{\sin^2\varphi + \mu_2^2\Lambda c}}\right) d\varphi + z_1(\pi/2 + \delta_2), \end{cases} \qquad \pi/2 - \delta_1 < \varphi < \pi/2 + \delta_2, \tag{38}$$

where

$$z_0 \in \mathbf{R}, \quad \Lambda < 0, \quad c > 0,$$

and δ_1, $\delta_2 \in (0, \pi/2)$ are defined as follows: $(x', z') = (\cos(\pi/2 - \delta_1), \sin(\pi/2 - \delta_1))$ is the tangent vector of C at the inflection point between a neck $(x(s_N^1), z(s_N^1))$ and the next bulge $(x(s_B^1), z(s_B^1))$

such that $z(s_N^1) < z(s_B^1)$ *holds. On the other hand,* $(x', z') = (\cos(\pi/2 + \delta_2), \sin(\pi/2 + \delta_2))$ *is the tangent vector of* C *at the inflection point between the bulge* $(x(s_B^1), z(s_B^1))$ *and the next neck* $(x(s_N^2), z(s_N^2))$ *such that* $z(s_B^1) < z(s_N^2)$ *holds.*

By combining (37) and (38) with their parallel translations, we obtain an embedded periodic surface along the vertical direction.

Proposition 5.4 (Representation formula for anisotropic unduloids) *An anisotropic unduloid is represented as follows:*

$$z = \pm \int^x \left(\frac{4x^2}{\mu_2^2 (c - \Lambda x^2)^2} - 1 \right)^{-1/2} dx.$$

Proof. Since $z' > 0$, we can represent $x = x(z)$. From $x_s = x_z z_s$ and $x_s^2 + z_s^2 = 1$, we obtain

$$z_s = (1 + x_z^2)^{-1/2}. \tag{39}$$

By substituting (39) for (24), we obtain

$$x_z = \pm \left(\frac{4x^2}{\mu_2^2 (c - \Lambda x^2)^2} - 1 \right)^{1/2}.$$

q.e.d.

The following result will serve to motivate results of the next section concerning the exceptional set of the Gauss map of a surface with constant anisotropic mean curvature.

Proposition 5.5 *The Gauss map* ν *of the an anisotropic unduloid is contained in a band around the equator. Specifically, the Gauss map satisfies*

$$\nu_3^2 \le M := 1 - \frac{4BN}{(B+N)^2} \left(\frac{(\mu_2|_{\nu_3=0})^{-1}}{||\mu_2^{-1}||_\infty} \right)^2.$$

Proof. From (24) and the fact that $c > 0$ holds on the anisotropic unduloid, we obtain

$$2z'/\mu_2 = c/x - \Lambda x \ge 2\sqrt{-\Lambda}\sqrt{c}.$$

We also find from (29) that

$$-\Lambda = \frac{2}{\mu_2|_{\nu_3=0}(B+N)}, \qquad c = \frac{1}{\mu_2|_{\nu_3=0}} \frac{2BN}{(B+N)}.$$

Using this and the fact that $\nu_3^2 = (x')^2 = 1 - (z')^2$, the result follows. **q.e.d.**

On the other hand, from Proposition 5.2, we see the following

Proposition 5.6 *The Gauss map of an anisotropic nodoid is surjective.*

6 The Gauss map

In contrast to the case of minimal surfaces, relatively little is known about constant mean curvature surfaces whose Gauss map is constrained to lie in some particular subset of the 2-sphere. Perhaps the best known result in this direction is that of Hoffman-Osserman-Schoen [6] which states that a complete CMC immersion whose Gauss map is contained in an open hemisphere must be flat. This result will be generalized below.

The goal here is to consider surfaces whose Gauss map lies in a possibly large subset of $S^2 \setminus \pm E_3$. Our result is motivated by considering domains of a fixed area in a one parameter family of unduloids which are deforming to a cylinder. As their Gaussian images are restricted to lie in a narrowing neighborhood of the equator, their Gauss curvatures are tending uniformly to zero.

