A Convex Formulation of Controller Synthesis for Piecewise-Affine Slab Systems Based on Invariant Sets

Sina Kaynama, Behzad Samadi, Luis Rodrigues

Abstract— **This paper presents a controller synthesis method to stabilize piecewise-affine (PWA) slab systems based on invariant sets. The method has two steps: the design of an attractive invariant set and the design of the controller parameters. Inspired by the theory of sliding modes, sufficient stabilization conditions are cast as a set of Linear Matrix Inequalities (LMIs) by proper choice of an invariant set which is a target sliding surface. While previous approaches to PWA controller synthesis are cast as Bilinear Matrix Inequalities (BMIs) that can, in some cases, be relaxed to LMIs at the cost of adding conservatism, our method leads naturally to a convex formulation. Furthermore, the LMIs obtained in this paper have lower dimension when compared to other methods because the dimension of the closed-loop state space is reduced. A numerical example on flutter suppression is included to demonstrate the effectiveness of the approach.**

I. INTRODUCTION

Piecewise-affine (PWA) systems are a special class of hybrid systems. PWA systems can also be used to approximate a wide variety of nonlinear systems. Several promising methods have emerged for analysis and synthesis of PWA control systems such as those proposed in [1]–[9] and references therein. Unfortunately, synthesis of PWA controllers naturally leads to non-convex problems. Solving these problems is therefore a non-trivial task. PWA slab systems [6] are a subclass of PWA systems where the regions partitioning the domain are slabs. Reference [2] shows that sufficient conditions for quadratic stabilization using piecewise-linear state feedback for PWA slab systems can be cast as a convex optimization problem. Unfortunately, if affine terms are included in the controller, the convex structure is apparently destroyed, making

it hard to solve the problem globally. However, under certain additional assumptions, reference [6] shows that one can develop algorithms to solve these non-convex problems with optimality guarantees. Reference [10] presents algorithms for state-feedback design of PWA systems based on LMIs. However, this approach is only applicable to discrete-time PWA systems.

To the best of our knowledge the only proposed methods that can formulate the PWA state feedback problem as a set of LMIs is presented in [11], [12]. The method in [11] shows that one can avoid solving the Bilinear Matrix Inequalities (BMIs) proposed in [6] by using a convex relaxation which leads to a set of LMIs. Unfortunately, using more conservative conditions may lead to infeasibility. In [12] a backstepping approach is developed for PWA systems in strict feedback form. Controller synthesis was formulated as a convex problem but one cannot control the way in which the trajectories converge to the origin. This limitation motivates the work that will be presented here. It will be shown that the synthesis procedure proposed in this paper leads to a convex problem in a reduced state space and the closed-loop trajectories converge to the origin along a desired direction.

The contribution of this paper is to use invariant set ideas to formulate the PWA synthesis problem as a set of LMIs. Inspired by the theory of sliding modes, sufficient stabilization conditions are cast directly as a set of LMIs by proper choice of an invariant set which is a target sliding surface. It is then shown that the conservatism of the proposed approach is less than the only other available convex approach. Furthermore, the dimension of the LMIs obtained in this paper is less than would be using other methods because the dimension of the state space is reduced, which further simplifies the synthesis problem.

We have considered active flutter suppression (AFS), an interesting and hard control problem in aerospace systems, as an application of our method. Flutter is inherently a nonlinear phenomenon. However, one can approximate this phenomenon by a PWA system using

S. Kaynama and L. Rodrigues are with the Department of Electrical and Computer Engineering, Concordia University, 1515 St. Catherine Street, Montreal, QC, Canada H3G 2W1 {s_kayn, luisrod}@encs.concordia.ca

B. Samadi is with the Department of Electrical Engineering, Amirkabir University of Technology, 424 Hafez Street, Tehran, Iran bsamadi@aut.ac.ir

for example the method detailed in [4]. Simulation results will show how the method proposed in this paper can actively suppress flutter in a wing section.

This paper is divided into five parts. The problem formulation is presented first. The proposed controller synthesis for PWA systems is explained in section III and then it will be shown in section IV that the proposed approach is less conservative than the other approaches. The proposed method is then applied to AFS in section V. Finally, after showing simulation results, conclusions will be drawn.

II. PRELIMINARIES

The dynamics of a PWA system can be written as

$$\dot{x}(t) = A_i x(t) + a_i + B_i u(t), \quad x(t) \in \mathcal{R}_i \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^p$ the control input and a subset of the state space $\mathcal{X} \subset \mathbb{R}^n$ is partitioned into M polytopic cells $\mathcal{R}_i$, $i \in \mathcal{I} = \{1, \ldots, M\}$ such that $\cup_{i=1}^M \overline{\mathcal{R}}_i = \mathcal{X}$, $\mathcal{R}_i \cap \mathcal{R}_j = \emptyset$ where $\overline{\mathcal{R}}_i$ denotes the closure of $\mathcal{R}_i$ (see [4] for generating such partition).

