RESEARCH ARTICLE

A note on the inverse spectral problem for symmetric doubly stochastic matrices

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The symmetric doubly stochastic inverse spectral problem is the problem of determining necessary and sufficient conditions for a real n-tuple to be the spectrum of an n × n symmetric doubly stochastic matrix. For n ≥ 4, this problem remains open though many partial results are known. In this note, we present a new family of necessary conditions for this problem by using some matrix trace inequalities. In addition, we prove that this family of new inequalities sharpen the existing known necessary conditions for the inverse spectral problem of nonnegative matrices. Finally, we prove that these necessary conditions are not sufficient for the case n = 3.

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1. Introduction

An n × n matrix A with real entries is said to be nonnegative if all of its entries are nonnegative. If, in addition, each row and column sum of A is equal to 1, then A is said to be a doubly stochastic matrix. Let In, Jn and Cn be the n × n identity matrix, the n × n matrix whose all entries are 1/n and the n × n matrix whose diagonal entries are all zeroes and whose off-diagonal entries are all equal to 1/n − 1 respectively, where n ≥ 2. For any n × n matrix A, let σ(A) denotes its spectrum as an n-tuple and Tr A represents the trace of A. In addition, if e_n = 1/√n (1, 1, · · · , 1)^T ∈ ℝn then clearly an n × n nonnegative matrix A is doubly stochastic if and only if Ae_n = e_n and e_n^T A = e_n^T or equivalently AJ_n = J_n A = J_n. As a result, the product of doubly stochastic matrices is doubly stochastic, and in particular, any doubly stochastic matrix taken to any power is also doubly stochastic.

In the present note, we are interested in the following problem.

Problem 1.1 The symmetric doubly stochastic inverse spectral problem is the problem of determining the necessary and sufficient conditions for a real n-tuple to be spectrum of an n × n symmetric doubly stochastic matrix.

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Although this problem has been studied extensively (see e.g. [1, 2, 4, 6, 10, 11]), it remains open for $n \geq 4$.

For an $n \times n$ nonnegative matrix $A$ with spectrum $\sigma(A) = (\lambda_1, \cdots, \lambda_n)$, and for any positive integer $k$, let the trace of $A^k$ be denoted by

$$s_k(\sigma) = s_k = \lambda^k_1 + \lambda^k_2 + \cdots + \lambda^k_n.$$ 

Then some obvious necessary conditions for the preceding problem are the following:

- $s_k \geq 0$ for all natural numbers $k$, which just simply means that the trace of the nonnegative matrix $A^k$ is nonnegative.
- $0 \leq |\lambda_i| \leq 1$ for $i = 2, \cdots, n$; as the Perron-Frobenius theorem insures (see e.g. [8]).
- $0 \leq s_1 = TrA \leq n$ for any $n \times n$ doubly stochastic matrix $A$ with $\sigma(A) = (1, \lambda_2, \cdots, \lambda_n)$. In particular, since $A^k$ is doubly stochastic and $s_k = TrA^k$, then $0 \leq s_k \leq n$ for all positive integers $k$. If, in addition, $A$ is symmetric and $k$ is even, then obviously $1 \leq s_k \leq n$.

Since doubly stochastic matrices are nonnegative, then less obvious necessary conditions for preceding problem are those concerning the nonnegative inverse eigenvalue problem which is the problem of determining the necessary and sufficient conditions for a real $n$-tuple to be spectrum of an $n \times n$ nonnegative matrix. The most popular ones are the (JLL) conditions [3, 7], which can be stated as follows:

**Theorem 1.2**: [3, 7] Let $A$ be an $n \times n$ nonnegative matrix with spectrum $\sigma(A) = (\lambda_1, \cdots, \lambda_n)$. Then

$$n^{k-1}s_{km} \geq s^k_m$$

for all positive integers $k, m$.

A refinement of (1) in the special case of $k = m = 2$, $n$ odd and $TrA = 0$, is due to Laffey and Meehan [5] and can be stated as follows.

