The eigenvalue for a class of singular \( p \)-Laplacian fractional differential equations involving the Riemann–Stieltjes integral boundary condition

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**Abstract**
In this paper, we are concerned with the eigenvalue problem of a class of singular \( p \)-Laplacian fractional differential equations involving the Riemann–Stieltjes integral boundary condition. The conditions for the existence of at least one positive solution is established together with the estimates of the lower and upper bounds of the solution at any instant of time. Our results are derived based on the method of upper and lower solutions and the Schauder fixed point theorem.

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1. Introduction

This paper deals with the eigenvalue problem for a class of singular \( p \)-Laplacian fractional differential equations (PFDE for short) involving the Riemann–Stieltjes integral boundary condition

\[
\begin{align*}
-\frac{d}{dt} \left( \varphi_p \left( \frac{d}{dt} x(t) \right) \right) &= f(t, x(t)), \quad t \in (0, 1), \\
x(0) &= 0, \quad \frac{d}{dt} x(0) = 0, \quad x(1) = \int_0^1 x(s) \, dA(s),
\end{align*}
\]

where \( \varphi_p \) and \( \varphi_p \) are the standard Riemann–Liouville derivatives with \( 1 < \alpha < 2 \), \( 0 < \beta < 1 \). \( A \) is a function of bounded variation and \( \int_0^1 x(s) \, dA(s) \) denotes the Riemann–Stieltjes integral of \( x \) with respect to \( A \), the \( p \)-Laplacian operator \( \varphi_p \) is defined as \( \varphi_p(s) = |s|^{p-2}s \), \( p > 1 \), \( f(t, x) : (0, 1) \times (0, +\infty) \to [0, +\infty) \) is continuous and may be singular at \( t = 0, 1 \) and \( x = 0 \).

Integral and derivative operators of fractional order can describe the characteristics exhibited in many complex processes and systems having long-memory in time, and for this reason many classical integer-order models for complex systems are being substituted by fractional order models. Fractional calculus also provides an excellent tool to describe the hereditary properties of materials and processes, particularly in viscoelasticity, electrochemistry and porous media (see \([1–5]\)). Many successful new applications of fractional calculus in various fields have also been reported recently. For example, Nieto and Pimentel \([18]\) extended a second-order thermostat model to the fractional model, Ding and Jiang \([19]\) used waveform...
relaxation methods to study some fractional functional differential equations models. For the basic theories of fractional calculus and some recent work in application, the reader is referred to Refs. [17,23–34].

Much of the work on fractional calculus deals with boundary value problems [6–9,20]. In particular, Ahmad and Nieto [20] considered a nonlinear Dirichlet boundary value problem of sequential fractional integro-differential equations in the sense of the Caputo fractional derivative, and the existence results are established by means of some standard tools of fixed point theory. On the other hand, some developments on the topic involving the p-Laplacian operator and complex boundary value conditions have been reported [10–13,21]. In [22], Li and Lin considered a Hadamard fractional boundary value problem with a p-Laplacian operator as below:

\[
\begin{aligned}
\mathcal{D}_t^\alpha \left( \varphi_p(\mathcal{D}_t^\beta x) \right)(t) &= f(t,x(t)), & 1 < t < \varepsilon, \\
x(1) &= x'(1) = x''(1) = 0, & \mathcal{D}_t^\alpha x(1) = \mathcal{D}_t^\beta x(e) = 0,
\end{aligned}
\]

where \( 2 < \alpha \leq 3 \), \( 1 < \beta < 2 \), \( \varphi_p(s) = |s|^{p-2}s \), \( p > 1 \), and \( f : [1, \varepsilon] \times [0, +\infty) \to [0, +\infty) \) is a positive continuous function. By using the Leray–Schauder type alternative and the Guo–Krasnoselskii fixed point theorem, the existence and the uniqueness of the positive solutions were established.

The aim of this paper is to deal with the eigenvalue problem for the PFDE involving the Riemann–Stieltjes integral boundary condition, which allows the boundary conditions to be quite general, by using the upper and lower solutions and the Schauder fixed point theorem, so as to determine the interval of eigenvalue for the existence of positive solutions.

2. Preliminaries and lemmas

Denote by \( C[0,1] \) the space of all continuous functions on \([0,1]\) with the usual norm \( ||x|| = \max_{t \in [0,1]} |x(t)| \). Indeed, \( C[0,1] \) is a Banach space with a partial order, namely for \( x, y \in C[0,1] \), \( x \leq y \iff x(t) \leq y(t) \) for \( t \in [0,1] \).

Now consider the linear fractional differential equation

\[
\begin{aligned}
\mathcal{D}_t^\alpha x(t) + h(t) &= 0, & t \in (0,1), \\
x(0) &= 0, & x(1) = \int_0^1 x(s)dA(s).
\end{aligned}
\] (2.1)

**Lemma 2.1** [14]. Given \( h(t) \in L^1[0,1] \), the problem

\[
\begin{aligned}
\mathcal{D}_t^\alpha x(t) + h(t) &= 0, & 0 < t < 1, \\
x(0) &= 0, & x(1) = 0
\end{aligned}
\] (2.2)

has the unique solution

\[
x(t) = \int_0^1 G(t,s)h(s)ds,
\] (2.3)

where \( G(t,s) \) is the Green function of (2.2) and is given by

\[
G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} 
[|t-1|]^{\alpha-1}, & 0 \leq t \leq s \leq 1, \\
|t-1|^{\alpha-1} - |t-s|^{\alpha-1}, & 0 \leq s \leq t \leq 1.
\end{cases}
\] (2.4)

