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Sub-solution approach for the Asymptotic Behavior of a Parabolic Variational Inequality Related to American Options Problem

Djaber Chemseddine Benchettah
Department of Mathematics,
Faculty of Science,
Badji-Mokhtar-Annaba University,
P.O. Box 12, 23000 Annaba, Algeria
E-mail: benchettah097@gmail.com, haiourm@yahoo.fr

Abstract

The paper deals with the semi-implicit time scheme combined with a finite element spatial approximation for a parabolic variational inequality related to American options problem. The convergence of the iterative scheme is established and a optimal L^∞-asymptotic behavior is given. Furthermore, the proposed approach is based on a Sub-solutions method.

AMS subject classification:
Keywords: Sub-solution approach, Semi-implicit scheme, Finite elements approximation, Parabolic variational inequality, American option, Asymptotic behaviour.

1. Introduction

The American options problem in a black scholes model with constant coefficients and without dividend may be solved by considering the following parabolic variational inequality (P.V.I) see [21]:

$$\begin{cases} \dfrac{\partial U}{\partial t} + AU \geq 0, \ U \geq \Psi & \text{for } (t,x) \in (0,T] \times \Omega \\ \left(\dfrac{\partial U}{\partial t} + AU\right)(U - \Psi) = 0 & \text{for } (t,x) \in (0,T] \times \Omega \\ U(t,x) = \Psi(x) & \text{for } (t,x) \in (0,T] \times \Gamma \end{cases} \tag{1.1}$$

with the initial condition

$$U(0,x)=\Psi(x), \quad x\in\Omega,$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N\geq 1$, whith smooth boundary $\Gamma.\Psi(x)$ denote the payoff.

The operator A is given by:

$$AU=-\sum_{i,j=1}^{N}\left(\frac{1}{2}\sum_{k=1}^{N}\sigma_{ik}\sigma_{jk}\right)\frac{\partial^2 U}{\partial x_i\partial x_j}+\sum_{j=1}^{N}\left(\frac{1}{2}\sum_{i=1}^{N}\sigma_{ij}^2-r\right)\frac{\partial U}{\partial x_j}+rU. \tag{1.2}$$

To discretize (1.1) by the finite element method, we write the parabolic variational inequality related to american options problem in a more compact form.

Assuming that there exixts a C^2 function $\widehat{\psi}$ such that $\widehat{\psi}=\Psi$ on Γ, after setting $u=U-\widehat{\psi}$, we can reformulate the problem (1.1) to the following P.V.I:

Find $u\in\mathbb{K}$ solution of

$$\left(\frac{\partial u}{\partial t},v-u\right)+a(u,v-u)\geq(f,v-u),\ \ \forall v\in\mathbb{K}, \tag{1.3}$$

where $\mathbb{K}$ is a closed convex set defined as follows:

$$\mathbb{K}=\left\{\begin{array}{c} v(t,x)\in L^2([0,T],H_0^1(\Omega)),\quad v(t,x)\geq\psi(x),\\ v(0,x)=\psi(x)\text{ in }\Omega,\quad v=0\ \text{ on }\Gamma.\end{array}\right\}. \tag{1.4}$$

$$f\ \text{ is a regular function, } f=-A\widehat{\psi}\text{ in } L^2(\Omega). \tag{1.5}$$

$$\psi\text{ a positive obstacle of } W^{1,2}(\Omega),\text{ where }\psi=\Psi-\widehat{\psi}. \tag{1.6}$$

the function ψ has at least the same regularity as Ψ, and $\widehat{\psi}$ is a quite smooth function.

The bilinear form associated with operator A is given by:

$$a(u,v)=\int_{\Omega}\left(\sum_{i,j=1}^{N}a_{i,j}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}+\sum_{j=1}^{N}b_j\frac{\partial u}{\partial x_j}v+a_0uv\right)dx, \tag{1.7}$$

and the coefficients: $a_{i,j}, b_j, a_0$, where $1\leq i\leq N$, $1\leq j\leq N$ are satisfy the following conditions:

$$a_{i,j}=a_{j,i}=\frac{1}{2}\sum_{k=1}^{N}\sigma_{ik}\sigma_{jk};\ a_0=r\geq\beta>0,\text{ where }\beta\text{ is constant,} \tag{1.8}$$

$$\sum_{i,j=1}^{N}a_{i,j}\varepsilon_i\varepsilon_j\geq\gamma\,|\varepsilon|^2;\ \ \varepsilon\in\mathbb{R}^N,\ \ \gamma>0. \tag{1.9}$$

$$b_j=\frac{1}{2}\sum_{i=1}^{N}\sigma_{ij}^2-r. \tag{1.10}$$

For more detail on the parabolic variational inequality related to American options problem (see [1, 16, 22]).

The formalization of the American options problem as variational inequality and its discretization by numerical methods, appeared only rather tardily in the article of Jaillet, Lamberton and Lapeyre see [17]. A little later, the book of Wilmott, Dewynne and Howison see [22] has made it much more accessible the pricing by L.C.P from American Option problem.Then, the article of Xiao lan Zhang see [21], and the article of Feng, Linetsky, Morales, Nocedal see [16]. For the problems at free boundary several numerical results have was obtained for parabolic and elliptic variational and quasi-variational inequality see [2–11].