A slab region is defined as

$$\mathcal{R}_i = \{x \mid \beta_i < \lambda^T x < \beta_{i+1}\} \quad (2)$$

where $\lambda \in \mathbb{R}^n$, $\lambda \neq 0$ and β_i, $\beta_{i+1} \in \mathbb{R}$, $i = 1, \ldots, M$. The slab region $\mathcal{R}_i$ can also be cast as a degenerate ellipsoid

$$\mathcal{R}_i = \{x \mid \|L_i x + l_i\| < 1\} \quad (3)$$

where

$$L_i = 2\lambda^T / (\beta_{i+1} - \beta_i), \quad (4)$$
$$l_i = -(\beta_{i+1} + \beta_i)/(\beta_{i+1} - \beta_i). \quad (5)$$

A PWA system whose regions are slabs is called a PWA slab system, ref [6]. Using a PWA control input of the form

$$u = K_i x(t) + k_i, \quad x(t) \in \mathcal{R}_i \quad (6)$$

into system (1) yields the closed-loop dynamics

$$\dot{x}(t) = \overline{A}_i x(t) + \overline{a}_i, \quad x(t) \in \mathcal{R}_i, \quad (7)$$

where

$$\overline{A}_i = A_i + B_i K_i, \quad \overline{a}_i = a_i + B_i k_i \quad (8)$$

The following lemma gives sufficient conditions for asymptotic stability of closed-loop PWA slab systems.

Lemma 1: [6] Consider the PWA slab system (1). If there exist $Q = Q^T > 0$, $\alpha > 0$, $\mu_i > 0$ satisfying

$$\Omega_{i0} + \alpha Q < 0 \qquad \text{if } 0 \in \overline{\mathcal{R}}_i, \quad (9)$$

$$\Omega_i = \begin{bmatrix} \Omega_{i1} & \Omega_{i2} \\ \Omega_{i3} & \Omega_{i4} \end{bmatrix} < 0 \qquad \text{otherwise} \quad (10)$$

where

$$\Omega_{i0} = \overline{A}_i Q + Q \overline{A}_i^T$$
$$\Omega_{i1} = \overline{A}_i Q + Q \overline{A}_i^T + \alpha Q - \mu_i \overline{a}_i \overline{a}_i^T$$
$$\Omega_{i2} = -\mu_i \overline{a}_i l_i^T + Q L_i^T$$
$$\Omega_{i3} = -\mu_i l_i \overline{a}_i^T + L_i Q$$
$$\Omega_{i4} = \mu_i (1 - l_i l_i^T)$$

with L_i and l_i defined in (4) and (5), then the origin is an exponentially stable equilibrium point.

Proof: See reference [6]. ■

Note that the constraint (10) is nonconvex. The nonconvexity of BMIs (10) is due to the existence of the term

$$-\mu_i \overline{a}_i \overline{a}_i^T \quad (11)$$

which includes a product of unknown gains k_i. Therefore controller synthesis for PWA slab systems is a non-convex problem. The method proposed in [11], shows that one can avoid solving the BMIs (10) by ignoring the negative definite term (11), which is a convex relaxation. Note that, conditions of Lemma 1 are sufficient conditions and therefore conservatism has been already introduced to the problem and by ignoring the negative definite term (11), [11] adds more conservatism to the problem. Unfortunately, the resulting conditions may lead to infeasibility. Motivated by the drawbacks of existing methods, the next section presents a convex formulation of the synthesis problem using an invariant set approach.

III. CONTROLLER SYNTHESIS

Consider the following class of PWA slab systems

$$\dot{x}(t) = A_i x(t) + a_i + \begin{bmatrix} 0 \\ B_{2_i} \end{bmatrix} u(t), \quad x(t) \in \mathcal{R}_i \quad (12)$$

where $u \in \mathbb{R}^p$, $B_{2_i} \in \mathbb{R}^{m \times p}$ and $m \in \mathcal{M} = \{1, \cdots, n-1\}$. In other words, any PWA system belongs to the mentioned class as long as there exists at least one zero row in its B matrix structure.

We can rewrite equations (12) in the following form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11_i} & A_{12_i} \\ A_{21_i} & A_{22_i} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} a_{1_i} \\ a_{2_i} \end{bmatrix} + \begin{bmatrix} 0 \\ B_{2_i} \end{bmatrix} u, x(t) \in \mathcal{R}_i \quad (13)$$

where $x_1 \in \mathbb{R}^{n-m}$, $x_2 \in \mathbb{R}^m$. Assume further that in this class of PWA systems, the slab regions are only functions of x_1. Therefore, the definition of slab regions (3) can be rewritten as

$$\mathcal{R}_i = \{x \mid \|L_i x + l_i\| = \|\begin{bmatrix} L_{1i} & 0 \end{bmatrix} x + l_i\| \\ = \|L_{1i} x_1 + l_i\| < 1\} \quad (14)$$

where $L_{1i}^T \in \mathbb{R}^{n-m}$.

Remark 1: Note that when the slab regions are a function of x_2, no state linear transformation can make the system fall into the class (13).

This paper proposes a new method to formulate PWA controller synthesis for system (13) as a convex feasibility problem. The main result is presented in the next theorem.