Consider the problem

\[
\begin{aligned}
\mathcal{D}_t^\alpha x(t) &= 0, & 0 < t < 1, \\
x(0) &= 0, & x(1) = 1,
\end{aligned}
\] (2.5)

by the property of the Riemann–Liouville fractional integral and derivative operators, the unique solution of the problem (2.5) is \( t^{\alpha-1} \). Defining \( \mathcal{G}_A(s) = \int_0^s G(t,s)dA(t) \), as in [15,16], we can get that the Green function for the nonlocal FDE (2.1) is given by

\[
J(t,s) = \frac{t^{\alpha-1}}{1-C} \mathcal{G}_A(s) + G(t,s), \quad C = \int_0^1 t^{\alpha-1}dA(t).
\] (2.6)

**Lemma 2.2** [16]. Let \( 0 \leq C < 1 \) and \( \mathcal{G}_A(s) \geq 0 \) for \( s \in [0,1] \), then the Green function defined by (2.6) satisfies:

1. \( J(t,s) > 0 \) for all \( t, s \in (0,1). \)
2. There exist two constants \( c, c^\prime \) such that

\[
c \cdot t^{\alpha-1} \mathcal{G}_A(s) \leq J(t,s) \leq c^\prime t^{\alpha-1} \leq c^\prime, \quad t, s \in [0,1],
\] (2.7)
Lemma 2.3. The associated linear PFDE (2.8) has the unique positive solution
\[ x(t) = \int_0^1 J(t,s) \left( \int_0^s b(s - \tau)^{\beta - 1} h(\tau) d\tau \right)^{q^{-1}} ds. \] (2.9)

Proof. In fact, let \( w = D_t^\alpha x, \nu = \varphi_p(w) \). Then the solution of the initial value problem
\[
\begin{aligned}
D_t^\alpha \nu(t) + h(t) &= 0, \quad t \in (0,1), \\
\nu(0) &= 0
\end{aligned}
\] (2.10)
is given by \( \nu(t) = C_1 t^{\beta - 1} - L^p h(t), \quad t \in [0,1] \). From the relations \( \nu(0) = 0, \ 0 < \beta \leq 1 \), we get \( C_1 = 0 \), and consequently
\[ \nu(t) = - L^p h(t), \quad t \in [0,1]. \] (2.11)

Noting that \( D_t^\alpha w = \nu, \ w = \varphi_p^{-1}(\nu) \), we have from (2.11) that the solution of (2.8) satisfies
\[
\begin{aligned}
D_t^\alpha x(t) &= \varphi_p^{-1}(- L^p h(t)), \quad t \in (0,1), \\
x(0) &= 0, \quad x(1) = \int_0^1 x(s) dA(s).
\end{aligned}
\] (2.12)

By (2.6), the solution of Eq. (2.12) can be written as
\[ x(t) = - \int_0^1 J(t,s) \varphi_p^{-1}(- L^p h(s)) ds, \quad t \in [0,1]. \]

Since \( h(s) \geq 0, \ s \in [0,1] \), we have \( \varphi_p^{-1}(- L^p h(s)) = - (L^p h(s))^{q^{-1}}, \ s \in [0,1] \), which implies that the solution of Eq. (2.8) is
\[ x(t) = \int_0^1 J(t,s) \left( \int_0^s b(s - \tau)^{\beta - 1} h(\tau) d\tau \right)^{q^{-1}} ds, \quad t \in [0,1]. \]

Definition 2.3. A continuous function \( \Psi(t) \) is called a lower solution of the PFDE (1.1) if it satisfies
\[
\begin{aligned}
- D_t^\alpha \left( \varphi_p(\mathcal{D}_t^\alpha \Psi) \right)(t) &\leq J(t, \Psi(t)), \quad t \in (0,1), \\
\Psi(0) &\geq 0, \quad D_t^\alpha \Psi(0) \geq 0, \quad \Psi(1) \geq \int_0^1 \Psi(s) dA(s).
\end{aligned}
\]

Definition 2.4. A continuous function \( \Phi(t) \) is called an upper solution of the PFDE (1.1) if it satisfies
\[
\begin{aligned}
- D_t^\alpha \left( \varphi_p(\mathcal{D}_t^\alpha \Phi) \right)(t) &\geq J(t, \Phi(t)), \quad t \in (0,1), \\
\Phi(0) &\leq 0, \quad D_t^\alpha \Phi(0) \leq 0, \quad \Phi(1) \leq \int_0^1 \Phi(s) dA(s).
\end{aligned}
\]

Remark 2.1. Assume \( 0 \leq C < 1 \) and \( \mathcal{G}_0(s) \geq 0 \) for \( s \in [0,1] \), and \( x \in C([0,1], \mathbb{R}) \) satisfies
\[ x(0) = 0, \quad x(1) = \int_0^1 x(s) dA(s) \]
and \(- \mathcal{D}_t^\alpha x(t) \geq 0\) for any \( t \in [0,1] \). Then \( x(t) \geq 0, \ t \in [0,1] \).