In this paper, we establish an L^{∞}-asymptotic behavior using the semi-implicit discretization scheme combined with a finite element spatial approximation and subsolution method. This method is introduced in P. Cortey-dumont [14] , [15] , M. Boulbrachene [5] , [8] , and M. Boulbrachene, P. Cortey-dumont [7], which characterizes the continuous solution (resp. the discrete solution) as the least upper bound of the set of continuous subsolutions (resp. the discrete subsolution) will also be crucial to prove the asymptotic behavior in uniform norm for the variational inequality related to American options problem. The approximation method developed in this paper stands on the construction a sequence of continuous subsolutions denoted $\left(\beta^{k}\right)_{k\geq 1}$ such that

$$\beta^{k} \leq u^{k} \text{ and } \left\|\beta^{k}-u_{h}^{k}\right\|_{\infty} \leq Ch^{2} \mid \ln h \mid^{2}, \forall k=1,\ldots,n,$$

and the construction a sequence of discrete subsolution $\left(\alpha_{h}^{k}\right)_{n\geq 1}$ such that

$$\alpha_{h}^{k} \leq u_{h}^{k} \text{ and } \left\|\alpha_{h}^{k}-u^{k}\right\|_{\infty} \leq Ch^{2} \mid \ln h \mid^{2}, \forall k=1,\ldots,n,$$

we obtain

$$\left\|u^{k}-u_{h}^{k}\right\|_{\infty} \leq Ch^{2} \mid \ln h \mid^{2}, \forall k=1,\ldots,n,$$

and the estimate of convergence of the continuous and discrete iterative scheme

$$\left\|u^{k}-u^{\infty}\right\|_{\infty} \leq \left(\frac{1}{1+\beta\Delta t}\right)^{k} \left\|u^{0}-u^{\infty}\right\|_{\infty},$$

$$\left\|u_{h}^{k}-u_{h}^{\infty}\right\|_{\infty} \leq \left(\frac{1}{1+\beta\Delta t}\right)^{k} \left\|u_{h}^{0}-u_{h}^{\infty}\right\|_{\infty}.$$

In this situation we establish an L^{∞}-asymptotic behavior that is:

$$\| u_{h}^{n} - u^{\infty} \|_{\infty} \leq C\left[h^{2} \mid \ln h \mid^{2} + \left(\frac{1}{1+\beta\Delta t}\right)^{n}\right].$$

It is worth mentioning that the construction of second membre f and the approximation method presented in this paper is entirely different from the one developed in [2].

The outline of the paper is as follows. In section 2, we transform the parabolic variational inequality into a non-coercive elliptic variational inequality by the semi-implicit scheme, then we associate with the continuous V.I a fixed point mapping and we use that in proving the existence of a unique continous solution, and we prove some related qualitative properties. In Section 3, we discretize the parabolic variational inequality by the semi-implicit Euler discretization scheme combined with a finite element method, then we associate with the discrete V.I a fixed point mapping and we use that in proving the existence of a unique discrete solution, and we give analogous qualitative properties for the discrete problem. In section 4, we introduce two auxiliary problems which allow us to define sequences of continuous and discrete subsolution. Finally, in Section 5, we establish the asymptotic behavior in L^{∞}-norm for the American Options problem.

2. Study of the continuous problem

2.1. The time discretization

Now, we apply the semi-implicit scheme of the continuous parabolic variational inequality (1.3), we have for $u^k \in \mathbb{K}$ and $k = 1, \ldots, n$

$$\left(\frac{u^k - u^{k-1}}{\Delta t}, v - u^k\right) + a(u^k, v - u^k) \geq (f, v - u^k), \quad \forall v \in \mathbb{K}, \tag{2.1}$$

where $\Delta t = \dfrac{T}{n}$, $t_k = k\Delta t$.

The inequality (2.1) is equivalent to

$$b(u^k, v - u^k) \geq (f + \lambda u^{k-1}, v - u^k), \quad \forall v \in \mathbb{K}, \tag{2.2}$$

such that

$$\begin{cases} b(u^k, v - u^k) = \lambda \left(u^k, v - u^k\right) + a(u^k, v - u^k), \\ \lambda = \dfrac{1}{\Delta t} = \dfrac{n}{T}. \end{cases} \tag{2.3}$$

Notation:

$(.,.)$ denotes the inner product in $L^2(\Omega)$.

$$\|.\|_{L^\infty} = \|.\|_\infty, \ \|.\|_{H^1} = \|.\|_1 \text{ and } \|.\|_{L^2} = \|.\|_2 .$$

2.2. Existence and uniqueness for the continuous P.V.I

Using the preceding assumptions, we shall prove the existence of a unique solution for problem (2.2) by means of the Banach fixed point theorem.

2.2.1 Fixed Point Mapping Associated with the continuous P.V.I (2.2)

We define the following mapping

$$T : L^\infty_+(\Omega) \to L^\infty_+(\Omega) \tag{2.4}$$

$$w \to T(w) = \xi,$$

such that ξ is the solution of the following problem:

$$b(\xi, v - \xi) \geq (f + \lambda w, v - \xi), \quad \forall v \in \mathbb{K}, \tag{2.5}$$

where $L^{\infty}_{+}(\Omega)$ is the positive cone of $L^{\infty}(\Omega)$.

2.2.2 Iterative continuous algorithm

We choose u^0 as the solution of the following continuous equation

$$b(u^0, v) = (F, v), \quad \forall v \in \mathbb{K}, \tag{2.6}$$

where F is a regular function given.

Now we give the following continuous algorithm:

$$u^k = Tu^{k-1}, \quad k = 1, \ldots, n, \quad u^k \in \mathbb{K}, \tag{2.7}$$

where ,u^k is the solution of the continuous P.V.I (2.2).

2.2.3 A monotonicity property of the continuous solution

Let $F = f + \lambda u^{k-1}$, $\tilde{F} = f + \lambda \tilde{u}^{k-1}$ be two right-hand sides and $u^k = \partial(F, \psi)$, $\tilde{u}^k = \partial\left(\tilde{F}, \tilde{\psi}\right)$ the correspending solutions to the continuous P.V.I (2.2), respectively.