Let Σ be a Riemann surface equipped with any conformal metric. Let $U_0 \subset U$ be relatively compact subdomains of Σ. Let $\omega = \omega(U_0, U)$ be the harmonic function on $U \setminus \bar{U}_0$ with boundary values 1 on ∂U_0 and 0 on ∂U. Note that $0 \leq \omega = \omega(U_0, U) \leq 1$ by the maximum principle. The number defined by

$$\frac{1}{\mu(U_0, U)} := \int_{U \setminus U_0} |\nabla \omega|^2 \, d\Sigma$$

is clearly a conformal invariant of the pair (U_0, U). The surface Σ is said to be *parabolic* if there exists an exhaustion $U_0 \subset U_1 \subset U_2 \subset \cdots$ with $1/\mu(U_0, U_j) \to 0$ as $j \to \infty$. This condition is independent of the choice of exhaustion. It is easy to see that the sequence $\{1/\mu(U_0, U_j)\}$ is strictly monotonically decreasing. The class of parabolic surfaces includes those which are conformally a compact surface minus a finite number of points.

Let $\mathcal{F}$ be a uniform parametric elliptic functional and let $X : \Sigma \to \mathbf{R}^3$ be a conformal immersion which is a critical point for $\mathcal{F}$ with a volume constraint. In particular, the immersion has constant anisotropic mean curvature Λ. Recall that the uniformity of $\mathcal{F}$ means that there exist constants σ and τ, such that

$$0 < \sigma \leq \langle A\xi, \xi \rangle \leq \tau < \infty \tag{40}$$

holds for all tangent vectors ξ with $|\xi| = 1$.

Let $X : \Sigma \to \mathbf{R}^3$ be a conformal immersion with constant anisotropic mean curvature Λ. Let ν be its Gauss map and assume that $\nu(\Sigma) \subset S^2 \setminus \pm E_3$. Define the multi-valued function $\alpha := \arg(\nu_1 + i\nu_2)$. Fix a base point $p_0 \in \Sigma$ and define the (possibly multi-valued) function

$$\hat{\alpha}(p) := \int_{p_0}^{p} \nu^* d\alpha,$$

where the integration is along any path from p_0 to p. We will be assuming that $\hat{\alpha}$ is a bounded function on a certain subset of Σ. Note that this means, in particular, that $\hat{\alpha}$ is single valued on the subset.

Theorem 6.1 *There exist constants $c = c(b - a, \tau\sigma^{-1})$ with the following property: Let $X : \Sigma \to \mathbf{R}^3$ be a conformal immersion with constant anisotropic mean curvature Λ for a parametric elliptic functional satisfying (40), such that the image Ω of its Gauss map satisfies*

$$1 > M := \sup_{\Omega} \nu_3^2. \tag{41}$$

Then, for any relatively compact subdomains $U_0 \subset U \subset \Sigma$ such that

$$-\infty < a < \hat{\alpha} < b < \infty \tag{42}$$

holds on U, there holds

$$\int_{U_0} |K|\, d\Sigma \le \frac{cM^{1/2}}{(1-M)^{3/4}} \left\{ \frac{1}{\mu(U_0,U)} \inf_{U_0 \subset V \subset U} \left(\frac{1}{\mu(U_0,V)} + \frac{1}{\mu(V,U)} \right) \right\}^{1/2} . \tag{43}$$

If we further restrict the range of the Gauss map to lie in a topologically simple subset, we obtain

Corollary 6.1 *Let Ω be a domain with compact closure in $S^2 \setminus \pm E_3$, with the property that the winding number of every closed curve in Ω with respect to $-E_3$ is zero. On Ω it is possible to define the function*

$$\alpha := \arctan(\nu_2/\nu_1) = \arg(\nu_1 + i\nu_2)$$

as a single valued continuous function such that there exist constants a, b with

$$-\infty < a < \alpha < b < \infty.$$

Set

$$1 > M := \sup_{\Omega} \nu_3^2.$$

There exist constants $c = c(b-a, \tau\sigma^{-1})$ with the following property: If $X : \Sigma \to \mathbf{R}^3$ is a conformal immersion with constant anisotropic mean curvature Λ, such that the image of its Gauss map is contained in a domain Ω as above, then for any relatively compact subdomains $U_0 \subset U \subset \Sigma$, (43) holds.

Proof of Theorem 6.1. By replacing the base point p_0 if necessary, we have

$$-\infty < -\delta < \hat{\alpha} < \delta < \infty, \qquad \delta := (b-a)/2. \tag{44}$$

By introducing coordinates (γ, α) in Ω as

$$\nu = (\sin\gamma\cos\alpha, \sin\gamma\sin\alpha, \cos\gamma),$$

the canonical metric on S^2 can be expressed as

$$d\underline{s}^2 = d\gamma^2 + \sin^2\gamma\, d\alpha^2, \tag{45}$$

where α is the same as above. Note that $d\alpha = d\hat{\alpha}$ on Σ.