In fact, let
\[ - \mathcal{D}_t^\alpha x(t) = h(t) \] (2.13)
and
\[ x(0) = 0, \quad x(1) = \int_0^1 x(s)dA(s). \]  \hspace{1cm} (2.14)

Then the unique solution of Eq. (2.13) subject to boundary condition (2.14) is
\[ x(t) = \int_0^1 f(t,s)h(s)ds. \]

By Lemma 2.2 (1) and noticing \( h(t) \geq 0 \), one gets the conclusion \( x(t) \geq 0, \ t \in [0,1] \).

**Lemma 2.4 (Schauder fixed point theorem).** Let \( T \) be a continuous and compact mapping of a Banach space \( E \) into itself, such that the set
\[ \{ x \in E : x = \sigma T x, \text{ for some } 0 \leq \sigma \leq 1 \} \]
is bounded. Then \( T \) has a fixed point.

### 3. Main results

To establish the existence of a solution to the boundary value problem (1.1), we need to make the following assumptions.

**{(H0)}** \( A \) is a function of bounded variation satisfying \( G_0(s) \geq 0 \) for \( s \in [0,1] \) and \( 0 \leq C < 1 \).

**{(H1)}** \( f : (0,1) \times (0, +\infty) \to (0, +\infty) \) is continuous and is non-increasing in \( x > 0 \); and for all \( r \in (0,1) \), there exists a constant \( \alpha > 0 \) such that, for any \( (t,x) \in (0,1) \times (0, +\infty) \),
\[ f(t,rx) \leq r^{-\alpha}f(t,x). \]  \hspace{1cm} (3.1)

**Remark 3.1.** For all \( r > 1 \), we have \( \frac{1}{r} \in (0,1) \) and \( y = rx \in (0, \infty) \), given \( x \in (0, \infty) \), and thus from **{(H1)}**, we have
\[ f\left( t, \frac{1}{r} y \right) \leq \left( \frac{1}{r} \right)^{-\alpha} f(t,y) \]
using \( y = rx \), we have
\[ f(t,x) \leq \left( \frac{1}{r} \right)^{-\alpha} f(t,rx), \]
which give
\[ f(t,rx) \geq r^{-\alpha}f(t,x). \]  \hspace{1cm} (3.2)

Now denote
\[ g(s) = f(s,s^{\mu-1}), \text{ for any } s \in (0,1) \]
and introduce the Lebesgue space \( L^2(0,1) \) \((0 < \mu < 1)\) with the norm denoted by
\[ \|g\|_{L^2} = \left( \int_0^1 g^2(s)ds \right)^{\frac{1}{2}}. \]

**Theorem 3.1.** Suppose **{(H0)}** and **{(H1)}** hold, and there exists a constant \( 0 < \mu < \beta \) such that
\[ g \in L^2(0,1). \]  \hspace{1cm} (3.3)

Then there exists a constant \( \lambda^* > 0 \) such that the PFDE (1.1) has at least one positive solution \( w(t) \) for any \( \lambda \in (\lambda^*, +\infty) \), and moreover there exist two constants \( 0 < \lambda < 1 \) and \( L > 1 \) such that \( L^{2-\alpha} \leq w(t) \leq L^{2-\alpha} \).

**Proof.** Let \( E = C[0,1] \), and define a subset \( P \) of \( E \) as follows:
\[ P = \left\{ x \in E : \text{there exists a constant } 0 < \lambda < 1 \text{ such that } l^{2-\alpha} \leq x(t) \leq L^{2-\alpha}, \ t \in [0,1] \right\}. \]

Clearly, \( P \) is a nonempty set since \( t^{2-\alpha} \in P \). Now define the operator \( T \) in \( E \):
We assert that $T_x$ is well defined and $T_x(P) \subset P$. In fact, for any $x \in P$, there exists a positive number $0 < l_x < 1$ such that $l_x t^{x-1} \leq x(t) \leq l_x^{-1} t^{x-1}$, $t \in [0, 1]$, and thus, by Lemma 2.2, (H1), (3.1) and (3.3) and the Hölder inequality and noticing $0 < \mu < 1$, one gets
\begin{align}
T_x(t) &= \lambda \int_0^1 J(t, s) \left( \int_0^s b(s - \tau)^{\frac{1}{\beta} - 1} f(\tau, x(\tau)) d\tau \right)^{q-1} ds, \quad t \in [0, 1].
\end{align}

On the other hand, from (3.2) and (2.7), we have
\begin{align}
(T_x)(t) &= \lambda \int_0^1 J(t, s) \left( \int_0^s b(s - \tau)^{\frac{1}{\beta} - 1} f(\tau, x(\tau)) d\tau \right)^{q-1} ds
\leq \lambda \int_0^1 J(t, s) \left( \int_0^s b(s - \tau)^{\frac{1}{\beta} - 1} f(\tau, l_x^{-1} t^{-1}) d\tau \right)^{q-1} ds
\leq \lambda l_x^{x-1} b^{1-\beta} c t^{x-1} \int_0^1 \left[ \left( \int_0^s (s - \tau)^{1-\mu} g(\tau) d\tau \right)^{1-\mu} \left( \int_0^s g(\tau) d\tau \right)^{\mu} \right]^{q-1} ds
\leq \lambda l_x^{x-1} b^{1-\beta} c t^{x-1} \left[ \left( 1 - \frac{\mu}{\beta} \right) \left( 1 - \frac{1}{q} \right) \||g||_{L^q} \right]^{q-1} \left( \int_0^s (s - \tau)^{1-\mu} g(\tau) d\tau \right)^{q-1} ds.
\end{align}