Lemma 2.1. If $F \geq \tilde{F}$ and $\psi \geq \tilde{\psi}$ then $u^k \geq \tilde{u}^k$.

Proof. This proof is an adaptation of [5] .

Starting from u^0 and $\tilde{u}^0$ solutions to equation (2.6) with right-hand sides F and $\tilde{F}$, respectively. Then $u^0 \geq \tilde{u}^0$ implies $f + \lambda u^0 \geq f + \lambda \tilde{u}^0$ and $\psi \geq \tilde{\psi}$.

Therefore, applying standard comparison result in coercive VI (see [12]), we get

$$u^1 \geq \tilde{u}^1.$$

Now assume that

$$u^{k-1} \geq \tilde{u}^{k-1},$$

it follows that

$$f + u^{k-1} \geq f + \tilde{u}^{k-1},$$

and

$$\psi \geq \tilde{\psi}.$$

Finally, we get

$$u^k \geq \tilde{u}^k.$$

This completes the proof. ■

2.2.4 A lipschitz dependence property of the continuous solution

Proposition 2.2. Under the above notations and conditions of lemma 2.1, we have

$$\left\|u^k - \tilde{u}^k\right\|_\infty \leq \left\|F - \tilde{F}\right\|_\infty . \tag{2.8}$$

Proof. This proof is an adaptation of [5] . Let

$$\Phi = \left\|F - \tilde{F}\right\|_\infty .$$

it is easy to see that

$$F \leq \tilde{F} + \left\|F - \tilde{F}\right\|_\infty .$$

$$F \leq \tilde{F} + \Phi.$$

and

$$\psi \leq \tilde{\psi}.$$

Using Lemma 2.1, we have

$$\partial\left(F, \psi\right) \leq \partial\left(\tilde{F} + \Phi, \tilde{\psi}\right).$$

$$\leq \partial\left(\tilde{F} + \Phi, \tilde{\psi} + \Phi\right) = \partial\left(\tilde{F}, \tilde{\psi}\right) + \Phi.$$

where

$$\tilde{\psi} \leq \tilde{\psi} + \Phi.$$

Thus,

$$\partial\left(F, \psi\right) \leq \partial\left(\tilde{F}, \tilde{\psi}\right) + \Phi.$$

Therefore

$$u^k \leq \tilde{u}^k + \Phi.$$

Interchanging the roles of F and $\tilde{F}$, we similarly get

$$\tilde{u}^k \leq u^k + \Phi.$$

Therefore,

$$\left\|u^k - \tilde{u}^k\right\|_\infty \leq \left\|F - \tilde{F}\right\|_\infty .$$

This completes the proof. ■

Proposition 2.3. Under conditions of Lemma 2.1 and assumption (1.8), the mapping T is a contraction in $L^\infty(\Omega)$, i.e.,

$$\left\|T(w) - T(\tilde{w})\right\|_\infty \leq \frac{1}{1 + \beta\Delta t}\left\|w - \tilde{w}\right\|_\infty \tag{2.9}$$

Therefore, T admits a unique fixed point, which coincides with the solution of continuous V.I (2.2).

Proof. For w, $\tilde{w}$ in $L^{\infty}(\Omega)$, we consider $\xi = T(w) = \partial(f + \lambda w, \psi)$ and $\tilde{\xi} = T(\tilde{w}) = \partial(f + \lambda\tilde{w}, \tilde{\psi})$ solution to continuous variational inequality (2.2) with right-hand sid $F = f + \lambda w$, $\tilde{F} = f + \lambda\tilde{w}$.

Now, setting

$$\phi = \frac{1}{\lambda + \beta} \left\| F - \tilde{F} \right\|_{\infty}.$$

Then, it is clear that

$$\begin{aligned} F &\le \tilde{F} + \left\| F - \tilde{F} \right\|_{\infty}. \\ &\le \tilde{F} + \frac{a_0 + \lambda}{\beta + \lambda} \left\| F - \tilde{F} \right\|_{\infty}. \end{aligned}$$

(because $a_0 \ge \beta > 0$).

$$\le \tilde{F} + (a_0 + \lambda)\phi.$$

Using to Lemma 2.1, it follows that

$$\partial(F, \psi) \le \partial(\tilde{F} + (a_0 + \lambda)\phi, \tilde{\psi}).$$

On the other hand, one has

$$\partial(\tilde{F}, \tilde{\psi}) + \phi = \partial(\tilde{F} + (a_0 + \lambda)\phi, \tilde{\psi} + \phi).$$

Indeed, $\tilde{\xi} + \phi$ is solution of

$$b(\tilde{\xi} + \phi, (v + \phi) - (\tilde{\xi} + \phi)) \ge (\tilde{F} + (a_0 + \lambda)\phi, (v + \phi) - (\tilde{\xi} + \phi))$$

$$\tilde{\xi} + \phi \ge \psi + \phi, \ v + \phi \ge \psi + \phi, \ \forall v \in \mathbb{K}$$

Thus

$$\partial(F, \psi) \le \partial(\tilde{F} + (a_0 + \lambda)\phi, \tilde{\psi}) \le \partial(\tilde{F} + (a_0 + \lambda)\phi, \tilde{\psi} + \phi).$$

hence

$$\xi \le \tilde{\xi} + \phi.$$

Similarly, interchanging the roles of w and $\tilde{w}$, we also get

$$\tilde{\xi} \le \xi + \phi.$$

Finally, this yields

$$\begin{aligned} \|T(w) - T(\tilde{w})\|_{\infty} &\le \frac{1}{\lambda + \beta} \left\| F - \tilde{F} \right\|_{\infty} \\ &\le \frac{1}{\lambda + \beta} \|f + \lambda w - f - \lambda\tilde{w}\|_{\infty} \\ &\le \frac{\lambda}{\lambda + \beta} \|w - \tilde{w}\|_{\infty} \end{aligned}$$

$$\|T(w) - T(\tilde{w})\|_\infty \leq \frac{1}{1+\beta\Delta t}\|w - \tilde{w}\|_\infty .$$