Theorem 1: Given with invertible matrix $S_2 B_{2_i}$, the PWA controller

$$
\begin{aligned}
u = & - (S_2 B_{2_i})^{-1}[S_1(A_{11_i}x_1 + A_{12_i}x_2 + a_{1_i}) \\
& + S_2(A_{21_i}x_1 + A_{22_i}x_2 + a_{2_i}) \\
& + \gamma \frac{S_1 x_1 + S_2 x_2}{\|S_1 x_1 + S_2 x_2\|}],
\end{aligned} \tag{15}
$$

for $x \in \mathcal{R}_i$, $i = 1, \ldots, M$, exponentially stabilizes system (13) if given $\gamma > 0$ and $\alpha > 0$, there exist $Q = Q^T > 0$, $\mu_i > 0$, and $Y = S_1 Q$, satisfying the following LMIs

$$
\omega_{i0} + \alpha Q < 0 \qquad \text{if } 0 \in \overline{\mathcal{R}}_i, \tag{16}
$$

$$
\omega_i = \begin{bmatrix} \omega_{i1} & \omega_{i2} \\ \omega_{i3} & \omega_{i4} \end{bmatrix} < 0 \qquad \text{otherwise} \tag{17}
$$

$$
\begin{aligned}
\omega_{i0} &= A_{11_i}Q + QA_{11_i}^T - A_{12_i}S_2^\dagger Y - Y^T (S_2^\dagger)^T A_{12_i}^T \\
\omega_{i1} &= A_{11_i}Q + QA_{11_i}^T - A_{12_i}S_2^\dagger Y - Y^T (S_2^\dagger)^T A_{12_i}^T \\
&\quad + \alpha Q - \mu_i a_{1_i} a_{1_i}^T \\
\omega_{i2} &= -\mu_i a_{1_i} l_i^T + Q L_{1i}^T \\
\omega_{i3} &= -\mu_i l_i a_{1_i}^T + L_{1i}Q \\
\omega_{i4} &= \mu_i (1 - l_i l_i^T)
\end{aligned}
$$

where

$$
S_2^\dagger = S_2^T (S_2 S_2^T)^{-1} \tag{18}
$$

Proof: Consider a surface of the form

$$
\sigma(x) = Sx = 0 \tag{19}
$$

where

$$
S = \begin{bmatrix} S_1 & S_2 \end{bmatrix} \tag{20}
$$

with $S_1 \in \mathbb{R}^{p \times (n-m)}$ and $S_2 \in \mathbb{R}^{p \times m}$. In order to make $\sigma(x) = 0$ an attractive invariant set, we define a candidate Lyapunov function of the form

$$
V(\sigma(x)) = \frac{1}{2}\sigma^T(x)\sigma(x). \tag{21}
$$

Note that, although $V(\sigma(x))$ is implicitly based on $x(t)$, it is not a Lyapunov function for x, but it is rather a Lyapunov function for $\sigma(x)$. As a function of $\sigma(x)$, $V(\sigma(x))$ is obviously positive definite because it is a norm. In order to have finite-time convergence

to $\sigma(x) = 0$, according to [13] and [14] one needs to ensure

$$
\dot{V}(\sigma(x)) \leq -\mu \|\sigma(x)\| \tag{22}
$$

where $\mu > 0$. Note that the Lie derivative of the Lyapunov function in (21) is

$$
\dot{V}(\sigma(x)) = \frac{\partial V(\sigma(x))}{\partial \sigma(x)}\dot{\sigma}(x) = \sigma^T(x)\dot{\sigma}(x). \tag{23}
$$

Designing $\sigma(x)$ such that

$$
\dot{\sigma}(x) = -\gamma \left(\frac{\sigma(x)}{\|\sigma(x)\|} \right) \tag{24}
$$

with $\gamma \geq \mu > 0$, the decreasing rate of the Lyapunov function in (21) will be

$$
\begin{aligned}
\dot{V}(\sigma(x)) &= -\gamma \sigma^T(x) \left(\frac{\sigma(x)}{\|\sigma(x)\|} \right) \\
&= -\gamma \|\sigma(x)\| \leq -\mu \|\sigma(x)\|,
\end{aligned} \tag{25}
$$

which verifies (22). Using (13), (19) and (20) we can write

$$
\begin{aligned}
\dot{\sigma}(x) = S\dot{x} = & S_1(A_{11_i}x_1 + A_{12_i}x_2 + a_{1_i}) \\
& + S_2(A_{21_i}x_1 + A_{22_i}x_2 + a_{2_i}) + (S_2 B_{2_i})u
\end{aligned} \tag{26}
$$

Since $S_2 B_{2_i}$ is invertible, the control law (15) ensures that (25) is verified and therefore makes the target surface $\sigma(x) = 0$ an attractive invariant set. We now show that the trajectories converge to this target surface in finite time. Observe that (25) is equivalent to

$$
\dot{V}(\sigma(x)) = -\gamma \sqrt{2} V^{\frac{1}{2}}(\sigma(x)) \tag{27}
$$

for the Lyapunov function defined in (21). This is a differential equation. Assuming $V(\sigma(x(t_0)))$ as the initial condition, the solution to (27) can be found as

$$
V^{\frac{1}{2}}(\sigma(x(t))) = V^{\frac{1}{2}}(\sigma(x(t_0))) - \frac{\sqrt{2}\gamma}{2}(t - t_0). \tag{28}
$$