Choose
\begin{align}
\tilde{l}_x &= \min \left\{ \frac{1}{2}, \left( \lambda l_x^{x-1} b^{1-\beta} c \left( 1 - \frac{\mu}{\beta} \right) \left( 1 - \frac{1}{q} \right) \||g||_{L^q} \right)^{q-1} \right\}^{-1},
\times \lambda l_x^{x-1} b^{1-\beta} c \int_0^1 G_A(s) \left( \int_0^s (s - \tau)^{1-\mu} g(\tau) d\tau \right)^{q-1} ds.
\end{align}

then it follows from (3.5)-(3.7) that
\begin{align}
\tilde{l}_x t^{x-1} \leq (T_x)(t) \leq l_x^{-1} t^{x-1},
\end{align}

which implies that $T_x$ is well defined and $T_x(P) \subset P$. Furthermore, comparing (3.4) and (2.9), the right hand side of (3.4) is exactly the same as the right hand side of (2.9) if the $h(t)$ in (2.9) is taken as $\lambda f(t, x(t))$. Hence as the left hand side of (2.9), i.e. $x(t)$ satisfies equation (2.8) according to Lemma 2.3, the left hand side of (3.4), i.e. $T_x(t)$ must also satisfy Eq. (2.8) with $h(\tau)$ replaced by $\lambda f(t, x(\tau))$, namely
\begin{align}
\begin{cases}
-\partial_t^\gamma \left( \partial_t^\gamma \partial_x^a (T_x(t)) \right) = \lambda f(t, x(t)), \quad t \in (0, 1),
(T_x)(0) = 0, \quad \partial_t^\gamma (T_x)(0) = 0, \quad (T_x)(1) = \int_0^1 (T_x)(s) d\lambda(s).
\end{cases}
\end{align}

Next we shall devote to finding the upper and lower solutions of the PFDE (1.1). Firstly, let
\begin{align}
e(t) &= \int_0^1 J(t, s) \left( \int_0^s b(s - \tau)^{\frac{1}{\beta} - 1} g(\tau) d\tau \right)^{q-1} ds.
\end{align}

By Lemma 2.2, we have
\begin{align}
e(t) &\geq c_t^{x-1} \int_0^1 G_A(s) \left( \int_0^s b(s - \tau)^{\frac{1}{\beta} - 1} g(\tau) d\tau \right)^{q-1} ds, \quad \forall t \in [0, 1]
\end{align}
and consequently there exists a constant $\lambda_1 \geq 1$ such that
\[\lambda_1 e(t) \geq t^{z-1}, \quad \forall t \in [0, 1].\]  
(3.9)

On the other hand, by (H1), we know that the operator $T_x$ is decreasing, and thus for any $\lambda > \lambda_1$, from (3.5), we have
\[
\int_0^1 J(t, s) \left( \int_0^s b(s - \tau)^{\beta-1} f(\tau, \lambda e(\tau))d\tau \right) q^{-1} ds \leq \int_0^1 J(t, s) \left( \int_0^s b(s - \tau)^{\beta-1} f(\tau, \lambda_1 e(\tau))d\tau \right) q^{-1} ds
\]
\[
\leq \int_0^1 J(t, s) \left( \int_0^s b(s - \tau)^{\beta-1} f(\tau, t^{z-1})d\tau \right) q^{-1} ds
\]
\[
= \int_0^1 J(t, s) \left( \int_0^s b(s - \tau)^{\beta-1} g(\tau)d\tau \right) q^{-1} ds < +\infty
\]
and
\[
e(t) \leq c b^{\beta-1} \int_0^1 \left( \int_0^s (s - \tau)^{\beta-1} g(\tau)d\tau \right) q^{-1} ds \leq c b^{\beta-1} \left( \frac{1 - \mu}{\beta - \mu} \right)^{(q-1)(1-\mu)} ||g||_{L^q} < +\infty.
\]
Now let $\rho = c b^{\beta-1} \left( \frac{1 - \mu}{\beta - \mu} \right)^{(q-1)(1-\mu)} ||g||_{L^q} + 1$ and take
\[\lambda^* = \max \left\{ \lambda_1, \left[ c, \rho^{-\left(q-1\right)} \int_0^1 G_a(s) \left( \int_0^s b(s - \tau)^{\beta-1} f(\tau, 1)d\tau \right) q^{-1} ds \right]^{\frac{1}{q-1}} \right\}.
\]
Then by (2.7) and (3.2), we obtain
\[
+\infty > \lambda^* \int_0^1 J(t, s) \left( \int_0^s b(s - \tau)^{\beta-1} f(\tau, \lambda^* e(\tau))d\tau \right) q^{-1} ds
\]
\[
\geq (\lambda^*)^{-\left(q-1\right)} c t^{z-1} \int_0^1 G_a(s) \left( \int_0^s b(s - \tau)^{\beta-1} f(\tau, e(\tau))d\tau \right) q^{-1} ds
\]
\[
\geq (\lambda^*)^{-\left(q-1\right)} c t^{z-1} \int_0^1 G_a(s) \left( \int_0^s b(s - \tau)^{\beta-1} f(\tau, \rho)d\tau \right) q^{-1} ds
\]
\[
\geq (\lambda^*)^{-\left(q-1\right)} c \rho^{-\left(q-1\right)} t^{z-1} \int_0^1 G_a(s) \left( \int_0^s b(s - \tau)^{\beta-1} f(\tau, 1)d\tau \right) q^{-1} ds
\]
\[
\geq t^{z-1}, \quad \forall t \in [0, 1].
\]
Let
\[
\Phi(t) = \lambda^* e(t), \quad \Psi(t) = \lambda^* \int_0^1 J(t, s) \left( \int_0^s b(s - \tau)^{\beta-1} f(\tau, \lambda^* e(\tau))d\tau \right) q^{-1} ds,
\]
then
\[
\Phi(t) = T_{\lambda^*} (t^{z-1}), \quad \Psi(t) = T_{\lambda^*} (\Phi(t)).
\]
(3.11)