This completes the proof. ■

Proposition 2.4. Under the conditions of Proposition 2.2 and the previous hypotheses, we have the following estimate of geometric convergence

$$\left\|u^k - u^\infty\right\|_\infty \leq \left(\frac{1}{1+\beta\Delta t}\right)^k \left\|u^0 - u^\infty\right\|_\infty , \tag{2.10}$$

where u^∞ is stationary solution to the following continuous V.I.

$$b(u^\infty, v - u^\infty) \geq (f + \lambda u^\infty, v - u^\infty), \quad \forall v \in \mathbb{K}. \tag{2.11}$$

Proof. For $k = 1$, we have

$$u^\infty = Tu^\infty,$$

and

$$\left\|u^1 - u^\infty\right\|_\infty = \left\|Tu^0 - Tu^\infty\right\|_\infty \leq \left(\frac{1}{1+\beta\Delta t}\right)\left\|u^0 - u^\infty\right\|_\infty .$$

We assume that, for step k

$$\left\|u^k - u^\infty\right\|_\infty \leq \left(\frac{1}{1+\beta\Delta t}\right)^k \left\|u^0 - u^\infty\right\|_\infty ,$$

then, we use the Bensoussan-Lions algorithm

$$\left\|u^{k+1} - u^\infty\right\|_\infty = \left\|Tu^k - Tu^\infty\right\|_\infty \leq \left(\frac{1}{1+\beta\Delta t}\right)\left\|u^k - u^\infty\right\|_\infty ,$$

thus

$$\begin{aligned}\left\|u^{k+1} - u^\infty\right\|_\infty &\leq \left(\frac{1}{1+\beta\Delta t}\right)\left(\frac{1}{1+\beta\Delta t}\right)^k \left\|u^0 - u^\infty\right\|_\infty . \\ &\leq \left(\frac{1}{1+\beta\Delta t}\right)^{k+1} \left\|u^0 - u^\infty\right\|_\infty .\end{aligned}$$

This completes the proof. ■

2.2.5 The notion of continuous subsolution

Definition 2.5. $\omega^k \in \mathbb{K}$ is said to be a subsolution for the continuous P.V.I (2.2) if

$$b(\omega^k, v) \leq (f + \lambda\omega^{k-1}, v), \quad \forall v \in \mathbb{K}, \tag{2.12}$$

Let $\mathbb{X}$ be the set of such subsolutions.

Theorem 2.6. The solution of the continuous P.V.I (2.2) is the maximum elements of the set $\mathbb{X}$.

Proof. It is an easy adaptation of [15] . ■

3. The study of discrete problem

We decomposed Ω into triangles and let τ_h denotes the set of all those elements, where $h > 0$ is the mesh size. We assume that the family τ_h is regular and quasi-uniform. We consider the usual basis of affine functions φ_i, $i = \{1, \ldots, m(h)\}$ defined by $\varphi_i(M_j) = \delta_{ij}$, where M_j is a vertex of the considered triangulation.We introduce the following discrete spaces $\mathbb{V}^h$ of finite element constructed from polynomials of degree 1.

$$\mathbb{V}^h = \{v_h \in L^2(0, T, H_0^1(\Omega)) \cap C(0, T, H_0^1(\bar{\Omega})), \text{ such that } v_h \mid_k \in P_1, k \in \tau_h\}, \quad (3.1)$$

and

$$\mathbb{K}^h = \{v_h \in V^h, \ v_h \geq r_h\psi_h, \ v_h(0, x) = r_h\psi_h \text{ in } \Omega, \quad v_h = 0 \text{ on } \Gamma\}. \quad (3.2)$$

We consider r_h be the usual interpolation operator defined by

$$v_h \in L^2(0, T, H_0^1(\Omega)) \cap C(0, T, H_0^1(\bar{\Omega})), \ r_h v = \sum_{i=1}^{m(h)} v(M_i)\varphi_i(x). \quad (3.3)$$

The discrete maximum principle assumption (dmp) $(cf.$[13]): We assume the matrix whose generic coefficients $a(\varphi_i, \varphi_j)$ are M-matrix.

Theorem 3.1. [$cf.$[20]] Let us assume that the bilinear form $a(.,.)$ is weakly coercive in $\mathbb{K}^h \subset H_0^1(\Omega)$, there exists two constants $\alpha > 0$ and $\lambda > 0$ such that:

$$a(u_h, u_h) + \lambda \|u_h\|_2 \geq \alpha \|u_h\|_1 . \quad (3.4)$$

3.1. Discretization

We discretize the space $H_0^1(\Omega)$ by a space discretization of finite dimensional $\mathbb{K}^h \in H_0^1(\Omega)$ constructed from polynomials of degree 1. In a second step, we discretize the problem with respect to time using the semi-implicit scheme. For this, we search a sequence of elements $u_h^k \in \mathbb{K}^h$ which approaches $u^k(t_k)$, $t_k = k\Delta t$, with initial data $u_h^0 = u_{0h}$.

