Decreasing rearrangement and a fuzzy variational problem

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Abstract

In this note we introduce a variational problem with respect to an integrable fuzzy set \( f \). The energy functional is maximized over a deleted \( \sigma \)-algebra. Using the decreasing rearrangement of \( f \) we prove that the admissible set can be replaced by the more convenient set of cuts of \( f \). Finally an special case is considered where the variational problem can be transformed into a one dimensional setting.

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1. Introduction

It often happens that we have a set \( E \) and \( f \) is a characteristic property of it. The problem is that often elements of \( E \) have property \( f \) with some degree and we want to make a good partition of \( E \). In [1] the author introduced the concept of *separating power of a fuzzy set* which gives in many cases a good partition, called the *max-separating partition*. This concept is interesting in two different aspects. Firstly, it allows to participate the referential universe by means of the considered fuzzy set. Secondly, it gives a measure of the entropy of this fuzzy set. It is the only measure of fuzziness which is not pointwise defined [2]. Separating power has applications in problems of Decision-Aid in Medicine. It has also proved to be useful in defuzzification problems [3]. For example, in diagnosis problems, it may be wanted and sufficient, in a first step, to know the subclass of the most plausible syndroms, instead

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of getting one unique syndrom. Some applications of separating power in ecological system has been investigated in [4]. In [5] the author uses the decreasing rearrangement of functions to prove that the measure of separation of a function is invariant through the corresponding decreasing rearrangement function, and moreover it is possible to recover the initial max-separating partition by a knowledge of the max-separating partition of the decreasing rearrangement by an algorithmic way.

The purpose of the present paper is to generalize the results of [5]. Our interest is pure mathematical. The techniques used in [5] are not applicable in this generalized situation. We put the problem in a variational setting. More precisely, the problem of making a good separation for a fuzzy set is done through maximizing an energy functional where the admissible set is a deleted \(\sigma\)-algebra (of a measure space), which is a rather large set. We prove that the admissible set can be replaced by a much smaller and definitely more functional, numerically speaking, subset; namely, the set of all cuts of the fuzzy set in question. We also show that in the case that the measure space is nonatomic and separable, then the maximization problem can be reduced to a similar problem in a one dimensional setting, where it is then that the techniques of [5] are applicable.

2. Definitions and notation

**Definition.** Let \((\Omega, \Sigma, \mu)\) denote a finite measure space. Let \(f : \Omega \to \mathbb{R}\) be a measurable function. The distribution function of \(f\) with respect to \(\mu\), denoted \(\lambda_{f,\mu}(\alpha)\), is defined by

\[
\lambda_{f,\mu}(\alpha) = \mu(\{x \in \Omega : f(x) \geq \alpha\}) \equiv \mu\{f \geq \alpha\},
\]

for every \(\alpha \in \mathbb{R}\).

**Definition.** Let \((\Omega, \Sigma, \mu)\) and \((\Omega', \Sigma', \mu')\) be two measure spaces with \(\mu(\Omega) = \mu'(\Omega')\). Let \(f : \Omega \to \mathbb{R}\) and \(g : \Omega' \to \mathbb{R}\) be two measurable functions. We say \(f\) is a rearrangement of \(g\) whenever

\[
\lambda_{f,\mu}(\alpha) = \lambda_{g,\mu'}(\alpha),
\]

for every \(\alpha \in \mathbb{R}\). In this case we write \(f \sim g\).

**Definition.** Let \((\Omega, \Sigma, \mu)\) be a finite measure space. Let \(f : \Omega \to \mathbb{R}\) be a measurable function. The (essentially) unique decreasing rearrangement of \(f\) is a real valued function, denoted \(f_{\mu}^\Delta \equiv f^\Delta\), which is defined on \([0, \mu(\Omega)]\) by

\[
f^\Delta(s) = \sup\{\alpha \in \mathbb{R} : \lambda_{f,\mu}(\alpha) \geq s\}.
\]

**Remark 1.** It is well known that \(f^\Delta\) is right continuous and when \([0, \mu(\Omega)]\) is endowed with Lebesgue measure, \(f^\Delta \sim f\).