**Theorem 1.3**: [5] For $n$ odd, let $A$ be an $n \times n$ nonnegative matrix with $TrA = 0$ and let its spectrum be $\sigma(A) = (\lambda_1, \cdots, \lambda_n)$. Then

$$(n - 1)s_4 \geq s^2_2.$$  

As far as we know, necessary conditions for Problem 1.1 have never been dealt with before and our intention in this paper is to deal with this issue. More specifically, we use some matrix trace inequalities to present some necessary conditions for Problem 1.1, which sharpen the inequalities of the preceding two theorems for some special class of symmetric doubly matrices.

2. Main results

The main result of this paper is the following theorem, which gives a family of necessary conditions for Problem 1.1.
Theorem 2.1: Let $A$ be an $n \times n$ symmetric doubly stochastic matrix with spectrum $\sigma(A) = (1, \lambda_2, \cdots, \lambda_n)$, and $s_k = \text{Tr} A^k$. Then

$$
\left(1 + (n-1) \left(a - \frac{1-a}{n-1}\right)^{\frac{2k}{k+1}}\right)^{\frac{2k-4}{k+1}} s_{2km} \geq \left(a - \frac{1-a}{n-1}\right) s_m + \frac{n(1-a)}{n-1} 2^k
$$

(3)

for all positive integers $k, m$ and for all $0 \leq a \leq 1$.

Proof: Recall that Hölder’s inequality for complex matrices (see e.g. [12]), is given by

$$
|\text{Tr} M^* N| \leq (\text{Tr} |M|^p)^{1/p} (\text{Tr} |N|^q)^{1/q}, \quad 1 \leq p, q \leq \infty, \quad 1/p + 1/q = 1
$$

(4)

where $|M| = (M^*M)^{1/2}$. If $B = A^m$, then as mentioned earlier $B$ is symmetric doubly stochastic. Now using inequality (4) for $M = aI_n + (1-a)C_n$ where $0 \leq a \leq 1$, and $N = B$ with $p = \frac{2k}{k+1}$ and $q = 2k$, we obtain

$$
|\text{Tr}(aI_n + (1-a)C_n)^T B| \leq \left(\text{Tr}(|aI_n + (1-a)C_n|^2)^{\frac{1}{2k}}\right)^{\frac{2k-4}{k+1}} \left(\text{Tr}(|B|^2)^k\right)^{1/2k}.
$$

(5)

Now it is easy to check that $C_n = \frac{n}{n-1} J_n - \frac{1}{n-1} I_n$ so that $\sigma(C_n) = (1, -\frac{1}{n-1}, \cdots, -\frac{1}{n-1})$ as $\sigma(J_n) = (1, 0, \cdots, 0)$. From the fact that the matrix $aI_n + (1-a)C_n$ is symmetric and the spectral theorem for symmetric matrices, we have $|aI_n + (1-a)C_n|^2 = (aI_n + (1-a)C_n)^2 = U^T \text{diag} \left(1, (a - \frac{1-a}{n-1})^2, \cdots, (a - \frac{1-a}{n-1})^2\right) U$ for some orthogonal matrix $U$. Thus we obtain

$$
\text{Tr}(|aI_n + (1-a)C_n|^2)^{\frac{1}{2k}} = 1 + (n-1) \left(a - \frac{1-a}{n-1}\right)^{\frac{2k}{k+1}}.
$$

Noticing that $|B|^2 = B^2$, and $(aI_n + (1-a)C_n)^T B = (aI_n + \frac{n(1-a)}{n-1} J_n - \frac{1-a}{n-1} I_n) B = (a - \frac{1-a}{n-1}) B + \frac{n(1-a)}{n-1} J_n$, we see that inequality (5) becomes

$$
\left(a - \frac{1-a}{n-1}\right) \text{Tr} B + \frac{n(1-a)}{n-1} \leq \left(1 + (n-1) \left(a - \frac{1-a}{n-1}\right)^{\frac{2k}{k+1}}\right)^{\frac{2k-4}{k+1}} \left(\text{Tr}(B^2)^k\right)^{1/2k}.
$$