We apply the finite element method to approximate inequality (1.3), and the semi-discrete P.V.I takes the form of

$$\left(\frac{\partial u_h}{\partial t}, v_h - u_h\right) + a(u_h, v_h - u_h) \geq (f, v_h - u_h), \quad \forall v_h \in \mathbb{K}^h. \quad (3.5)$$

Now, we apply the semi-implicit scheme on the semi-discrete problem (3.5); for any $k = 1, \ldots, n$, we have for $v_h \in \mathbb{K}^h$

$$\left(\frac{u_h^k - u_h^{k-1}}{\Delta t}, v_h - u_h^k\right) + a(u_h^k, v_h - u_h^k) \geq (f, v_h - u_h^k). \quad (3.6)$$

Then, problem (3.6) can be refomulated into the following coercive discrete elliptic variational inequality:

$$b(u_h^k, v_h - u_h^k) \geq (f + \lambda u_h^{k-1}, v_h - u_h^k), \quad \forall v_h \in \mathbb{K}^h. \tag{3.7}$$

such that

$$\begin{cases} b(u_h^k, v_h - u_h^k) = \lambda \left(u_h^k, v_h - u_h^k\right) + a(u_h^k, v_h - u_h^k), \\ \lambda = \dfrac{1}{\Delta t} = \dfrac{n}{T}. \end{cases} \tag{3.8}$$

3.2. Existence and uniqueness for the discrete V.I

In a similar way to that of Section 2, we shall characterize the solution of discrete V.I as the unique fixed point of a contraction. To that end, we need a monotonicity result and a lipschitz dependence property for the discrete V.I.

3.2.1 Fixed Point Mapping Associated with the discrete V.I (3.7)

We define the following mapping

$$T_h : L_+^\infty(\Omega) \to \mathbb{K}^h \tag{3.9}$$

$$w \to T_h(w) = \xi_h,$$

such that ξ_h is the solution of the following problem:

$$b(\xi_h, v_h - \xi_h) \geq (f + \lambda w, v_h - \xi_h), \quad \forall v_h \in \mathbb{K}^h, \tag{3.10}$$

where $L_+^\infty(\Omega)$ is the positive cone of $L^\infty(\Omega)$.

3.2.2 Iterative discrete algorithm

We choose u_h^0 as the solution of the following discrete equation

$$b(u_h^0, v_h) = (F, v_h), \quad \forall v_h \in \mathbb{K}^h, \tag{3.11}$$

where F is a regular function given.

Now we give the following discrete algorithm:

$$u_h^k = T_h u_h^{k-1}, \quad k = 1, \ldots, n, \quad u_h^k \in \mathbb{K}^h, \tag{3.12}$$

where u_h^k is the solution of the discrete P.V.I (3.7).

Remark 3.2. Under the D.M.P., the qualitative properties and results established in the section 2 are conserved in the discrete case.Their respective proofs will be omitted as they are very similar to their continuous analogous ones.

3.2.3 A monotonicity property of the discrete solution

Let $F = f + \lambda u_h^{k-1}$, $\tilde{F} = f + \lambda \tilde{u}_h^{k-1}$ be two right-hand sides and $u_h^k = \partial_h (F, r_h\psi)$, $\tilde{u}_h^k = \partial_h \left(\tilde{F}, r_h\tilde{\psi}\right)$ the correspending solutions to the discrete P.V.I (3.7), respectively.

Lemma 3.3. **[**cf[5]**]** Under the D.M.P, If $F \geq \tilde{F}$ and $r_h\psi \geq r_h\tilde{\psi}$ then $u_h^k \geq \tilde{u}_h^k$.

3.2.4 A lipschitz dependence property of the discrete solution

Proposition 3.4. **[**cf[5]**]** Under the above notations and conditions of Lemma 3.3, we have

$$\|u_h^k - \tilde{u}_h^k\|_\infty \leq \left\|F - \tilde{F}\right\|_\infty . \tag{3.13}$$

Proposition 3.5. **[**cf[2]**]** Under conditions of Lemma 3.3 and assumption (1.8), the mapping T_h is a contraction in $L^\infty(\Omega)$, i.e.,

$$\|T_h(w) - T_h(\tilde{w})\|_\infty \leq \frac{1}{1+\beta\Delta t} \|w - \tilde{w}\|_\infty . \tag{3.14}$$

Therefore, T_h admits a unique fixed point, which coincides with the solution of discrete P.V.I (3.7).

Proposition 3.6. **[**cf[2]**]** Under the conditions of Proposition 2.4 and the previous hypotheses, we have the following estimate of geometric convergence

$$\|u_h^k - u_h^\infty\|_\infty \leq \left(\frac{1}{1+\beta\Delta t}\right)^k \|u_h^0 - u_h^\infty\|_\infty , \tag{3.15}$$

where u_h^∞ is stationary solution to the following discrete V.I

$$b(u_h^\infty, v_h - u_h^\infty) \geq (f + \lambda u_h^\infty, v_h - u_h^\infty), \ \ \forall v_h \in \mathbb{K}^h. \tag{3.16}$$

3.2.5 The notion of discrete subsolution

Definition 3.7. **[**Cf[15]**]** $\omega_h^k \in \mathbb{K}^h$ is said to be a subsolution for the discrete V.I (2.2) if

$$b(\omega_h^k, \varphi_i) \leq (f + \lambda\omega_h^{k-1}, \varphi_i), \ \forall i \in \{1, \ldots, m(h)\} . \tag{3.17}$$

Let $\mathbb{X}_h$ be the set of such discrete subsolutions.

Theorem 3.8. **[**Cf[15]**]** Under the **D.M.P.**, the solution of the discrete.P.V.I (3.7) is the maximum elements of the set $\mathbb{X}_h$.

4. Error estimates in the L^∞-norm

In this section, we introduce two auxiliary problems which allow us to define sequences of continuous and discrete subsolution.