The energy $E[\nu]$ of a map ν from a surface Σ into $(\Omega, d\sigma^2)$ can be expressed as

$$\begin{aligned} E[\nu] : &= (1/2) \int_\Sigma \langle A \cdot d\nu, d\nu \rangle\, d\Sigma \\ &= (1/2) \int_\Sigma \left(\langle A\nabla\gamma, \nabla\gamma \rangle + (\sin^2\gamma) \langle A\nabla\hat{\alpha}, \nabla\hat{\alpha} \rangle \right) d\Sigma. \end{aligned}$$

From this, one can derive the following equations for the critical points of the energy of a map into $(\Omega, d\underline{s}^2)$:

$$\operatorname{div}(A\nabla\gamma) = (\cos\gamma\sin\gamma)\langle A\nabla\hat{\alpha}, \nabla\hat{\alpha}\rangle, \tag{46}$$

$$\operatorname{div}((\sin^2\gamma)A\nabla\hat{\alpha}) = 0. \tag{47}$$

Let ζ be a smooth function with $0 \le \zeta \le 1$, which is identically 1 on U_0 and has compact support in U. From (47), we have

$$\begin{aligned} 0 &= \oint_{\partial U} \zeta^2(\sin^2\gamma)\hat{\alpha} * (A\nabla\hat{\alpha})^\flat \\ &= \int_U \zeta^2(\sin^2\gamma)\langle A\nabla\hat{\alpha}, \nabla\hat{\alpha}\rangle + 2\zeta\hat{\alpha}(\sin^2\gamma)\langle\nabla\zeta, A\nabla\hat{\alpha}\rangle \, d\Sigma. \end{aligned}$$

From this, we can rearrange terms to obtain

$$\begin{aligned} \sigma\int_U \zeta^2(\sin^2\gamma)|\nabla\hat{\alpha}|^2 \, d\Sigma &\le 2\tau\int_U |\zeta\hat{\alpha}(\sin^2\gamma)||\nabla\zeta||\nabla\hat{\alpha}| \, d\Sigma \\ &\le \epsilon^2\tau\int_U \zeta^2(\sin^2\gamma)|\nabla\hat{\alpha}|^2 \, d\Sigma + \epsilon^{-2}\tau\int_U \hat{\alpha}^2(\sin^2\gamma)|\nabla\zeta|^2 \, d\Sigma \\ &\le \epsilon^2\tau\int_U \zeta^2(\sin^2\gamma)|\nabla\hat{\alpha}|^2 \, d\Sigma + \epsilon^{-2}\tau\int_U \hat{\alpha}^2|\nabla\zeta|^2 \, d\Sigma. \end{aligned} \tag{48}$$

Now let ζ be a sufficiently smooth approximation to the piecewise differentiable function

$$\omega_1(z) = \begin{cases} 1, & \text{in } U_0, \\ \omega(U_0, U), & \text{in } U \setminus U_0. \end{cases}$$

By taking $\epsilon^2\tau := (1/2)\sigma$ in (48), we obtain, using (44),

$$(1 - M)\int_{U_0} |\nabla\hat{\alpha}|^2 \, d\Sigma \le \int_{U_0} (\sin^2\gamma)|\nabla\hat{\alpha}|^2 \, d\Sigma \le \frac{4(\tau\sigma^{-1})^2\delta^2}{\mu(U_0, U)}. \tag{49}$$

In a similar way, we will bound the L^2 norm of $|\nabla\gamma|$. Using (46), we have

$$\begin{aligned} 0 &= \oint_U \zeta^2(\cos\gamma) * (A\nabla\gamma)^\flat \\ &= \int_U -\zeta^2(\sin\gamma)\langle A\nabla\gamma, \nabla\gamma\rangle + \zeta^2(\sin\gamma)(\cos^2\gamma)\langle\nabla\hat{\alpha}, A\nabla\hat{\alpha}\rangle + 2\zeta(\cos\gamma)\langle\nabla\zeta, A\nabla\gamma\rangle \, d\Sigma. \end{aligned}$$

Recall that $0 < \sqrt{1 - M} \le \sin\gamma$ and $|\cos\gamma| \le \sqrt{M}$ hold on Ω. We have

$$\begin{aligned} \sigma\sqrt{1 - M}\int_U \zeta^2|\nabla\gamma|^2 \, d\Sigma &\le \tau M\int_U \zeta^2|\nabla\hat{\alpha}|^2 \, d\Sigma \\ &\quad + \epsilon^2\tau\int_U \zeta^2|\nabla\gamma|^2 \, d\Sigma + \epsilon^{-2}\tau M\int_U |\nabla\zeta|^2 \, d\Sigma. \end{aligned}$$