It follows from (3.9) and (3.10) that for any $t \in [0, 1],$
\[
\Phi(t) = \lambda^* \int_0^1 J(t, s) \left( \int_0^s b(s - \tau)^{\beta-1} g(\tau)d\tau \right) q^{-1} ds \geq \lambda_1 e(t) \geq t^{z-1};
\]
\[
\Psi(t) = \lambda^* \int_0^1 J(t, s) \left( \int_0^s b(s - \tau)^{\beta-1} f(\tau, \lambda^* e(\tau))d\tau \right) q^{-1} ds \geq t^{z-1}.
\]
(3.12)

Moreover, by (3.8) and (3.11), we know
\[
\Phi(0) = 0, \quad T_{\lambda^*} \Phi(0) = 0, \quad \Phi(1) = \int_0^1 \Phi(s)dA(s),
\]
\[
\Psi(0) = 0, \quad T_{\lambda^*} \Psi(0) = 0, \quad \Psi(1) = \int_0^1 \Psi(s)dA(s).
\]
(3.13)

Proceeding as in (3.5)-(3.7), we get that $\Phi(t), \Psi(t) \in P$. By (3.11) and (3.12), we have
\[
t^{z-1} \leq \Psi(t) = (T_{\lambda^*} \Phi)(t), \quad t^{z-1} \leq \Phi(t), \quad \forall t \in [0, 1],
\]
(3.14)
which implies
\[ \Psi(t) = (T_x \Phi)(t) = \lambda^* \int_0^1 J(t,s) \left( \int_0^s b(s-\tau)^{q-1} f(\tau, \varphi(\tau)) d\tau \right)^{q-1} ds \leq \lambda^* \int_0^1 J(t,s) \left( \int_0^s b(s-\tau)^{q-1} g(\tau) d\tau \right)^{q-1} ds = \Phi(t), \quad \forall t \in [0,1]. \quad (3.15) \]

Thus, taking account of \( f \) being non-increasing in \( x > 0 \), and by (3.11), (3.8), (3.14) and (3.15), we have
\[ D^p_t \left( \varphi_p \left( D^q_t \Psi \right) \right)(t) + \lambda^* f(t, \Psi(t)) = D^p_t \left( \varphi_p \left( D^q_t (T_x \Phi) \right) \right)(t) + \lambda^* f(t, \Psi(t)) \]
\[ \geq D^p_t \left( \varphi_p \left( D^q_t (T_x \Phi) \right) \right)(t) + \lambda^* f(t, \Phi(t)) - \lambda^* f(t, \Phi(t)) = 0, \quad \lambda^* f(t, \Phi(t)) - \lambda^* f(t, \Phi(t)) = 0, \quad (3.16) \]
\[ D^p_t \left( \varphi_p \left( D^q_t \Phi \right) \right)(t) + \lambda^* f(t, \Phi(t)) = D^p_t \left( \varphi_p \left( D^q_t (T^2 \Phi) \right) \right)(t) + \lambda^* f(t, \Phi(t)) - \lambda^* f(t, \Phi(t)) = 0, \quad (3.17) \]

It follows from (3.13) and (3.15)–(3.17) that \( \Psi(t) \), \( \Phi(t) \) are upper and lower solutions of the PFDE (1.1), and \( \Psi(t), \Phi(t) \in P \).

Now let us define a function
\[ F(t, x) = \begin{cases} f(t, \Psi(t)), & x < \Psi(t), \\ f(t, x(t)), & \Psi(t) \leq x \leq \Phi(t), \\ f(t, \Phi(t)), & x > \Phi(t). \end{cases} \quad (3.18) \]

It then follows from (H1) and (3.18) that \( F : [0,1] \times [0, +\infty) \to [0, +\infty) \) is continuous.

We now show that the fractional boundary value problem
\[
\begin{aligned}
-\frac{d}{dt} \left( \varphi_p \left( D^q_t x \right) \right)(t) &= \lambda^* F(t, x(t)), & t \in (0,1), \\
x(0) &= 0, \quad D^q_t x(0) = 0, \quad x(1) = \int_0^1 x(s) dA(s) \\
\end{aligned}
\]

has a positive solution.

Define the operator \( \mathcal{D}_x \) by
\[ \mathcal{D}_x(t) = \lambda^* \int_0^1 J(t,s) \left( \int_0^s b(s-\tau)^{q-1} F(\tau, x(\tau)) d\tau \right)^{q-1} ds, \quad t \in [0,1]. \quad (3.20) \]

Then \( \mathcal{D}_x : C[0,1] \to C[0,1] \), and a fixed point of the operator \( \mathcal{D}_x \) is a solution of the PFDE (3.19).