The measure space \((\Omega, \Sigma, \mu)\) is called nonatomic if for every \(U \in \Sigma\) with \(\mu(U) > 0\) there exists \(V \in \Sigma\) with \(V \subset U\) and \(0 < \mu(V) < \mu(U)\). The measure space \((\Omega, \Sigma, \mu)\) is called separable if there is a sequence \(\{U_n\}_{n=1}^\infty\) of measurable sets such that for every \(V \in \Sigma\) and \(\epsilon > 0\) there exists \(n \in \mathbb{N}\) such that

\[
\mu(V \setminus U_n) + \mu(U_n \setminus V) < \epsilon.
\]

It is a standard result that any finite separable nonatomic measure space is isomorphic to an interval of \(\mathbb{R}\), in the sense we describe now. Let \((\Omega, \Sigma, \mu)\) and \((\Omega', \Sigma', \mu')\) be two measure spaces, and let their respective families of null sets be \(\mathcal{N}\) and \(\mathcal{N}'\). Regard two members of \(\Sigma\) as equivalent if their symmetric difference lies in \(\mathcal{N}\), and write \(\Sigma/\mathcal{N}\) for the space of equivalence classes. Similarly define
An isomorphism from \((\Omega, \Sigma, \mu)\) to \((\Omega', \Sigma', \mu')\) is a bijection \(\Phi : \Sigma/N \rightarrow \Sigma'/N'\) having the properties

\[
\Phi(U \setminus V) = \Phi(U) \setminus \Phi(V)
\]

\[
\Phi \left( \bigcup_{n=1}^{\infty} V_n \right) = \bigcup_{n=1}^{\infty} \Phi(V_n)
\]

\[
\mu'(\Phi(U)) = \mu(U)
\]

for all \(U, V, V_n \in \Sigma\), where we have abused notation by failing to distinguish between measurable sets and their equivalence classes.

**Isomorphism theorem.** Let \((\Omega, \Sigma, \mu)\) be a finite separable nonatomic measure space. Then \((\Omega, \Sigma, \mu)\) is isomorphic to the interval \((0, \mu(\Omega))\) with Lebesgue measure.

A proof of the Isomorphic Theorem may be found in [6]. Note that any Lebesgue measurable set in \(\mathbb{R}^N\), with any measure that is absolutely continuous with respect to Lebesgue measure, is separable and nonatomic.

### 3. Preliminaries

In this section we will state and/or prove some lemmas which are essential in our analysis. The following result is standard; see for example [7].

**Lemma 1.** Let \((\Omega, \Sigma, \mu)\) and \((\Omega', \Sigma', \mu')\) be two measure spaces with \(\mu(\Omega) = \mu'(\Omega')\). Let \(f : \Omega \rightarrow \mathbb{R}\) and \(g : \Omega' \rightarrow \mathbb{R}\) be two measurable functions with \(f \sim g\). Suppose \(\phi : \mathbb{R} \rightarrow \mathbb{R}\) is a Borel measurable function. Then \(\phi \circ f \sim \phi \circ g\). Moreover, if \(\phi \circ f\) is integrable, so is \(\phi \circ g\) and

\[
\int_\Omega \phi \circ f \, d\mu = \int_{\Omega'} \phi \circ g \, d\mu'.
\]  

(3.1)

**Remark 2.** Note that by taking respectively \(\phi(x) = |x|^p\), where \(p \in [1, \infty)\), and \(\phi(x) = x\), in Lemma 1, we obtain

\[
\|f\|_{p, \mu} = \|g\|_{p, \mu'}.
\]  

(3.2)

\[
\int_\Omega f \, d\mu = \int_{\Omega'} g \, d\mu'.
\]

In particular we get

\[
\int_\Omega f \, d\mu = \int_0^{\mu(\Omega)} f^\Delta.
\]  