We take $\tau\epsilon^2 := (1/2)\sigma\sqrt{1 - M}$. Choose a domain V with $U_0 \subset\subset V \subset\subset U$. Letting ζ be a sufficiently smooth approximation to the function

$$\omega_2(z) = \begin{cases} 1, & \text{in } U_0, \\ \omega(U_0, V), & \text{in } V \setminus U_0, \\ 0, & \text{in } U \setminus V, \end{cases}$$

we have, using (49) with U replaced by V,

$$\begin{aligned}\sigma\sqrt{1-M}\int_{U_0}|\nabla\gamma|^2\,d\Sigma &\leq 2\left(\tau M\int_V|\nabla\hat{\alpha}|^2\,d\Sigma+\frac{2\tau^2M}{\sigma\sqrt{1-M}\mu(U_0,V)}\right)\\ &\leq 4\tau\left(\frac{2(\tau\sigma^{-1})^2M\delta^2}{(1-M)\mu(V,U)}+\frac{\tau\sigma^{-1}M}{\sqrt{1-M}\mu(U_0,V)}\right).\end{aligned}$$

The last inequality can be expressed as

$$\begin{aligned}\int_{U_0}|\nabla\gamma|^2\,d\Sigma &\leq 4\left(\frac{2(\tau\sigma^{-1})^3M\delta^2}{(1-M)^{3/2}\mu(V,U)}+\frac{(\tau\sigma^{-1})^2M}{(1-M)\mu(U_0,V)}\right)\\ &\leq C_1\frac{M}{(1-M)^{3/2}}\left(\frac{1}{\mu(V,U)}+\frac{1}{\mu(U_0,V)}\right),\end{aligned}\tag{50}$$

where C_1 depends only on $\delta(=(b-a)/2)$ and $\tau\sigma^{-1}$.

In order to relate these estimates to the curvature, note that the area element for the metric (45) is given by

$$dA=(\sin\gamma)\,d\gamma\wedge d\hat{\alpha}.$$

From this it follows that

$$K d\Sigma=\nu^*dA=(\sin\gamma)\langle J\nabla\gamma,\nabla\hat{\alpha}\rangle\,d\Sigma,$$

where J is the almost complex structure on Σ. Hence, using (49) and (50), we have

$$\begin{aligned}\int_{U_0}|K|\,d\Sigma &\leq \left(\int_{U_0}|\nabla\gamma|^2\,d\Sigma\right)^{1/2}\left(\int_{U_0}(\sin^2\gamma)|\nabla\hat{\alpha}|^2\,d\Sigma\right)^{1/2}\\ &\leq 4\left(\frac{2(\tau\sigma^{-1})^3M\delta^2}{(1-M)^{3/2}\mu(V,U)}+\frac{(\tau\sigma^{-1})^2M}{(1-M)\mu(U_0,V)}\right)^{1/2}\left(\frac{(\tau\sigma^{-1})^2\delta^2}{\mu(U_0,U)}\right)^{1/2}\\ &\leq C_2\frac{M^{1/2}}{(1-M)^{3/4}}\left(\frac{1}{\mu(V,U)}+\frac{1}{\mu(U_0,V)}\right)^{1/2}\left(\frac{1}{\mu(U_0,U)}\right)^{1/2},\end{aligned}$$

where C_2 depends only on $\delta(=(b-a)/2)$ and $\tau\sigma^{-1}$. **q.e.d.**

We would like to remove the condition that $\nu^*d\alpha$ have no periods. Let A be a bordered Riemann surface. (The boundary may consist of a finite number of curves but no points.) Let $\{\gamma_1,\cdots,\gamma_N\}$ be a minimal set of generators for the fundamental group of A. There exist harmonic 1-forms $\{\eta_1,\cdots,\eta_N\}$ on A with

$$\int_{\gamma_i}\eta_j=\delta_{ij}.$$

For any closed one form ξ on A, there is a function u, unique up to an additive constant, with

$$\xi=du+\sum_i c_i\eta_i,$$

where

$$c_i=\int_{\gamma_i}\xi.$$

We define the oscillation of ξ on A to be the oscillation of u on A, i.e.