One now can see that

$$
\exists t_c \in \mathbb{R}, \quad \text{such that} \quad V(\sigma(x(t_c))) = 0 \tag{29}
$$

where $t_c \geq t_0$ is the finite time of convergence to the surface. In fact, replacing $V(\sigma(x(t_c))) = 0$ in (28) yields

$$
t_c = \sqrt{2}\gamma^{-1}V^{\frac{1}{2}}(\sigma(x(t_0))) + t_0. \tag{30}
$$

Furthermore, the differential equation (27) implies that

$$
\dot{V}(\sigma(x(t_c))) = -\gamma\sqrt{2}V^{\frac{1}{2}}(\sigma(x(t_c))) = 0, \tag{31}
$$

which yields

$$
V^{\frac{1}{2}}(\sigma(x(t))) = 0, \qquad \forall t > tc \tag{32}
$$

and therefore

$$V(\sigma(x(t))) = 0, \qquad \forall t \geq tc. \qquad (33)$$

Since the trajectories converge in finite time to the surface $\sigma(x) = 0$ and remain on that surface for all future times, using (19) and (20), for $t \geq t_c$ we can write

$$S_1 x_1 + S_2 x_2 = 0. \qquad (34)$$

Assuming

$$x_2 = S_2^T Z \qquad (35)$$

where $Z \in \mathbb{R}^p$, we can rewrite (34) as

$$Z = -(S_2 S_2^T)^{-1} S_1 x_1 \qquad (36)$$

Hence

$$x_2 = -S_2^\dagger S_1 x_1 \qquad (37)$$

where

$$S_2^\dagger = S_2^T (S_2 S_2^T)^{-1} \qquad (38)$$

is the pseudo-inverse of the matrix S_2. Therefore, using (34) and (37) we can rewrite the dynamics of the PWA system (13) for $t \geq t_c$ as

$$x_2 = -S_2^\dagger S_1 x_1 \qquad (39)$$

$$\dot{x}_1 = (A_{11_i} - A_{12_i} S_2^\dagger S_1) x_1 + a_{1_i}, \quad x \in \mathcal{R}_i. \qquad (40)$$

Due to (39), if $x_1(t)$ exponentially converges to the origin, then $x_2(t)$ will also exponentially converge to the origin. Therefore, exponential stability of the reduced order system (40) ensures that the PWA slab system (13) is exponentially stable under the control law (15). However, exponential stability of the reduced order system (40) is guaranteed if the LMIs (16)–(17) hold, based on Lemma 1 using

$$\overline{A}_i := (A_{11_i} - A_{12_i} S_2^\dagger S_1) \qquad (41)$$

$$\overline{a}_i := a_{1_i} \qquad (42)$$

This finishes the proof ∎

Remark 2: As one can see, Theorem 1 results in a set of LMIs. Moreover no relaxation is used in the proof. In fact since (42) is always a constant vector (in each region), the term (11) is known, which makes the problem convex.

Remark 3: Note that the conditions of Theorem 1 are not influenced by the choice of S_2 and therefore S_2 can be any arbitrary matrix as long as (18) exists.

Remark 4: Theorem 1 reduces the complexity of the LMIs that must be solved because transforming the closed-loop stability problem for system (13) into a stability problem for system (40) makes the dimension

of the closed-loop state space smaller than the dimension of the open-loop state-space. The other methods in the literature of continuous-time PWA systems do not perform this transformation and therefore are more complex because of two reasons: i) they lead to BMIs (10) and ii) the dimension of the state space is larger.

Remark 5: Note that the dimension of the sliding surface (19) depends on the number of the inputs, p. In other words, the more inputs that the systems has, the less dimension the sliding surface will have. For instance, for a second order system with two inputs, the sliding surface will be a point, while if there was only one input the sliding surface would be a line. Since the number of the inputs does not directly affect the derivation of the control laws and they only affect the dimension of the sliding surface, stability of the systems might be preserved even in case of the failure of some inputs to the system.

Remark 6: Stability of system (40) might also be achieved with piecewise Lyapunov function. However, note that both piecewise and global Lyapunov functions are used for slab systems and they both are sufficient conditions, see [6].

IV. CONSERVATISM OF THE APPROACH AND COMPARISON

As it was mentioned in the previous sections, relaxation approach in [11] shows that one can avoid solving BMIs (10) by ignoring the negative definite term (11). The following lemma gives sufficient conditions for asymptotic stability of closed-loop PWA slab system (1) using relaxation method.

Lemma 2: [11] Consider PWA slab system (1) and PWA state feedbacks (6), if there exist $P = P^T > 0$, $\epsilon > 0$, $\zeta_i > 0$ satisfying

$$\Gamma_{i0} + \epsilon P < 0 \qquad \text{if } 0 \in \overline{\mathcal{R}}_i, \qquad (43)$$

$$\Gamma_i = \begin{bmatrix} \Gamma_{i1} & \Gamma_{i2} \\ \Gamma_{i3} & \Gamma_{i4} \end{bmatrix} < 0 \qquad \text{otherwise} \qquad (44)$$

where

$$\Gamma_{i0} = \overline{A}_i P + P \overline{A}_i^T$$

$$\Gamma_{i1} = \overline{A}_i P + P \overline{A}_i^T + \epsilon P$$

$$\Gamma_{i2} = -\zeta_i \overline{a}_i l_i^T + P L_i^T$$

$$\Gamma_{i3} = -\zeta_i l_i \overline{a}_i^T + L_i P$$

$$\Gamma_{i4} = \zeta_i (1 - l_i l_i^T)$$

with L_i, l_i, $\overline{A}_i$ and $\overline{a}_i$ defined in (4), (5), and (8), then the origin is an exponentially stable equilibrium point.