4.1. Two Auxiliary Sequences of Variational inequality

Let $\bar{u}^k$ be the solution of the continuous V.I:

$$b(\bar{u}^k, v - \bar{u}^k) \geqq (f + \lambda u_h^{k-1}, v - \bar{u}^k) \; \forall v \in \mathbb{K}, \tag{4.1}$$

where u_h^k is defined in (3.12). Let $\bar{u}_h^k$ be the solution of the discrete V.I:

$$b(\bar{u}_h^k, v_h - \bar{u}_h^k) \geqq (f + \lambda u^{k-1}, v_h - \bar{u}_h^k) \; \forall v \in \mathbb{K}^h, \tag{4.2}$$

where u^k is defined in (2.7).

Lemma 4.1. **[**cf[15]**]** There exists a constant C independent of h, and k such that

$$\| \bar{u}^k - u_h^k \|_\infty \leq Ch^2 \mid \ln h \mid^2, \tag{4.3}$$

and

$$\| \bar{u}_h^k - u^k \|_\infty \leq Ch^2 \mid \ln h \mid^2 . \tag{4.4}$$

Now, we shall estimate the error in the L^∞-norm between the u^k and u_h^k defined in (2.7) and (3.12), respectively.

Theorem 4.2.

$$\left\| u^k - u_h^k \right\|_\infty \leq Ch^2 \mid \ln h \mid^2 . \tag{4.5}$$

the following lemma plays crucial role in proving the theorem (4) .

Lemma 4.3. There exists a sequence of continuous subsolutions $(\beta^k)_{k\geq 1}$ such that

$$\beta^k \leq u^k, \forall k = 1, \ldots, n, \tag{4.6}$$

and

$$\left\| \beta^k - u_h^k \right\|_\infty \leq Ch^2 \mid \ln h \mid^2, \tag{4.7}$$

and a sequence of discrete subsolutions $(\alpha_h^k)_{k\geq 1}$ such that

$$\alpha_h^k \leq u_h^k, \forall k = 1, \ldots, n, \tag{4.8}$$

and

$$\left\| \alpha_h^k - u^k \right\|_\infty \leq Ch^2 \mid \ln h \mid^2 . \tag{4.9}$$

Proof. Consider the continuous V.I

$$b(\bar{u}^1, v - \bar{u}^1) \geqq (f + \lambda u_h^0, v - \bar{u}^1) \; \forall v \in \mathbb{K}.$$

Then, as $\bar{u}^1$ is solution to a continuous V.I, it is also a subsolution, i.e.,

$$b(\bar{u}^1, v) \leq (f + \lambda u_h^0, v) \; \forall v \in \mathbb{K}.$$

and

$$b(\bar{u}^1, v) \leq (f + \lambda u_h^0 - \lambda u^0 + \lambda u^0, v)\ \forall v \in \mathbb{K}.$$

we have

$$\left\| u_h^0 - u^0 \right\|_\infty \leq Ch^2 \mid \ln h \mid \quad \text{(see [19])} \tag{4.10}$$

Then

$$b(\bar{u}^1, v) \leq (f + \lambda \left\| u_h^0 - u^0 \right\|_\infty + \lambda u^0, v)\ \forall v \in \mathbb{K}.$$

and using (4.10) , we get

$$b(\bar{u}^1, v) \leq (f + \lambda Ch^2 \mid \ln h \mid + \lambda u^0, v)\ \forall v \in \mathbb{K}.$$

So, $\bar{u}^1$ is a subsolution for the continuous V.I whose solution is

$$\bar{U}^1 = \partial(f + \lambda Ch^2 \mid \ln h \mid + \lambda u^0).$$

Then, as $u^1 = \partial(f + \lambda u^0)$ making use of Proposition (1), we have

$$\begin{aligned} \left\| \bar{U}^1 - u^1 \right\|_\infty &\leq \left\| f + \lambda Ch^2 \mid \ln h \mid + \lambda u^0 - f - \lambda u^0 \right\|_\infty \\ &\leq Ch^2 \mid \ln h \mid, \end{aligned}$$

and, due to Theorem (1), we have

$$\bar{u}^1 \leq \bar{U}^1 \leq u^1 + Ch^2 \mid \ln h \mid .$$

Now putting

$$\beta^1 = \bar{u}^1 - Ch^2 \mid \ln h \mid,$$

we have

$$\beta^1 \leq u^1 \tag{4.11}$$

and

$$\begin{aligned} \left\| \beta^1 - u_h^1 \right\|_\infty &\leq \left\| \bar{u}^1 - Ch^2 \mid \ln h \mid - u_h^1 \right\|_\infty \qquad (4.12) \\ &\leq \left\| \bar{u}^1 - u_h^1 \right\|_\infty + Ch^2 \mid \ln h \mid \\ &\leq Ch^2 \mid \ln h \mid^2 + Ch^2 \mid \ln h \mid \\ &\leq Ch^2 \mid \ln h \mid^2 . \end{aligned}$$

Consider the discrete of discrete V.I

$$b(\bar{u}_h^1, v_h - \bar{u}_h^1) \geqq (f + \lambda u^0, v_h - \bar{u}_h^1)\ \forall v_h \in \mathbb{K}^h.$$

Then, as $\bar{u}_h^1$ is solution to discrete V.I, it is also a subsolution i.e.,

$$b(\bar{u}_h^1, \varphi_s) \leq (f + \lambda u^0, \varphi_s) \ \forall \varphi_s, s = 1, \ldots, m(h)$$

and

$$b(\bar{u}_h^1, \varphi_s) \leq (f + \lambda u^0 - \lambda u_h^0 + \lambda u_h^0, \varphi_s) \ \forall \varphi_s, s = 1, \ldots, m(h)$$