$$\mathrm{osc}_A\xi:=\sup_A u-\inf_A u.$$

Lemma 6.1 *Let A be a Riemann surface which is conformally an annulus $\{1 < |z| < R\}$. Let ξ be a smooth closed 1-form on the closure of A and let P be the period of ξ over a generator of $\pi_1(A)$. Let $\exp : \mathbf{C} \to \mathbf{C}^*$ denote the exponential map, and for n a positive integer and y_0 real, let $R_n = \{0 \le x \le \ln R,\ y_0 \le y \le y_0 + 2n\pi\}$. Finally let f be any function on R_n with $df = \exp^* \xi$. Then*

$$\mathrm{osc}_{R_n} f \le \mathrm{osc}_A(\xi) + n|P|.$$

(Here the oscillation of a function f means $\sup f - \inf f$*.)*

Proof. Let p and q be any points in R_n and let Γ be a path from p to q consisting of one horizontal and one vertical line segment. On R_n we have

$$\exp^* \xi = \exp^* du + P(\frac{dy}{2\pi}).$$

Therefore

$$f(q) - f(p) = \int_\Gamma \exp^* \xi = u(q) - u(p) + P \int_\Gamma \frac{dy}{2\pi}\ ,$$

and so

$$|f(q) - f(p)| \le \mathrm{osc}_A(u) + n|P|\ ,$$

from which the result follows. (Note that to obtain the last inequality, we have used that dy has a sign on vertical vectors). **q.e.d.**

Corollary 6.2 *Let A be a Riemann surface which is conformally an annulus and let $X : A \to \mathbf{R}^3$ be an immersion with constant anisotropic mean curvature with respect to a functional satisfying (40). Assume that the variation of $d\alpha$ over any positively oriented generator of $\pi_1(A)$ is $2\pi k$ and that the Gauss map satisfies (41). Then for any concentric annulus $A_0 \subset A$, there holds*

$$\int_{A_0} |K|\, d\Sigma \le \frac{cM^{1/2}}{(1-M)^{3/4}} \cdot I(A_0, A),$$

where $I(A_0, A)$ is a conformal invariant of the pair (A_0, A) and c depends only on $\tau\sigma^{-1}$, $\mathrm{osc}_A(\nu^* d\alpha)$ and k.

Remark 6.1 The value of $I(A_0, A)$ is given in the proof. These numbers are bounded below away from zero.

Remark 6.2 The corollary gives an upper bound on how many times the Gauss map can "vertically" wrap A_0 over $\{\nu_3^2 < M\}$ in terms of the conformal invariant $I(A_0, A)$. The anisotropic unduloids of the previous section provide examples of surfaces whose Gauss maps multiple cover a band about the equator.

Proof of Corollary 6.2. We may assume that A is equivalent to $\{1 < |z| < R\} \subset \mathbf{C}$ and that $A_0 \subset \{1 + \epsilon < |z| < R - \epsilon\}$ for some $\epsilon > 0$. Let $U = \{0 < x < \ln R,\ -2\pi < y < 2\pi\}$, $U_0 = \{\ln(1+\epsilon) < x < \ln(R-\epsilon),\ -\pi < y < \pi\}$.

By the previous lemma, we obtain a bound on $\mathrm{osc}_{\mathrm{U}}\hat{\alpha}$ which depends only on $\mathrm{osc}_A(\nu^* d\alpha)$ and k. By applying Theorem 6.1, we obtain (43). Set

$$I_1 = \left\{ \frac{1}{\mu(U_0, U)} \inf_{U_0 \subset V \subset U} \left(\frac{1}{\mu(U_0, V)} + \frac{1}{\mu(V, U)} \right) \right\}^{1/2} \quad .$$

We then define $I(A_0, A)$ to be the infimum of all such upper bounds I_1 as (U_0, U) range over all such coverings. **q.e.d.**

Corollary 6.3 *There exist constants $c = c(b - a, \tau\sigma^{-1})$ with the following property: Let $G \subset \mathbf{C}$ be a domain and let $X : G \to \mathbf{R}^3$ be a conformal immersion with constant anisotropic mean curvature with respect to a uniform parametric elliptic functional. Assume that the image of its Gauss map is contained in a region Ω as in the previous theorem. If G compactly contains a disc $D_R = D_R(z_0) = \{|z - z_0| < R\}$, then for any $r < R$, there holds,*

$$\int_{D_r} |K| \, d\Sigma \leq \frac{cM^{1/2}}{(1 - M)^{3/4} \log(R/r)}.$$

Proof. The result is obtained from Corollary 6.1 by noting that $1/\mu(D_r, D_R) = 2\pi / \log(R/r)$ and

$$\inf_{r<\rho<R} \left\{ f(\rho) := \frac{1}{\log(\rho/r)} + \frac{1}{\log(R/\rho)} \right\} = f(\sqrt{rR}) = \frac{4}{\log(R/r)}.$$

q.e.d.