Replacing $B$ by $A^m$, we conclude that

$$
\left(a - \frac{1-a}{n-1}\right) \text{Tr} A^m + \frac{n(1-a)}{n-1} \leq \left(1 + (n-1) \left(a - \frac{1-a}{n-1}\right)^{\frac{2k}{k+1}}\right)^{\frac{2k-4}{k+1}} \left(\text{Tr} A^{2km}\right)^{1/2k}.
$$

By noting that $\text{Tr} A^m = s_m$ and $\text{Tr} A^{2km} = s_{2km}$, we obtain inequality (3). □

Remark 1: Inequality (3) for $a = 1$ becomes $n^{2k-1} s_{2km} \geq s_m^{2k}$, which is the same as inequality (1) when replacing $k$ with $2k$.

Next we explore the direction in which inequality (3) gives an improvement of inequality (1). By replacing $k$ with $2k$, inequality (1) can be rewritten as

$$
s_{2km} \geq \frac{1}{n^{2k-1}} s_m^{2k}.
$$

(6)
On the other hand, inequality (3) can also be rewritten as

$$s_{2km} \geq \left(1 + (n - 1) \left(a - \frac{1 - a}{n - 1}\right) \right)^{\frac{2k}{2k-1}} \left(a - \frac{1 - a}{n - 1}\right)^{s_m + \frac{n(1 - a)}{n - 1}} 2^k$$

(7)

Comparing the right-hand sides of both (6) and (7), we obtain the following interesting consequence regarding the sharpnesses of inequalities (6) and (7).

**Theorem 2.2:** Let $A$ be an $n \times n$ symmetric doubly stochastic matrix with spectrum $\sigma(A) = (1, \lambda_2, \cdots, \lambda_n)$ and $s_m = TrA^m$. For any $0 \leq a \leq 1$ and for any positive integers $k$ and $m$ then

- for $0 \leq s_m \leq \frac{n^{2k-1}(1 - \frac{a - 1}{n - 1})}{(1 + (n - 1)(\frac{a - 1}{n - 1})^{\frac{2k}{2k-1}})} - n^{\frac{2k-1}{2k}}(\frac{a - 1}{n - 1})$ it holds that

$$s_{2km} \geq \left(1 + (n - 1) \left(a - \frac{1 - a}{n - 1}\right) \right)^{\frac{2k}{2k-1}} \left(a - \frac{1 - a}{n - 1}\right)^{s_m + \frac{n(1 - a)}{n - 1}} 2^k \geq \frac{1}{n^{2k-1}} 2^k s_m.$$

- for $\frac{n^{2k-1}(1 - \frac{a - 1}{n - 1})}{(1 + (n - 1)(\frac{a - 1}{n - 1})^{\frac{2k}{2k-1}})} \leq s_m \leq n$, it holds that $s_{2km} \geq \frac{1}{n^{2k-1}} s_m 2^k$

$$\geq \left(1 + (n - 1) \left(a - \frac{1 - a}{n - 1}\right) \right)^{\frac{2k}{2k-1}} \left(a - \frac{1 - a}{n - 1}\right)^{s_m + \frac{n(1 - a)}{n - 1}} 2^k.$$