(3.3)

**Lemma 2.** Let \(f : (\Omega, \Sigma, \mu) \rightarrow \mathbb{R}\) be a measurable function and \(U\) be a cut of \(f\); that is,

\[
U = \{x \in \Omega : f(x) \geq \gamma\},
\]

for some \(\gamma\). Then

\[
\int_U f \, d\mu = \gamma \mu(U) + \int_0^{\mu(\Omega)} (f^\Delta - \gamma)^+.
\]  

(3.4)
Proof. Note that
\[
\int_{U} f \, d\mu = \int_{U} ((f - \gamma)^+ + \gamma) \, d\mu = \gamma \mu(U) + \int_{U} (f - \gamma)^+ \, d\mu
\]
\[
\quad = \gamma \mu(U) + \int_{\mu(U)}^{\mu(\Omega)} ((f - \gamma)^+)^\Delta.
\]
It is clear that we derive (3.4) if we show that
\[
((f - \gamma)^+)^\Delta = (f^\Delta - \gamma)^+.
\] (3.5)
Since \( f \sim f^\Delta \), it follows that \( f - \gamma \sim f^\Delta - \gamma \). Thus from Lemma 1, by setting \( \phi(t) = \frac{t + |t|}{2} \) we have \((f - \gamma)^+ \sim (f^\Delta - \gamma)^+ \). Thus
\[
((f - \gamma)^+)^\Delta = ((f^\Delta - \gamma)^+)^\Delta = (f^\Delta - \gamma)^+,
\]
where the last equality follows from the fact that \((f^\Delta - \gamma)^+ \) is a decreasing function. Thus we derive (3.5). □

Lemma 3. Let \( f : (\Omega, \Sigma, \mu) \to \mathbb{R} \) be an integrable function. Then
\[
\int_{U} f \, d\mu \leq \int_{\mu(U)}^{\mu(\Omega)} f^\Delta_d \mu.
\] (3.6)
Moreover, equality holds in (3.6) if and only if \( U \) is a cut of \( f \).

Proof. Let \( f_1 \) and \( f_2 \) be the negative and the positive parts of \( f \), so \( f = f_1 - f_2 \). Thus, \( f^\Delta = f_1^\Delta + (-f_2)^\Delta \). Set
\[
G = \{(x, s) \in (\Omega \times \mathbb{R} : 0 \leq s \leq f_1(x)\}
\]
\[
\hat{G}(s) = \{ x \in \Omega : f_1(x) \geq s \}.
\]
Then by Fubini’s theorem we have
\[
\int_{U} f_1 \, d\mu = \int_{0}^{\infty} \int_{0}^{\infty} \chi_G(x, s) \, ds \, d\mu = \int_{0}^{\infty} \int_{U} \chi_G(x, s) \, d\mu \, ds
\]
\[
\quad = \int_{0}^{\infty} \mu(\hat{G}(s) \cap U) \, ds \leq \int_{0}^{\infty} |\{ t \in (0, \mu(U)) : f_1^\Delta(s) \geq s \}| \, ds
\]
\[
\quad = \int_{\mu(U)}^{\mu(\Omega)} f_1^\Delta.
\]
The same argument applies with \( U, f_1 \) and \( \mu(U) \) replaced by \( \Omega \setminus U, f_2 \) and \( \mu(\Omega) - \mu(U) \), respectively. We obtain
\[
-\int_{U} f_2 \, d\mu = -\int_{\Omega} f_2 + \int_{\Omega \setminus U} f_2 \leq -\int_{0}^{\mu(\Omega)} f_2^\Delta + \int_{\mu(\Omega) - \mu(U)}^{\mu(\Omega)} f_2^\Delta
\]
\[
\quad = -\int_{\mu(\Omega) - \mu(U)}^{\mu(\Omega)} f_2^\Delta = \int_{0}^{\mu(U)} (-f_2)^\Delta.
\]
Hence (3.6) follows.