Corollary 6.4 *Let Σ be a parabolic Riemann surface. Let $X : \Sigma \to \mathbf{R}^3$ be a conformal immersion with constant anisotropic mean curvature with respect to a uniform parametric elliptic functional. Assume that the image of its Gauss map is contained in a region $S^2 \setminus \pm\{E_3\}$ and that the Gauss map satisfies (42) in Theorem 6.1. Then the surface is flat. More precisely, $X(\Sigma)$ is a right cylinder $C \times \mathbf{R}$ over a certain curve C in a plane which is parallel to the x_3-axis.*

Proof. From (49) in the proof of Theorem 6.1, we obtain

$$\int_{U_0} (\sin^2 \gamma) |\nabla \hat{\alpha}|^2 \, d\Sigma \leq \text{constant}/\mu(U_0, U),$$

for any $U_0 \subset U \subset \Sigma$. Note that

$$\sin \gamma = 0 \Longleftrightarrow \nu = \pm E_3.$$

So, in the present case, $\sin \gamma \neq 0$ at any point. If Σ is parabolic we can let U exhaust Σ and conclude that α is a constant. This implies the conclusion. **q.e.d.**

Corollary 6.5 *Let Ω be a domain with compact closure in $S^2 \setminus \pm E_3$, with the property that the winding number of every closed curve in Ω with respect to $-E_3$ is zero. There exists a constant $c = c(b - a, M)$ with the following property. For any CMC immersion $X : \Sigma \to \mathbf{R}^3$ whose Gauss map is contained in Ω, and for any relatively compact subdomain $U \subset \Sigma$ and simply connected domain $U_0 \subset U$, if*

$$1/\mu(U_0, U) + \inf_{U_0 \subset V \subset U} \left(\frac{1}{\mu(U_0, V)} + \frac{1}{\mu(V, U)} \right) \leq c \tag{51}$$

holds, then U_0 is strongly stable.

Proof. Note that we can choose $A \equiv 1$ and $\sigma = \tau = 1$. By combining (46), (49) and (50), we obtain that the energy of the Gauss map has an upper bound as in the left hand side of (51):

$$\begin{aligned} E[\nu] &= (1/2)\int_{U_0} |d\nu|^2 \, d\Sigma \\ &\leq 2\Big[\frac{\delta^2}{\mu(U_0,U)} + \frac{2M\delta^2}{(1-M)^{3/2}\mu(V,U)} + \frac{M}{(1-M)\mu(U_0,V)}\Big], \end{aligned}$$

where $\delta := (b-a)/2$. A result of Ruchert [12] (which follows from the Faber-Krahn inequality) states that, for a simply connected domain U_0 in a CMC surface,

$$E[\nu] = (1/2)\int_{U_0} |d\nu|^2 \, d\Sigma < 2\pi$$

implies strong stability. **q.e.d.**

We conclude this section by noting that the Bernstein type theorem of Hoffman-Osserman-Schoen [6] can be extended to certain surfaces with constant anisotropic mean curvature.

Theorem 6.2 *Let $X : \Sigma \to \mathbf{R}^3$ be an isometric immersion of a complete surface with constant anisotropic mean curvature with respect to a parametric elliptic functional satisfying (20). If the image of the Gauss map is contained in a closed hemisphere then the surface is flat.*

Proof. The proof is essentially the same as that used to prove the result of [6]. If the Gauss map is contained in a hemisphere $\langle \nu, a\rangle \geq 0$, $a \in S^2$, then the function $\langle \nu, a\rangle$ is a non negative solution of $L[u] = 0$. If $\langle \nu, a\rangle$ is not identically zero, it follows that the surface is globally strongly stable. By Theorem 4.1, the surface Σ would have to be compact, but this is impossible since the Gauss map of a compact surface is surjective.