On the other hand, from the definition of \( F \) and the fact that the function \( f(t, x) \) is nonincreasing in \( x \), we obtain
\[ f(t, \Phi(t)) = f(t, x(t)) \leq f(t, \Psi(t)) \]
provided that \( \Psi(t) \leq x \leq \Phi(t) \), \( f(t, x(t)) = f(t, \Psi(t)) \) provided that \( x < \Psi(t) \), and \( f(t, x(t)) = f(t, \Phi(t)) \) provided that \( x > \Phi(t) \). So we have
\[ f(t, \Phi(t)) \leq f(t, x(t)) \leq f(t, \Psi(t)), \quad \forall x \in E. \quad (3.21) \]

Furthermore, by (3.14) and (3.21), we have
\[ f(t, \Phi(t)) \leq f(t, x(t)) \leq f(t, t^{q-1}) = g(t), \quad \forall x \in E, \quad (3.22) \]

It follows from (2.7) and (3.22) that for any \( x \in E \),
\[ D_x x(t) = \lambda^* \int_0^1 J(t,s) \left( \int_0^1 b(s-\tau)^{q-1} F(\tau, x(\tau)) d\tau \right)^{q-1} ds \]
\[ \leq \lambda^* c b^{q-1} \int_0^1 \left( \int_0^1 b(s-\tau)^{q-1} g(\tau) d\tau \right)^{q-1} ds \]
\[ \leq \lambda^* c b^{q-1} \left( \frac{1 - \mu}{\beta - \mu} \right)^{(q-1)(1-\mu)} ||g||_b^{q-1} < +\infty, \quad (3.23) \]

namely, the operator \( \mathcal{D}_x \) is uniformly bounded.

On the other hand, let \( \Omega \subset E \) be bounded. As the function \( J(t,s) \) is uniformly continuous on \([0,1] \times [0,1] \), \( \mathcal{D}_x(\Omega) \) is equicontinuous. By the Arzela–Ascoli theorem, we have \( \mathcal{D}_x : E \to E \) is completely continuous. Moreover, (3.23) implies that (2.15) holds, thus, by using the Schauder fixed point theorem, \( \mathcal{D}_x \) has at least one fixed point \( w \) such that \( w = \mathcal{D}_x w \).

Now we prove
\[ \Psi(t) \leq w(t) \leq \Phi(t), \quad t \in [0,1]. \]
Since \( w \) is a fixed point of \( D_x \), by (3.13) and (3.19), we have
\[
\begin{align*}
    w(0) &= 0, \quad D_t^x w(0) = 0, \quad w(1) = \int_0^1 w(s) dA(s), \\
    \Phi(0) &= 0, \quad D_t^x \Phi(0) = 0, \quad \Phi(1) = \int_0^1 \Phi(s) dA(s).
\end{align*}
\] (3.24)

From (3.8), (3.11), (3.22) and noticing that \( w \) is a fixed point of \( D_x \), we also have
\[
D_t^x \left( \varphi_p (D_t^x \Phi) \right) (t) - D_t^x \left( \varphi_p (D_t^x w) \right) (t) = D_t^x \left( \varphi_p (D_t^x \Phi) - \varphi_p (D_t^x w) \right) (t) = -\dot{x}f(t, t^{x-1}) + \dot{\lambda}F(t, w(t)) \leq 0, \quad \forall t \in [0, 1].
\]

Let \( z(t) = \varphi_p (D_t^x \Phi(t)) - \varphi_p (D_t^x w(t)) \). Then
\[
\begin{align*}
    D_t^x z(t) &= D_t^x \left( \varphi_p (D_t^x \Phi(t)) \right) - D_t^x \left( \varphi_p (D_t^x w(t)) \right) \leq 0, \quad t \in [0, 1], \\
    D_t^x z(0) &= D_t^x \left( \varphi_p (D_t^x \Phi(0)) \right) - D_t^x \left( \varphi_p (D_t^x w(0)) \right) = 0.
\end{align*}
\]

It follows from (2.10) and (2.11) that
\[
z(t) \leq 0
\]
and
\[
\varphi_p (D_t^x \Phi(t)) - \varphi_p (D_t^x w(t)) \leq 0.
\]

Noticing that \( \varphi_p \) is monotone increasing, we have
\[
D_t^x \Phi(t) \leq D_t^x w(t), \quad \text{i.e.}, \quad D_t^x (\Phi - w)(t) \leq 0.
\]

It follows from Remark 2.1 and (3.24) that
\[
\Phi(t) - w(t) \geq 0.
\]

Thus we have \( w(t) \leq \Phi(t) \) on \([0, 1]\). By the same way, we also have \( w(t) \geq \Psi(t) \) on \([0, 1]\). So
\[
\Psi(t) \leq w(t) \leq \Phi(t), \quad t \in [0, 1].
\] (3.25)

Consequently, \( F(t, w(t)) = f(t, w(t)) \), \( t \in [0, 1] \). Hence \( w(t) \) is a positive solution of the problem (1.1).

Finally, by (3.25) and \( \Phi, \Psi \in P \), we have
\[
l_{\theta} t^{x-1} \leq \Psi(t) \leq w(t) \leq \Phi(t) \leq l_{\theta}^{-1} t^{x-1}. \quad \square
\]

In order to illustrate the validity of our results, we give the following example.