Then

$$b(\bar{u}_h^1, \varphi_s) \leq (f + \lambda \left\| u^0 - u_h^0 \right\|_\infty + \lambda u_h^0, \varphi_s) \ \forall \varphi_s, s = 1, \ldots, m(h)$$

and using (4.10) , we get

$$b(\bar{u}_h^1, \varphi_s) \leq (f + \lambda C h^2 \mid \ln h \mid + \lambda u_h^0, \varphi_s) \ \forall \varphi_s, s = 1, \ldots, m(h)$$

So, $\bar{u}_h^1$ is a subsolution for the system of V.I whose solution is

$$\bar{U}_h^1 = \partial_h (f + \lambda C h^2 \mid \ln h \mid + \lambda u_h^0).$$

Then, as $u_h^1 = \partial_h (f + \lambda u_h^0)$ making use of Proposition (4), we have

$$\begin{aligned} \left\| \bar{U}_h^1 - u_h^1 \right\|_\infty &\leq \left\| f + \lambda C h^2 \mid \ln h \mid + \lambda u_h^0 - f - \lambda u_h^0 \right\|_\infty \\ &\leq C h^2 \mid \ln h \mid . \end{aligned}$$

and, due to Theorem (3), we have

$$\bar{u}_h^1 \leq \bar{U}_h^1 \leq u_h^1 + C h^2 \mid \ln h \mid .$$

Now putting

$$\alpha_h^1 = \bar{u}_h^1 - C h^2 \mid \ln h \mid,$$

we have

$$\alpha_h^1 \leq u_h^1, \tag{4.13}$$

and

$$\begin{aligned} \left\| \alpha_h^1 - u^1 \right\|_\infty &\leq \left\| \bar{u}_h^1 - C h^2 \mid \ln h \mid - u^1 \right\|_\infty \qquad (4.14) \\ &\leq \left\| \bar{u}_h^1 - u^1 \right\|_\infty + C h^2 \mid \ln h \mid \\ &\leq C h^2 \mid \ln h \mid^2 + C h^2 \mid \ln h \mid \\ &\leq C h^2 \mid \ln h \mid^2 . \end{aligned}$$

Thus, combining (4.11),(4.12) and (4.13),(4.14) we get

$$\begin{aligned} u^1 &\leq \alpha_h^1 + Ch^2 \mid \ln h \mid^2 \\ &\leq u_h^1 + Ch^2 \mid \ln h \mid^2 \\ &\leq \beta^1 + Ch^2 \mid \ln h \mid^2 \\ &\leq u^1 + Ch^2 \mid \ln h \mid^2 \end{aligned}$$

Therefore

$$\left\| u^1 - u_h^1 \right\|_\infty \leq Ch^2 \mid \ln h \mid^2 .$$

Step k. we assume that

$$\left\| u^{k-1} - u_h^{k-1} \right\|_\infty \leq Ch^2 \mid \ln h \mid^2, \tag{4.15}$$

and we prove that

$$\left\| u^k - u_h^k \right\|_\infty \leq Ch^2 \mid \ln h \mid^2 .$$

For that, consider the continuous V.I

$$b(\bar{u}^k, v - \bar{u}^k) \geqq (f + \lambda u_h^{k-1}, v - \bar{u}^k), \ \forall v \in \mathbb{K}.$$

Then, as $\bar{u}^k$ is solution to a continuous V.I, it is also a subsolution i.e.,

$$b(\bar{u}^k, v) \leq (f + \lambda u_h^{k-1}, v), \ \forall v \in \mathbb{K},$$

and

$$b(\bar{u}^k, v) \leq (f + \lambda u_h^{k-1} - \lambda u^{k-1} + \lambda u^{k-1}, v), \ \forall v \in \mathbb{K}.$$

Then

$$b(\bar{u}^k, v) \leq (f + \lambda \left\| u_h^{k-1} - u^{k-1} \right\|_\infty + \lambda u^{k-1}, v), \ \forall v \in \mathbb{K}.$$

and, using (4.15), we get

$$b(\bar{u}^k, v) \leq (f + \lambda Ch^2 \mid \ln h \mid^2 + \lambda u^{k-1}, v), \ \forall v \in \mathbb{K}.$$

Let $\bar{u}^k$ subsolution for the continuous V.I, whose solution is

$$\bar{U}^k = \partial(f + \lambda Ch^2 \mid \ln h \mid^2 + \lambda u^{k-1}).$$

Then, as $u^k = \partial(f + \lambda u^{k-1})$, making use of Proposition (2), we have

$$\begin{aligned} \left\| \bar{U}^k - u^k \right\|_\infty &\leq \left\| f + \lambda Ch^2 \mid \ln h \mid^2 + \lambda u^{k-1} - f - \lambda u^{k-1} \right\|_\infty \\ &\leq Ch^2 \mid \ln h \mid^2, \end{aligned}$$

and, due to Theorem (1), we have

$$\bar{u}^k \leq \bar{U}^k \leq u^k + Ch^2 \mid \ln h \mid^2 .$$

Now putting

$$\beta^k = \bar{u}^k - Ch^2 \mid \ln h \mid^2,$$

we have

$$\beta^k \leq u^k, \tag{4.16}$$

and

$$\begin{aligned} \left\| \beta^k - u_h^k \right\|_\infty &\leq \left\| \bar{u}^k - Ch^2 \mid \ln h \mid^2 - u_h^k \right\|_\infty \qquad (4.17) \\ &\leq \left\| \bar{u}^k - u_h^k \right\|_\infty + Ch^2 \mid \ln h \mid^2 \\ &\leq Ch^2 \mid \ln h \mid^2 + Ch^2 \mid \ln h \mid^2 \\ &\leq Ch^2 \mid \ln h \mid^2 . \end{aligned}$$