Proof: See reference [11]. ∎

Therefore, using relaxation approach, one might be able to control (12) by using PWA state feedbacks (6). Note that, Theorem 1 and Lemma 2 use sufficient conditions of Lemma 1 and consequently, they both are conservative approaches. However, the following theorem shows that the proposed approach is less conservative than relaxation approach for PWA slab system (12).

Theorem 2: For every $P = P^T > 0$, $\epsilon > 0$, $\zeta_i > 0$ satisfying (43), (44) and stabilizing PWA system (12) with PWA state feedbacks (6), there exist $Q = Q^T > 0$, $\alpha > 0$, $\mu_i > 0$ and $Y = S_1 Q$ satisfying (16), (17) and stabilizing (12) with PWA state feedbacks (15). The reverse is not true.

Proof: Suppose (43) and (44) hold. Since Γ is negative definite and symmetric, Γ_{i_1}, the leading principal minor of Γ must also be negative definite:

$$\Gamma_{i_1} = \overline{A}_i P + P \overline{A}_i^T + \epsilon P < 0$$

or equivalently

$$\Gamma_{i_1} = A_i P + B_i H + H^T B_i^T + P A_i^T + \epsilon P < 0 \quad (45)$$

where

$$H = K_i P.$$

One can rewrite P as in the following

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \quad (46)$$

where $P_{11} = P_{11}^T \in \mathbb{R}^{(n-m) \times (n-m)} > 0$, $P_{12} = P_{21}^T \in \mathbb{R}^{(n-m) \times (m)}$, and $P_{22} = P_{22}^T \in \mathbb{R}^{m \times m}$. Therefore, (45) can be rewritten as

$$\Gamma_{i_1} = \begin{bmatrix} A_{11_i} P_{11} + A_{12_i} P_{21} & A_{11_i} P_{12} + A_{12_i} P_{22} \\ A_{21_i} P_{11} + A_{22_i} P_{21} & A_{21_i} P_{12} + A_{22_i} P_{22} \end{bmatrix}$$
$$+ \begin{bmatrix} A_{11_i} P_{11} + A_{12_i} P_{21} & A_{11_i} P_{12} + A_{12_i} P_{22} \\ A_{21_i} P_{11} + A_{22_i} P_{21} & A_{21_i} P_{12} + A_{22_i} P_{22} \end{bmatrix}^T$$
$$+ \begin{bmatrix} 0 \\ B_{2_i} H \end{bmatrix} + \begin{bmatrix} 0 \\ B_{2_i} H \end{bmatrix}^T + \epsilon \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} < 0. \quad (47)$$

Note that, Γ_{i_1} is symmetric. Therefore if (47) holds, then the leading principal minor of Γ_{i_1} must also be negative definite. Hence:

$$A_{11_i} P_{11} + A_{12_i} P_{21} + P_{11}^T A_{11_i}^T + P_{21}^T A_{12_i}^T + \epsilon P_{11} < 0 \quad (48)$$

or equivalently there exist $\epsilon > 0$, P_{11} and P_{21} that make (48) and as a result (47) and (45), negative definite. Now by defining

$$Q = P_{11}, \qquad Y = -S_2 P_{21} \quad \text{and} \qquad \alpha = \epsilon \quad (49)$$

or equivalently by substituting P_{11}, P_{21} and ϵ by Q, $-S_2^\dagger Y$ and α, respectively,

$$A_{11_i} Q + Q A_{11_i}^T - A_{12_i} S_2^\dagger Y - Y^T (S_2^\dagger)^T A_{12_i}^T + \alpha Q < 0 \quad (50)$$

and therefore ω_{i1}, the leading principal minor of the symmetric matrix ω in (17) will be negative definite or

$$\omega_{i1} = A_{11_i} Q + Q A_{11_i}^T - A_{12_i} S_2^\dagger Y - Y^T (S_2^\dagger)^T A_{12_i}^T + \alpha Q - \mu_i a_{1_i} a_{1_i}^T < 0 \quad (51)$$

since $-\mu_i a_{1_i} a_{1_i}^T < 0$ with

$$\mu_i = \zeta_i > 0. \quad (52)$$

Therefore, for any solution to LMIs (43) and (44), using (49), (51) and (52) there will also be a solution to LMIs (16) and (17). This finishes the first part of the proof.