Let us now suppose \( U \) is a cut of \( f \), so
\[
U = \{ x \in \Omega : f(x) \geq \gamma \},
\]
for some real $\gamma$. From Lemma 2 we have

$$\int_U f \, d\mu = \gamma \mu(U) + \int_0^{\mu(U)} (f^\Delta - \gamma)^+.$$  \hspace{1cm} (3.7)

Once again applying Lemma 2 to $f^\Delta : (0, \mu(\Omega)) \to \mathbb{R}$, we obtain

$$\int_{\{f^\Delta \geq \gamma\}} f^\Delta = \gamma |\{f^\Delta \geq \gamma\}| + \int_0^{\mu(U)} (f^\Delta - \gamma)^+.$$  \hspace{1cm} (3.8)

where $|A|$ denotes the one dimensional Lebesgue measure of $A$. Since $f \sim f^\Delta$, we infer $\mu(U) = |\{f^\Delta \geq \gamma\}|$, so by (3.7) and (3.8) we obtain

$$\int_U f \, d\mu = \int_{\{f^\Delta \geq \gamma\}} f^\Delta.$$  \hspace{1cm} (3.9)

Notice that $\{f^\Delta \geq \gamma\}$ is an interval of the form $(0, \beta)$ or $(0, \beta]$, for some $\beta > 0$. In either case we deduce $|\{f^\Delta \geq \gamma\}| = \beta$, hence $\mu(U) = \beta$. Thus from (3.9) we obtain

$$\int_U f \, d\mu = \int_0^{\mu(U)} f^\Delta,$$  \hspace{1cm} (3.10)

as desired.

Now we assume that (3.10) hold but $U$ is not a cut of $f$. Set $\delta = \text{ess inf}_U f$ and $\hat{\delta} = \text{ess sup}_U f$. Since $U$ is not a cut of $f$ there exists $E \subset \{f \geq \delta\}$ such that $\mu(E) > 0$ and $E \cap U = \emptyset$. Thus we can find $\alpha \in [\delta, \hat{\delta}]$ such that

$$\lambda_{f, \mu}(\alpha) > \lambda_{f^\Delta}(\alpha).$$  \hspace{1cm} (3.11)

From (3.11) we infer existence of $s \in (0, \mu(U))$ such that $f^\Delta(s) > (f\chi_U)^\Delta(s)$, so by the right-continuity of $f^\Delta$ we obtain

$$\int_0^{\mu(U)} (f\chi_U)^\Delta < \int_0^{\mu(U)} f^\Delta.$$  

Thus

$$\int_U f \, d\mu = \int_\Omega f\chi_U \, d\mu = \int_0^{\mu(U)} (f\chi_U)^\Delta$$

$$= \int_0^{\mu(U)} (f\chi_U)^\Delta < \int_0^{\mu(U)} f^\Delta,$$

as desired. \hspace{1cm} $\square$

4. Main results

In this section we assume that $(\Omega, \Sigma, \mu)$ is a probability space; that is, $\mu(\Omega) = 1$, and $f$ is a fuzzy subset of $\Omega$. This means that $\mu$ is a measurable function from $\Omega$ into $[0, 1]$. In addition, we assume that $f$ is integrable and has negligible level sets; that is, for every $\alpha \in [0, 1]$, the sets $\{f = \alpha\}$ have zero $\mu$-measure. By $\Sigma'$ we designate the deleted $\sigma$-algebra $\Sigma \setminus \{\emptyset, \Omega\}$. The symbol $\hat{\Sigma}$ indicates the subset of
\[ \Sigma' \] comprising cuts of \( f \). Hence for \( U \in \hat{\Sigma} \), \( U = \{ f \geq \alpha \} \), for some \( \alpha \in [0, 1] \). For \( A \in \Sigma' \), we set
\[ f_A = \frac{1}{\mu(A)} \int_A f \, d\mu. \]
We also define the energy of \( f \), \( \mathcal{E}_f : \Sigma' \to \mathbb{R} \), by
\[ \mathcal{E}_f(A) = f_A - f_{A^c}, \]
where \( A^c \) indicates the complement of \( A \). Let us now introduce our variational problem:
\[ \mathcal{P} : \sup_{A \in \hat{\Sigma}'} \mathcal{E}_f(A). \]
The solution set of \((\mathcal{P})\) is denoted \( S(f, \Sigma') \). We set \( \hat{\Sigma}' = \hat{\Sigma}' \backslash \{ \emptyset, \Omega \} \). The main result of this section is the following.