Consider the discrete V.I

$$b(\bar{u}_h^k, v_h - \bar{u}_h^k) \geqq (f + \lambda u^{k-1}, v_h - \bar{u}_h^k), \ \forall v_h \in \mathbb{K}^h.$$

Then, as $\bar{u}_h^k$ is solution to a discrete V.I, it is also a subsolution i.e.,

$$b(\bar{u}_h^k, \varphi_s) \leq (f + \lambda u^{k-1}, \varphi_s) \, \forall \varphi_s, s = 1, \ldots, m(h)$$

and

$$b(\bar{u}_h^k, \varphi_s) \leq (f + \lambda u^{k-1} - \lambda u_h^{k-1} + \lambda u_h^{k-1}, \varphi_s) \, \forall \varphi_s, s = 1, \ldots, m(h)$$

Then, we have

$$b(\bar{u}_h^k, \varphi_s) \leq (f + \lambda \left\| u^{k-1} - u_h^{k-1} \right\|_\infty + \lambda u_h^{k-1}, \varphi_s) \, \forall \varphi_s, s = 1, \ldots, m(h)$$

and, using (4.15), we get

$$b^i(\bar{u}_h^k, \varphi_s) \leq (f + \lambda Ch^2 \mid \ln h \mid^2 + \lambda u_h^{k-1}, \varphi_s) \, \forall \varphi_s, s = 1, \ldots, m(h).$$

Let $\bar{u}_h^k$ subsolution for the system of V.I, whose solution is

$$\bar{U}_h^k = \partial_h(f + \lambda Ch^2 \mid \ln h \mid^2 + \lambda u_h^{k-1}).$$

Then, as $u_h^k = \partial_h(f + \lambda u_h^{k-1})$ making use of Proposition (4), we have

$$\begin{aligned} \left\| \bar{U}_h^k - u_h^k \right\|_\infty &\leq \left\| f + \lambda Ch^2 \mid \ln h \mid^2 + \lambda u_h^{k-1} - f - \lambda u_h^{k-1} \right\|_\infty \\ &\leq Ch^2 \mid \ln h \mid^2 . \end{aligned}$$

and, due to Theorem 3, we have

$$\bar{u}_h^k \leq \bar{U}_h^k \leq u_h^k + Ch^2 \mid \ln h \mid^2 .$$

Now putting

$$\alpha_h^k = \bar{u}_h^k - Ch^2 \mid \ln h \mid^2,$$

we have

$$\alpha_h^k \leq u_h^k, \tag{4.18}$$

and

$$\begin{aligned} \left\|\alpha_h^k - u^k\right\|_\infty &\leq \left\|\bar{u}_h^k - Ch^2 \mid \ln h \mid^2 - u^k\right\|_\infty \\ &\leq \left\|\bar{u}_h^k - u^k\right\|_\infty + Ch^2 \mid \ln h \mid^2 \\ &\leq Ch^2 \mid \ln h \mid^2 + Ch^2 \mid \ln h \mid^2 \\ &\leq Ch^2 \mid \ln h \mid^2 . \end{aligned} \tag{4.19}$$

Thus, combining (4.16),(4.17) and (4.18),(4.19) we get

$$\begin{aligned} u^k &\leq \alpha_h^k + Ch^2 \mid \ln h \mid^2 \\ &\leq u_h^k + Ch^2 \mid \ln h \mid^2 \\ &\leq \beta^k + Ch^2 \mid \ln h \mid^2 \\ &\leq u^k + Ch^2 \mid \ln h \mid^2 \end{aligned}$$

Therefore

$$\left\|u^k - u_h^k\right\|_\infty \leq Ch^2 \mid \ln h \mid^2 .$$

■

5. Asymptotic behavior in L^∞-norm for the American Options problem

This section is devoted to the proof of principal result of the present paper, where we prove the theorem of the asymptotic behavior in L^∞-norm for parabolic variational inequality related to American options problem. More precisely, we evaluate the variation in L^∞ between $u_h(T, x)$, the discrete solution calculated at the moment $T = n\Delta t$ and u^∞, the stationary solution of continuous variational inequality.

Theorem 5.1. Under the results of the proposition (3) and theorem (4), we have

$$\left\| u_h^n - u^\infty \right\|_\infty \leq C \left[h^2 \mid \ln h \mid^2 + \left(\frac{1}{1 + \beta \Delta t} \right)^n \right]. \tag{5.1}$$

where C is a constant independent of h and n.

Proof. We have

$$u_h^k = u_h(t, x) \quad \text{for} \quad t \in](k-1)\Delta t; k\Delta t[, \tag{5.2}$$

thus

$$u_h^n = u_h(T, x), \tag{5.3}$$

then

$$\left\| u_h(T, x) - u^\infty \right\|_\infty = \left\| u_h^n - u^\infty \right\|_\infty \leq \left\| u_h^n - u^n \right\| + \left\| u^n - u^\infty \right\|.$$

Using the Theorem 5.1 and the Proposition 3.4, we have

$$\left\| u_h^n - u^\infty \right\|_\infty \leq C \left[h^2 \mid \ln h \mid^2 + \left(\frac{1}{1 + \beta \Delta t} \right)^n \right].$$

■

6. Conclusion

In this paper, we study the finite element approximation of the variational inequality related to American options problem. We establish an L^∞-asymptotic behavior similar to that in [2] using the subsolution method. The type of estimation that we get here enables us to locate the free boundary,thing crucial in practice of the American options. A future work we will consolidate our theoretical results by numerical simulation, where efficient numerical monotone algorithms will be treated.

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