If $\langle \nu, a\rangle \equiv 0$, then the image of the Gauss map is contained in a circle and so the Gauss curvature is identically zero. **q.e.d.**

7 Squared anisotropic mean curvature

Let $F : S^2 \to \mathbf{R}^+$ be a smooth function which satisfies the convexity condition, and let W be its Wulff shape. For a smooth, oriented immersion with Gauss map ν, consider the endomorphism of each tangent space defined by $-A(d\nu)$, where as before, $A := (D^2F + F1) \circ \nu$. Since both A and $d\nu$ are self-adjoint, the eigenvalues λ_1, λ_2 of $-A(d\nu)$ are real. We will refer to them as the *anisotropic principal curvatures.*

Courant and Hilbert define a plate as "an elastic two dimensional body, whose potential energy under deformations is given by a quadratic form in the principal curvatures of the plate." In analogy with the potential energy of a plate ([5]), we assign to each immersion of a compact surface Σ a potential energy whose density is a fixed positive definite quadratic form of the λ_i's. Such an energy can be expressed as

$$\mathcal{E}(a,b) = \int_\Sigma a(\lambda_1+\lambda_2)^2 + b\lambda_1\lambda_2 \, d\Sigma$$

for constants a, b which depend on the quadratic form.

Proposition 7.1 *If $a \neq 0$ and if only compactly supported variations of the immersion are considered, (or more generally, only $W_0^{2,2}$ variations are considered), then all the functionals $\mathcal{E}(a,b)$ are variationally equivalent to $\mathcal{E} := \mathcal{E}(1,0)$. In particular, they will all have the same critical points.*

Lemma 7.1 *Let Σ be a compact surface with smooth boundary and let M be a smooth surface. Let $f : \Sigma \to M$ be an immersion, and $f_t : \Sigma \to M$, $t \in (-\epsilon, \epsilon)$ be a smooth one parameter family of maps with $f_t \equiv f$ on $\partial\Sigma$. Let Ω_M denote the area form on M. Then*

$$\partial_t \Big(\int_\Sigma f_t^*(\Omega_M) \Big)_{t=0} = 0.$$

Proof. Because the boundary values of the maps are fixed, the tangent vector $\xi = \partial_t(f_t)_{t=0}$ lies in $\Gamma_0(f^*TM)$ = space of sections of the pull back of the tangent bundle of M which vanish on $\partial\Sigma$. It is well known that for any $p \in M$ there exists a 1-form θ on $M \setminus \{p\}$ which is a smooth solution of $d\theta = \Omega_M$. We assume first that f is not surjective and choose $p \in M \setminus f(\Sigma)$.

We have

$$\int_\Sigma f_t^* \Omega_M = \int_\Sigma f_t^*(d\theta) = \int_\Sigma d(f_t^*\theta) = \oint_{\partial\Sigma} f_t^*(\theta).$$

Let $\dot{c}$ be the unit tangent vector to $\partial\Sigma$. Note that

$$\oint_{\partial\Sigma} f_t^*(\theta) = \oint_{\partial\Sigma} \theta(f_*(\dot{c}) + t\nabla^M_{f_*\dot{c}}\xi + \mathcal{O}(t^2)).$$

The first order term in t is just

$$\oint_{\partial\Sigma} \theta(\nabla^M_{f_*\dot{c}}\xi). \tag{52}$$

Let $\theta^\#$ be the vector field associated to θ using the metric on M. Then,

$$0 = \dot{c}\langle \theta^\#, \xi\rangle = \langle \nabla^M_{f_*\dot{c}}\theta^\#, \xi\rangle + \langle \theta^\#, \nabla^M_{f_*\dot{c}}\xi\rangle,$$

shows that the integrand in (52) vanishes since ξ vanishes on the boundary.

The general case is easily treated using a partition of unity $\{\rho_i\}$ subordinate to an open cover $\{U_i\}$ of M such that on each U_i there is a solution θ_i of $d\theta_i = \rho_i\Omega_M$. The details are left to the reader. **q.e.d.**

Proof of Proposition 7.1. Since

$$\mathcal{E}(a,b) = \int_\Sigma a\Lambda^2 + b(K_\Sigma/K_W)\, d\Sigma,$$

it is enough to show that

$$\delta \int_\Sigma (K_\Sigma/K_W)\, d\Sigma = 0, \tag{53}$$

for all compactly supported variations.

Let $Y : S^2 \to \mathbf{R}^3$ be the embedding of the Wulff shape given by the inverse of the Gauss map of W. If $X : \Sigma \to \mathbf{R}^3$ is a smooth immersion of a compact, oriented surface with Gauss map ν, we let $f := Y \circ \nu$. If Ω is the area form on W and $\hat{\Omega}$ is the area form on S^2 then $Y^*\Omega = (1/K_W)\hat{\Omega}$ and so

$$f^*\Omega = \nu^*(1/K_W)\hat{\Omega} = \frac{K_\Sigma}{K_W}\, d\Sigma.$$

If X_t is a variation of X which fixes the boundary values up to first order, then we can apply Lemma 7.1 with $f_t := Y \circ \nu_t$ where ν_t is the corresponding variation of the Gauss map. This will show that (53) holds. **q.e.d.**