Now consider the following simple second order PWA slab system belongs to (12) or (13):

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad \text{if } x \in \mathcal{R}_1, \quad (53)$$

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad \text{otherwise} \quad (54)$$

where

$$\mathcal{R}_1 = \{x \in \mathbb{R}^2 \mid -2 < x_1 < 2\}.$$

Stability of closed-loop PWA system (53) and (54) with PWA state feedbacks (6) can not be achieved by relaxation method. In other words, conditions of Lemma 2 for such a system are not satisfied with any $P = P^T > 0$, $\epsilon > 0$, $\zeta_i > 0$. One can simply verify this fact by examining (48) which is the leading principal minor of the Γ_{i1}. Rewriting (48) for closed-loop PWA system (54) with the controller defined in (6) yields

$$(2 + \epsilon) P_{11} < 0 \quad (55)$$

with

$$A_{11_i} = 1, \qquad A_{12_i} = 0 \quad \text{and} \qquad P_{11} = P_{11}^T > 0.$$

Since $\epsilon > 0$, (55) implies

$$P_{11} < 0 \quad (56)$$

which contradicts with the positive definiteness of P_{11}. However, assuming

$$\alpha = 1, \qquad \mu_i = 3 \qquad \text{and} \qquad Q = 0.5, \quad (57)$$

we will have

$$\omega_{i0} + \alpha Q = -0.5 < 0 \qquad \text{if} \qquad x \in \mathcal{R}_1 \quad (58)$$

$$\omega_{i1} = -1.5 < 0 \qquad \text{for} \qquad x \notin \mathcal{R}_1. \qquad (59)$$

Note that, since ω_i is a symmetric matrix, (59) validates (16). Therefore conditions of (16) and (17) are satisfied. This counterexample finishes the proof. ∎

V. APPLICATION TO ACTIVE FLUTTER SUPRESSION

In this section, we consider the problem of Active Flutter Suppression (AFS) to demonstrate how the proposed method works. The flutter model is taken from [15] and is given by

$$\dot{x} = \begin{bmatrix} 0 & I \\ -M^{-1}(K_0 + K_\mu) & -M^{-1}(C_0 + C_\mu) \end{bmatrix} x$$
$$+ \begin{bmatrix} 0 \\ -M^{-1} \begin{bmatrix} 0 \\ x_2 k_\alpha(\alpha) \end{bmatrix} \end{bmatrix} + \begin{bmatrix} 0 \\ \mu M^{-1} \end{bmatrix} Bu \qquad (60)$$

where

$$C_\mu = \begin{bmatrix} \rho U b C_{L\alpha} & \rho U b^2 C_{L\alpha}(\frac{1}{2} - a) \\ \rho U b^2 C_{m\alpha} & -\rho U b^3 C_{m\alpha}(\frac{1}{2} - a) \end{bmatrix}$$

$$C_0 = \begin{bmatrix} C_h & 0 \\ 0 & C_\alpha \end{bmatrix}$$

$$M = \begin{bmatrix} m & m x_\alpha b \\ m x_\alpha b & I_\alpha \end{bmatrix}$$

$$K_0 = \begin{bmatrix} k_h & 0 \\ 0 & 0 \end{bmatrix}$$

$$K_\mu = \begin{bmatrix} 0 & \rho U^2 b C_{L\alpha} \\ 0 & -\rho U^2 b^2 C_{m\alpha} \end{bmatrix}$$

$$B = \begin{bmatrix} -\rho b C_{L\beta_1} & -\rho b C_{L\beta_2} \\ \rho b^2 C_{m\beta_1} & \rho b^2 C_{m\beta_2} \end{bmatrix} U^2$$

U is the airspeed, $C_{L\alpha}$ and $C_{m\alpha}$ are aerodynamic lift and moment coefficients, ρ is the air density, $C_{L\beta}$ and $C_{m\beta}$ are lift and moment coefficients per control surface deflection, respectively, m is the mass of the airfoil, I_α is the mass moment of inertia about the elastic axis, C_h and C_α are plunge and pitch structural damping coefficients, respectively, and L and M are the aerodynamic lift and moment about the elastic axis. Structural stiffness is represented by k_h and k_α for plunge and pitch motions, respectively. The source of nonlinearity is the torsional stiffness, which is

$$k_\alpha(\alpha) = k_{\alpha_0} + k_{\alpha_1}(\alpha) + k_{\alpha_2}(\alpha^2) + k_{\alpha_3}(\alpha^3) + k_{\alpha_4}(\alpha^4) \qquad (61)$$

All the model parameters are taken from [15] and are available in Appendix I. A controller is now designed for a PWA approximation of the system (60). Therefore, first we approximate the AFS nonlinear system by a PWA model using the method explained in [4]. The slab regions used for the approximation are defined by

$$\mathcal{R}_1 = \{x \in \mathbb{R}^4 \mid 0.6 < x_2 = \alpha < 1\}$$
$$\mathcal{R}_2 = \{x \in \mathbb{R}^4 \mid -1 < x_2 = \alpha < -0.6\}$$
$$\mathcal{R}_3 = \{x \in \mathbb{R}^4 \mid -0.6 < x_2 = \alpha < -0.2\}$$
$$\mathcal{R}_4 = \{x \in \mathbb{R}^4 \mid -0.2 < x_2 = \alpha < 0.2\}$$
$$\mathcal{R}_5 = \{x \in \mathbb{R}^4 \mid 0.2 < x_2 = \alpha < 0.6\}.$$

The dynamic equations in all regions will not be presented here for lack of space but, for example, the equations of the PWA model for AFS in $\mathcal{R}_5$ are