**Example 3.1.** Consider a singular PFDE involving the Riemann–Stieltjes integral boundary condition
\[
\begin{cases}
    -D_t^x \left( \varphi_p (D_t^x x) \right) (t) = \lambda \left( a_0(t) + \sum_{i=1}^n a_i(t) x^{-\gamma_i} \right), & t \in (0, 1), \\
    x(0) = 0, \quad D_t^x x(0) = 0, \quad x(1) = \int_0^1 x(s) dA(s).
\end{cases}
\] (3.26)

where \( \varphi_p(s) = |s|^{p-2} s, \quad p > 1 \), \( a_0(t) \) and \( a_i(t) \) are nonnegative and continuous on \((0, 1)\), \( \gamma_i \geq 0 \) for \( i = 1, 2, \ldots, n \), and
\[
A(t) = \begin{cases}
    0, & t \in [0, \frac{1}{2}], \\
    2, & t \in \left[ \frac{1}{2}, \frac{3}{4} \right], \\
    1, & t \in \left[ \frac{3}{4}, 1 \right].
\end{cases}
\]

Using Theorem 3.1 for the PFDE (3.26), we have the following conclusion.

**Corollary 3.1.** If there exists a constant \( 0 < \mu < \frac{1}{2} \) such that
\[
\int_0^1 \left[ a_0(t) + \sum_{i=1}^n a_i(t) t^{-\frac{\mu}{2}} \right] ds < +\infty.
\] (3.27)
Then there exists a constant $\lambda^* > 0$ such that for any $\lambda \in (\lambda^*, +\infty)$, the PFDE (3.26) has at least one positive solution $w(t)$, and there exist two constants, $0 < l < 1$ and $L > 1$, such that

$$lt^\frac{1}{2} \leq w(t) \leq Lt^\frac{1}{2}.$$  

**Proof.** According to the property of the Riemann–Stieltjes integral, the PFDE (3.26) becomes the following four-point boundary value problem with coefficients of both signs

$$\begin{cases}
-\mathcal{D}_t^\lambda \left( \varphi_x \left( \mathcal{D}_t^\lambda x \right) \right)(t) = \lambda \left( a(t) + \sum_{i=1}^n a_i(t)x^{-\gamma_i} \right), & t \in (0, 1), \\
x(0) = 0, \quad \mathcal{D}_t^\lambda x(0) = 0, \quad x(1) = 2x(\frac{1}{2}) - x(\frac{1}{4}).
\end{cases}$$

Let $\alpha = \frac{3}{2}$, $\beta = \frac{1}{2}$, $f(t, x) = a(t) + \sum_{i=1}^n a_i(t)x^{-\gamma_i}$, then

$$C = \int_0^1 t^{\alpha-1} \, d\lambda(t) = 2 \times \left( \frac{1}{2} \right)^{\frac{1}{2}} - \left( \frac{3}{4} \right)^{\frac{1}{2}} = 0.5482 < 1$$

and by simple computation, we have $G_0(s) > 0$, and so (H0) holds.

On the other hand, let $\epsilon = \max_{1 \leq i \leq n} \gamma_i$, then for all positive numbers $r < 1$, we have

$$f(t, rx) = a(t) + \sum_{i=1}^n a_i(t)(rx)^{-\gamma_i} \leq r^{-\epsilon} f(t, x)$$

and by (3.27), we also have $g(t) = a(t) + \sum_{i=1}^n a_i(t)t^{-\gamma_i}$.

So (H1) and (3.3) hold. By Theorem 3.1 there exists a constant $\lambda^* > 0$ such that for any $\lambda \in (\lambda^*, +\infty)$, the PFDE (3.26) has at least one positive solution $w(t)$, and there exist two constants, $0 < l < 1$ and $L > 1$, such that

$$lt^\frac{1}{2} \leq w(t) \leq Lt^\frac{1}{2}.$$  

$$\Box$$

**Remark 3.2.** The formula (3.27) not only implies that $a_i(t), \ i = 0, 1, \ldots, n$ can be singular on $[0, 1]$, but also $f$ can be singular at $x = 0$. So far, for the fractional differential equations involving $p$-Laplacian operator, no results have been obtained in existing work, especially, for the PFDE (1.1) in which $f$ has singularity at $t \in [0, 1]$ and $x = 0$.

**Remark 3.3.** There are a large number of functions that satisfy the conditions of Theorem 3.1. In particular, in Example 3.1, if $a_0(t)$ is nonnegative and continuous on $[0, 1]$ and $a_i(t) = t^\gamma$, $\gamma_i \geq \frac{\alpha}{2}$, then only $f$ is singular at $x = 0$. In this case, Corollary 3.1 still holds without the restriction of (3.27).