**Theorem 1.** With the notation as above we have
\[ \sup_{A \in \Sigma'} \mathcal{E}_f(A) = \sup_{A \in \hat{\Sigma}'} \mathcal{E}_f(A). \] (4.12)
Moreover, the following equality holds:
\[ S(f, \Sigma') = S(f, \hat{\Sigma}'), \] (4.13)
where \( S(f, \hat{\Sigma}') \) indicates the solution set of the variational problem \((\mathcal{P})\) with the admissible set \( \Sigma' \) replaced by \( \hat{\Sigma}' \).

**Proof.** We first prove (4.12). It suffices to show
\[ \sup_{A \in \Sigma'} \mathcal{E}_f(A) \leq \sup_{A \in \hat{\Sigma}'} \mathcal{E}_f(A). \] (4.14)
Let us fix \( A \in \Sigma' \). Note that \( \mathcal{E}_f(A) \) can be written as follows:
\[ \mathcal{E}_f(A) = \frac{1}{\mu(A)(1 - \mu(A))} \int_A f \, d\mu - \frac{1}{1 - \mu(A)} \| f \|_1. \]
So by setting \( \Gamma(A) = \mu(A)(1 - \mu(A)) \) and \( \hat{\Gamma}(A) = 1 - \mu(A) \) we deduce
\[ \mathcal{E}_f(A) = \frac{1}{\Gamma(A)} \int_A f \, d\mu - \frac{1}{\hat{\Gamma}(A)} \| f \|_1. \]
Therefore
\[ \mathcal{E}_f(A) = \frac{1}{\Gamma(A)} \int_A f \chi_A \, d\mu - \frac{1}{\hat{\Gamma}(A)} \| f \|_1 \]
\[ \leq \frac{1}{\Gamma(A)} \int_0^{\mu(A)} f^\Delta \, d\mu - \frac{1}{\hat{\Gamma}(A)} \| f^\Delta \|_1, \] (4.15)
where we have used Remark 2. Since \( f \) has negligible level sets it follows that there exists \( \beta > 0 \), not necessarily unique, such that \((0, \mu(A)) = \{ f^\Delta \geq \beta \} \). This in conjunction with (4.15) will give
\[ \mathcal{E}_f(A) \leq \frac{1}{\Gamma(A)} \int_{\{ f^\Delta \geq \beta \}} f^\Delta - \frac{1}{\hat{\Gamma}(A)} \| f^\Delta \|_1. \] (4.16)
We now set $\hat{A} = \{ f \geq \beta \}$, so

$$E_f(\hat{A}) = \frac{1}{\Gamma(\hat{A})} \int_{\hat{A}} f \, d\mu - \frac{1}{\Gamma(\hat{A})} \| f^\Delta \|_1,$$

(4.17)

where $\Gamma(\hat{A}) = \mu(\hat{A})(1 - \mu(\hat{A}))$ and $\hat{\Gamma}(\hat{A}) = 1 - \mu(\hat{A})$. Since $\mu(\hat{A}) = \mu(\{ f \geq \beta \}) = |\{ f^\Delta \geq \beta \}| = \mu(A)$, we deduce that $\Gamma(\hat{A}) = \Gamma(A)$. Therefore from Lemma 3, (4.16) and (4.17) we obtain

$$E_f(\hat{A}) \geq E_f(A).$$

Thus we have $E_f(A) \leq E_f(\hat{A})$, which proves (4.14).