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -293.27 & 2272.13 & -5.90 & -0.40 \\ 1885.94 & -69573.59 & 34.72 & 2.47 \end{bmatrix} x$$
$$+ \begin{bmatrix} 0 \\ 0 \\ -471.47 \\ 13972.34 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -7606.78 & -7642.55 \\ 14250.44 & 9021.92 \end{bmatrix} u. \qquad (62)$$

This approximation belongs to the class of PWA systems defined in (12). To design a controller, we first define the parameters

$$\gamma = 2 \qquad \alpha = 0.01. \qquad (63)$$

Note that, in this problem, S_2 can be any arbitrary square matrix of order two as long as $S_2 B_{2i}$ is invertible. For the simplicity we define

$$S_2 = B_{2_5}^{-1} = \begin{bmatrix} 0.0002 & 0.0002 \\ -0.0004 & -0.0002 \end{bmatrix} \qquad (64)$$

.

Using these parameters and solving LMIs (16) and (17), the matrix S_1 is obtained as

$$S_1 = \begin{bmatrix} 0.00017 & -0.0198 \\ -0.0017 & 0.0320 \end{bmatrix}. \qquad (65)$$

Therefore the sliding surface defined in (19) for the AFS problem is

$$\sigma(x) = \begin{bmatrix} 0.0017 & -0.0198 & 0.0002 & 0.0002 \\ -0.0017 & 0.0320 & -0.0004 & -0.0002 \end{bmatrix} x. \qquad (66)$$

After computing $\sigma(x)$ and using (15), we are able to derive control laws for all five regions. For example the
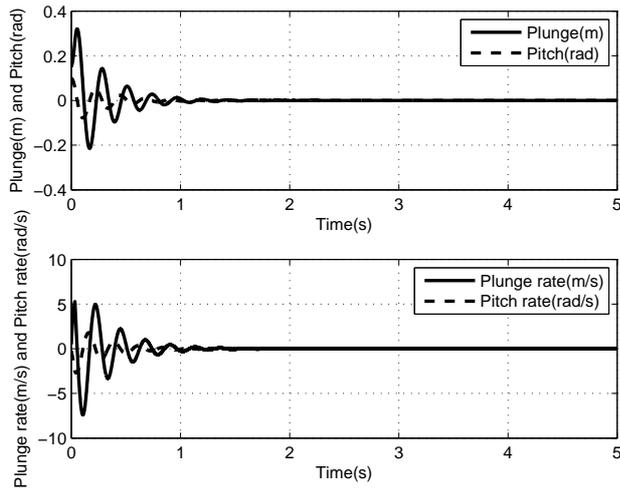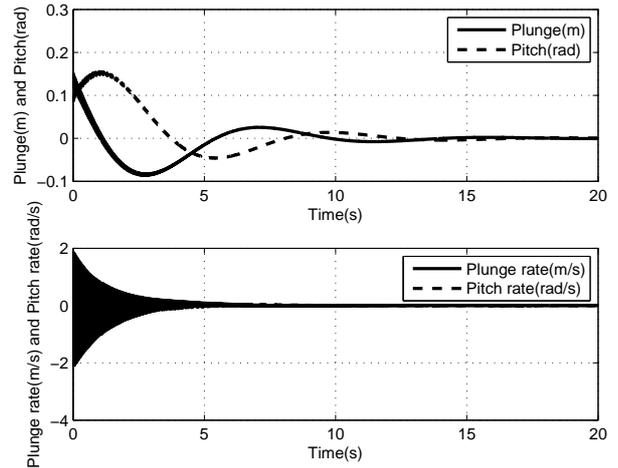
Fig. 1.   AFS state variables

Fig. 2.   AFS state variables

control input for the fifth region is

$$u = \begin{bmatrix} -0.292 & 12.691 & -0.007 & 0.019 \\ 0.252 & -12.334 & 0.006 & -0.031 \end{bmatrix} x$$

$$- \gamma \frac{\begin{bmatrix} 0.0017 & -0.0198 & 0.0002 & 0.0002 \\ -0.0017 & 0.0320 & -0.0004 & -0.0002 \end{bmatrix} x}{\left\| \begin{bmatrix} 0.0017 & -0.0198 & 0.0002 & 0.0002 \\ -0.0017 & 0.0320 & -0.0004 & -0.0002 \end{bmatrix} x \right\|}$$

$$+ \begin{bmatrix} -2.545 \\ 2.471 \end{bmatrix}$$

$$(67)$$

Fig. 1 shows simulation results for $x_0 = \begin{bmatrix} 0.15 & 0.1 & 0.5 & -0.2 \end{bmatrix}^T$ as an initial condition. It can be clearly seen in the figure that flutter was effectively suppressed as desired. Fig. 2 shows the simulation results of the same AFS problem controlled by the method proposed in [11]. As one can see the simulation results in Fig. 2 shows that the designed controller by relaxation approach are not applicable in real life due to the high frequency oscillatory behavior of the state variables.