To end this section, we demonstrate the calculation of $\lambda^*$ by considering the following specific case for Eq. (3.26),

$$\begin{cases}
-\mathcal{D}_t^\lambda \left( \varphi_x \left( \mathcal{D}_t^\lambda x \right) \right)(t) = \lambda x^{-\frac{1}{2}}(t), & t \in (0, 1), \\
x(0) = 0, \quad \mathcal{D}_t^\lambda x(0) = 0, \quad x(1) = 2x(\frac{1}{2}) - x(\frac{1}{4}).
\end{cases}$$  

(3.28)

Choose $0 < \mu = \frac{1}{4} < \frac{1}{2}, \ \epsilon = \frac{1}{6}$, then (3.27) and (H1) are satisfied, and the corresponding Green function is

$$G(t, s) = \begin{cases}
G_1(t, s) = \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{4})}, & 0 \leq t \leq s \leq 1, \\
G_2(t, s) = \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{4})}, & 0 \leq s \leq t \leq 1.
\end{cases}$$

Consequently, we have

$$G_0(s) = \begin{cases}
2G_1(\frac{1}{2}, s) - G_2(\frac{1}{2}, s), & 0 \leq s < \frac{1}{2}, \\
2G_1(\frac{1}{2}, s) - G_2(\frac{1}{2}, s), & \frac{1}{2} \leq s < \frac{3}{4}, \\
2G_1(\frac{1}{2}, s) - G_1(\frac{1}{2}, s), & \frac{3}{4} \leq s \leq 1.
\end{cases}$$

Clearly, $0 \leq G_0(s) \leq \frac{\sqrt{2}}{\Gamma(\frac{3}{4})}$, and $c_1 = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}-0.3482} \approx 2.2134$, $c^* = \frac{\sqrt{2}}{\Gamma(\frac{3}{4})} + \frac{1}{\Gamma(\frac{1}{2})} \approx 4.6601$, $b = \frac{1}{\Gamma(\frac{1}{2})} \approx 1.7725$. Noticing that
e(t) = \int_0^1 f(t, s) \left( \int_0^s 1.7725(s - \tau)^{1/2} |\tau|^{-1/2} d\tau \right) ds \\
\geq 2.2134^2 \int_0^1 G_a(s) \left( \int_0^s 1.7725(s - \tau)^{1/2} |\tau|^{-1/2} d\tau \right) ds \\
= 2.2134^2 \int_0^1 G_a(s) s^\lambda \left( \int_0^1 1.7725(1 - \tau)^{1/2} |\tau|^{-1/2} d\tau \right) ds \\
= 2.2134 \times 1.9341^2 \int_0^1 G_a(s) s^\lambda ds = 4.8306^2 \int_0^1 G_a(s) s^\lambda ds \\
= 4.8306^2 \left[ \left( \int_0^1 \left( \sqrt{2 - \frac{3}{2}} (1 - s)^2 \frac{3}{4} + \frac{3}{4} - s \right)^{1/2} s^\lambda ds \right) + \int_1^\frac{1}{2} \left( \sqrt{2 - \frac{3}{2}} (1 - s)^2 \frac{3}{4} + \frac{3}{4} - s \right)^{1/2} s^\lambda ds \right] \\
= 4.8306^2 (0.1665 + 0.3196 + 0.0748 + 0.2445 + 0.0755 + 0.0441) = 1.3806^2.

we have 1.3806^{-1}e(t) \geq \bar{t}^2, and thus (3.9) is satisfied by taking \lambda_1 = 1. Further

\int_0^1 G_a(s) \left( \int_0^s b(s - \tau)^{\beta-1} f(t, 1) d\tau \right)^{q-1} ds = \int_0^1 G_a(s) \left( \int_0^s 1.7725(s - \tau)^{1/2} |\tau|^{-1/2} d\tau \right) ds = (2 \times 1.7725)^{1/2} \int_0^1 G_a(s) s^\lambda ds \\
= (2 \times 1.7725)^{1/2} \left[ \left( \int_0^1 \left( \sqrt{2 - \frac{3}{2}} (1 - s)^2 \frac{3}{4} + \frac{3}{4} - s \right)^{1/2} s^\lambda ds \right) + \int_1^\frac{1}{2} \left( \sqrt{2 - \frac{3}{2}} (1 - s)^2 \frac{3}{4} + \frac{3}{4} - s \right)^{1/2} s^\lambda ds \right] \\
= 1.8828 \times (0.2738 + 0.2286 + 0.0733 - 0.2970) = 0.5247.

and

\rho = c' b^{\beta-1} \left( \frac{1 - \mu}{\beta - \mu} \right)^{(q-1)(1-\mu)} ||g||_{L^\beta}^{q-1} + 1 = 4.6601 \times 1.7725 \times \left( \frac{1 - \frac{1}{2}}{\frac{1}{2} - \frac{1}{3}} \right)^{1/2} \left( \int_0^1 t^{\lambda} dt \right)^{1/2} + 1 = 11.8394.

Thus

\left[ c, \rho^{-q-1} \right] \int_0^1 G_a(s) \left( \int_0^s b(s - \tau)^{\beta-1} f(t, 1) d\tau \right)^{q-1} ds \right]^{\frac{1}{q-1}} = \left[ 2.2134 \times 11.8394 \times 0.5247 \right]^{\frac{1}{q-1}} \approx 1.0634,

and hence

\bar{\lambda} = \max\{1, 1.0634\} = 1.0634.

So for any \bar{\lambda} > 1.0634, Eq. (3.28) has at least one positive solution w(t). Moreover by (3.14) and (3.23), we have

\bar{\lambda}^{-1} \leq \Psi(t) \leq \Phi(t) \leq \bar{\lambda} c' b^{\beta-1} \left( \frac{1 - \mu}{\beta - \mu} \right)^{(q-1)(1-\mu)} ||g||_{L^\beta}^{q-1} \bar{\lambda}^{-1},

consequently, the positive solution w(t) satisfies

\bar{t}^2 \leq w(t) \leq 11.5266\bar{t}^2.

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References