We now prove (4.13). It is obvious that we need only show that $\mathcal{S}(f, \Sigma') \subseteq \mathcal{S}(f, \hat{\Sigma}')$. To do this suppose $A \in \mathcal{S}(f, \Sigma')$ but is not a cut of $f$. Therefore we can apply Lemma 3, to obtain

$$E_f(A) = \sup_{B \in \Sigma'} E_f(B)$$

and by setting $A = \{ f \geq \beta \}$ we deduce from (4.18) that $E_f(A) < E_f(\hat{A})$. Since

$$E_f(\hat{A}) = \sup_{B \in \Sigma'} E_f(B)$$

we infer

$$\sup_{B \in \Sigma'} E_f(B) < \sup_{B \in \Sigma'} E_f(B),$$

which contradicts (4.12). \(\square\)

5. A special case

In this section we show that the discussions in the previous section can be transformed into a one dimensional setting once we assume that the probability space $(\Omega, \Sigma, \mu)$ is nonatomic and separable. Let us denote the $\sigma$-algebra of Lebesgue measurable subsets of $(0, 1)$ by $\mathcal{M}$. Let $\Phi$ denote an isomorphism from $(\Omega, \Sigma, \mu)$ to $(0, 1), \mathcal{M}, \mu_1)$, where $\mu_1$ denotes the one dimensional Lebesgue measure. For a measurable $E$ in $(0, 1)$ we write $|E|$ instead of $\mu_1(E)$. Let $f$ be a fuzzy subset of $\Omega$ which is integrable. Let $\Sigma'$ and $\mathcal{E}_f : \Sigma' \to \mathbb{R}$ be defined as in the previous section. Let $\mathcal{M}' = \mathcal{M}\setminus\{(0, 1), \emptyset\}$ and $\mathcal{F}_{f \circ \Phi^{-1}} : \mathcal{M}' \to \mathbb{R}$ by

$$\mathcal{F}_{f \circ \Phi^{-1}}(B) = \frac{1}{\gamma(B)} \int_B f \circ \Phi^{-1} - \frac{1}{\hat{\gamma}(B)} \| f \circ \Phi^{-1} \|_1,$$

where $\gamma(B) = |B|(1 - |B|)$ and $\hat{\gamma}(B) = 1 - |B|$. It is easily verified that $E_f(A) = \mathcal{F}_{f \circ \Phi^{-1}}(\Phi(A))$, for every $A \in \Sigma'$. It then follows that since $\Phi$ is an isomorphism we have

$$\sup_{A \in \Sigma'} E_f(A) = \sup_{B \in \mathcal{M}'} \mathcal{F}_{f \circ \Phi^{-1}}(B).$$

(5.20)
Moreover, if \( S(f \circ \Phi^{-1}, \mathcal{M}') \) denotes the solution set for the following variational problem,

\[
(\hat{P}): \sup_{B \in \mathcal{M}'} \mathcal{F}_{f \circ \Phi}(B),
\]

then we have \( S(f, \Sigma') = \Phi^{-1}(S(f \circ \Phi^{-1}, \mathcal{M}')) \). Now if \( \hat{\mathcal{M}}' \) denotes the subset of \( \mathcal{M}' \) comprising the cuts of \( f \circ \Phi' \) (note that the cuts of \( f \) are in a one-to-one correspondence with that of \( f \)), we can apply Theorem 1 to the variational problem \( (\hat{P}) \) to obtain the following.

**Theorem 2.** With the above notation, it follows that

\[
\sup_{A \in \mathcal{M}'} E_f(A) = \sup_{B \in \mathcal{M}'} \mathcal{F}_{f \circ \Phi^{-1}}(B)
\]

\[
S(f, \Sigma') = \Phi^{-1}(S(f \circ \Phi^{-1}, \hat{\mathcal{M}}')).
\]

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**References**