VI. Conclusion

This paper proposes a novel controller design method for PWA systems. The method has two steps: the design of an attractive invariant set and the design of the controller parameters. To the best of our knowledge, all previous approaches to continuous-time PWA controller synthesis are cast as non-convex BMIs that can, in some cases, be relaxed to LMIs at the cost of adding conservatism. Therefore, since our method leads naturally to a convex LMI formulation, it holds

the strong potential of being less conservative than previous approaches. Furthermore, the LMIs obtained in this paper have lower dimension when compared to other methods because the dimension of the closed-loop state space is reduced. Application to active control of flutter suppression, which is considered a hard problem in aerospace control, showed the effectiveness of the approach.

References

[1] M. Johansson, *Piecewise linear control systems: a computational approach.* Springer Verlag, 2003, vol. 284.

[2] A. Hassibi and S. Boyd, "Quadratic stabilization and control of piecewise-linear systems," in *American Control Conference, 1998. Proceedings of the 1998*, vol. 6. IEEE, 1998, pp. 3659–3664.

[3] S. Pettersson, "Analysis and design of hybrid systems," Ph.D. dissertation, Chalmers University of Technology, 1999.

[4] L. Rodrigues and J. How, "Automated control design for a piecewise-affine approximation of a class of nonlinear systems," in *American Control Conference, 2001. Proceedings of the 2001*, vol. 4. IEEE, 2001, pp. 3189–3194.

[5] L. Rodrigues, "Stability analysis of piecewise-affine systems using controlled invariant sets," *Systems & control letters*, vol. 53, no. 2, pp. 157–169, 2004.

[6] L. Rodrigues and S. Boyd, "Piecewise-affine state feedback for piecewise-affine slab systems using convex optimization," *Systems & Control Letters*, vol. 54, no. 9, pp. 835–853, 2005.

[7] G. F. Jianbin Qiu, Tiejun Zhang and H. Liu, "Piecewise affine model based h-infinity static output feedback control of constrained nonlinear processes," *IET Control Theory and Applications*, vol. 4, no. 11, pp. 2315–2330, Nov. 2010.

[8] A. Bemporad, G. Ferrari-Trecate, and M. Morari, "Observability and controllability of piecewise affine and hybrid systems," *Automatic Control, IEEE Transactions on*, vol. 45, no. 10, pp. 1864–1876, 2000.

[9] N. Van de Wouw and A. Pavlov, "Tracking and synchronisation for a class of pwa systems," *Automatica*, vol. 44, no. 11, pp. 2909–2915, 2008.

[10] D. Mignone, G. Ferrari-Trecate, and M. Morari, "Stability and stabilization of piecewise affine and hybrid systems: An lmi approach," in *Proc. 39th IEEE Conference on Decision and Control*, vol. 1.   IEEE, 2000, pp. 504–509.

[11] B. Samadi and L. Rodrigues, "Controller synthesis for piecewise affine slab differential inclusions: A duality-based convex optimization approach," in *Proc. 46th IEEE Conference on Decision and Control*.   IEEE, 2007, pp. 4999–5004.

[12] B. Samadi and L. Rodrigues, "Backstepping controller synthesis for piecewise affine systems: A sum of squares approach," in *Systems, Man and Cybernetics, 2007. ISIC. IEEE International Conference on*.   IEEE, 2007, pp. 58–63.

[13] H. Khalil and J. Grizzle, *Nonlinear systems*.   Prentice hall, 1992, vol. 3.

[14] V. Utkin, "Sliding mode control: Mathematical tools, design and applications," *Nonlinear and Optimal Control Theory*, pp. 289–347, 2008.

[15] S. Afkhami and H. Alighanbari, "Nonlinear control design of an airfoil with active flutter suppression in the presence of disturbance," *Control Theory & Applications, IET*, vol. 1, no. 6, pp. 1638–1649, 2007.

APPENDIX I

Parameters	Values
K_p	$\mathrm{diag}(0.5, 0.5)$
K_d	$\mathrm{diag}(0.05, 0.05)$
a	-0.45
U	$30\,\mathrm{m/s}$
M	$\begin{bmatrix} 12.387 & 0.418 \\ 0.418 & 0.065 \end{bmatrix}$
m	12.387
I_α	$0.065\,\mathrm{kgm^2}$
K_o	$\begin{bmatrix} 2844.4 & 0 \\ 0 & 0 \end{bmatrix}\,\mathrm{N/m}$
k_h	2844.4
K_μ	$\begin{bmatrix} 0 & 935.1 \\ 0 & -6.3 \end{bmatrix}\,\mathrm{kg/s^2}$
C_o	$\mathrm{diag}(27.43, 0.036)$
C_h	27.43
C_α	$0.036\,\mathrm{kgm^2/s}$
C_μ	$\begin{bmatrix} 31.17 & 3.99 \\ 0.21 & -0.027 \end{bmatrix}$
$K_\alpha(\alpha)$	$q\alpha$
q	$\begin{bmatrix} q_1 & q_2 & q_3 & q_4 & q_5 \end{bmatrix}^T$
q_1	2.82
q_2	-62.322
q_3	3709.71
q_4	-24195.6
q_5	48756.954
μ	176.609
$C_{L\beta_1}$	3.358
$C_{L\beta_2}$	3.458
$C_{m\beta_1}$	-0.635
$C_{m\beta_2}$	-0.735
ρ	$1.225\,\mathrm{kg/m^3}$
$C_{m\alpha}$	$(0.5 + a)C_{L\alpha}$
$C_{L\alpha}$	6.28