The functional

$$\mathcal{E}(X) = \int_\Sigma \Lambda^2 d\Sigma,$$

therefore has critical points which represent the equilibria of an anisotropic plate. They are also anisotropic analogues of Willmore surfaces. Note that the functional $\mathcal{E}$ is invariant under rescalings $(X \to rX)$ and translations of the immersion X. The Euler-Lagrange equation for this functional is derived as follows. Associated with the functional $\mathcal{F}$, there is a *Jacobi operator*

$$L[\psi] = \mathrm{div}_\Sigma((D^2F + F1)|_\nu \nabla\psi) + \langle (D^2F + F1)|_\nu d\nu, d\nu\rangle\psi.$$

From Proposition 2.1, this operator gives the infinitesimal change in Λ with respect to the normal variation $X_\epsilon = X + \epsilon\psi\nu + \mathcal{O}(\epsilon^2)$:

$$\delta\Lambda = L[\psi].$$

Using this, we compute the first variation of the functional $\mathcal{E}$ for compactly supported variation X_ϵ:

$$\delta\mathcal{E}(X) = 2\int_\Sigma (\Lambda\delta\Lambda - H\psi\Lambda^2)\, d\Sigma = 2\int_\Sigma (\Lambda L[\psi] - H\psi\Lambda^2)\, d\Sigma = 2\int_\Sigma \psi(L[\Lambda] - H\Lambda^2)\, d\Sigma.$$

Therefore $\delta\mathcal{E}(X) = 0$ for all compactly supported variations of X if and only if

$$L[\Lambda] - H\Lambda^2 = 0$$

holds.

Proposition 7.2 *The Wulff shape W minimizes $\mathcal{E}$ among all oriented, closed, genus zero surfaces in $\mathbf{R}^3$. It is the unique minimizer up to homothety and translation.*

Proof. Since the Wulff shape is convex, its Gauss map $N : W \to S^2$ is a diffeomorphism. Let $X : \Sigma \to \mathbf{R}^3$ be a smooth immersion with Gauss map ν. Let K_Σ denote the curvature of the induced metric and let K_W denote the curvature of the Wulff shape. We will also use K_W to denote the quantities $K_W(N^{-1}(\nu(p)))$, $p \in \Sigma$ and $K_W(N^{-1}(\nu))$, $\nu \in S^2$. We have shown previously, that

$$0 \le \Lambda^2 - 4K_\Sigma/K_W = (h_{11}/\mu_1 - h_{22}/\mu_2)^2 + 4h_{12}^2/\mu_1\mu_2$$

holds. Equality will occur at a point if and only if the operators A^{-1} and $d\nu$ are proportional. Reilly refers to such points as W-umbilics and shows ([11], Corollary 1 to Proposition 1) that any surface which is made up entirely of W-umbilics is homothetic to W.

We thus obtain when Σ is closed and genus$(\Sigma) = 0$,

$$\int_\Sigma \Lambda^2 d\Sigma \ge 4\int_\Sigma \frac{K_\Sigma}{K_W}\, d\Sigma = 4 \cdot \mathrm{degree}(\nu)\int_{S^2} \frac{1}{K_W} d\omega \ge 4\int_{S^2} \frac{1}{K_W} d\omega = 4 \cdot \mathrm{Area}(W),$$

where $d\omega$ is the area form on S^2. Since equality holds only for surfaces homothetic to W, this shows that W minimizes and, in particular, that W is a critical point of the functional $\mathcal{E}$. **q.e.d.**

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Added in Proof

Since the submission of this paper, the authors have obtained additional information concerning the anisotropic Delaunay surfaces which we wish to call to the Reader's attention.

If a rotationally symmetric integrand $F = F(\nu_3)$ is defined everywhere on S^2 and if the convexity condition holds everywhere, then the generating curve of an anisotropic catenoid is a graph over the whole of the vertical axis. This is Lemma 5.2 of our recent paper "*Anisotropic capillary surfaces with wetting energy*, preprint". This behavior is in contrast with Example 5.1 in the present paper.

Corollary 5.1 states that an anisotropic nodoid is either non-embedded or it is a compact embedded surface with genus 1. Lemma 4.1 (iv) in our recent paper "*Stability of anisotropic capillary surfaces between two parallel planes*, to appear in Calculus of Variations and Partial Differential Equations" shows that the second case does not occur.

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