**Proof:** It suffices to study the sign of the polynomial $p(s_m)$ in the variable $s_m$ given by $p(s_m) =$

$$\left(1 + (n - 1) \left(a - \frac{1 - a}{n - 1}\right) \right)^{\frac{2k}{2k-1}} \left(a - \frac{1 - a}{n - 1}\right)^{s_m + \frac{n(1 - a)}{n - 1}} 2^k - \frac{1}{n^{2k-1}} s_m 2^k$$

in the interval $[0, n]$ since as mentioned earlier, we have $0 \leq s_m \leq n$. Indeed, since we are dealing with positive quantities then a simple inspection shows that

$$\left(1 + (n - 1) \left(a - \frac{1 - a}{n - 1}\right) \right)^{\frac{2k}{2k-1}} \left(a - \frac{1 - a}{n - 1}\right)^{s_m + \frac{n(1 - a)}{n - 1}} 2^k \geq \frac{1}{n^{2k-1}} s_m 2^k$$

if and only if

$$\left(1 + (n - 1) \left(a - \frac{1 - a}{n - 1}\right) \right)^{\frac{2k}{2k-1}} s_m \leq n^{\frac{2k-1}{2k}} \left(a - \frac{1 - a}{n - 1}\right)^{s_m + \frac{n(1 - a)}{n - 1}}.$$

which after rearranging the terms yields the required result. $\square$

The importance of the first inequality in the preceding theorem lies in the fact that for each value of $a$ with $0 \leq a \leq 1$, we obtain a new inequality that sharpens inequality (1) for some special class of nonnegative matrices. This in turn means that we get a family of new inequalities that sharpens inequality (1) for the union
of those classes of nonnegative matrices. To illustrate this, we take the following example dealing with the case $a = 0$ and $k = 1$ and which seems to be of independent interest.

**Example 2.3** Let $A$ be an $n \times n$ symmetric doubly stochastic matrix with spectrum $\sigma(A) = (1, \lambda_2, \ldots, \lambda_n)$. Then

- $ns_{2m} \geq \frac{(n-s_m)^2}{n-1} \geq s_m^2$ for $0 \leq s_m \leq \frac{n(\sqrt{n-1}-1)}{n-2}$,
- $ns_{2m} \geq s_m^2 \geq \frac{(n-s_m)^2}{n-1}$ for $\frac{n(\sqrt{n-1}-1)}{n-2} \leq s_m \leq n$,

for all positive integers $m$. It is worth noting here that for large $n$ the quantity $\frac{n(\sqrt{n-1}-1)}{n-2}$ gets larger so that the sharpness of inequality (6) over that of (7) gets improved significantly as $n$ tends to infinity.

Similarly, we shall show the direction in which inequality (2) has been improved. Inequality (2) can be rewritten as

$$s_4 \geq \frac{s_2^2}{n-1}. \quad (8)$$

On the other hand, inequality (3) for $m = 2$ and $k = 1$ can be written as

$$s_4 \geq \left(1 + (n-1) \left(a - \frac{1-a}{n-1}\right)^2 \right)^{-1} \left( (a - \frac{1-a}{n-1}) s_2 + \frac{n(1-a)}{n-1} \right)^2. \quad (9)$$

Comparing the right-hand sides of both (8) and (9), we obtain the following theorem.

**Theorem 2.4**: Let $A$ be an $n \times n$ symmetric doubly stochastic matrix with spectrum $\sigma(A) = (1, \lambda_2, \ldots, \lambda_n)$. Then for $0 \leq s_2 \leq \frac{n(2a-1)(1-a)}{n-1} + \sqrt{\left( \frac{n(na-1)(1-a)}{n-1} \right)^2 + \frac{n^2(1-a)^2}{n-1}}$, we have

$$s_4 \geq \left(1 + (n-1) \left(a - \frac{1-a}{n-1}\right)^2 \right)^{-1} \left( (a - \frac{1-a}{n-1}) s_2 + \frac{n(1-a)}{n-1} \right)^2 \geq \frac{s_2^2}{n-1}.$$

**Proof**: It suffices to study the sign of the quadratic $q(s_2)$ in the variable $s_2$ given by

$$q(s_2) = \left(1 + (n-1) \left(a - \frac{1-a}{n-1}\right)^2 \right)^{-1} \left( (a - \frac{1-a}{n-1}) s_2 + \frac{n(1-a)}{n-1} \right)^2 - \frac{s_2^2}{n-1}$$

in the interval $[0, n]$. It can be easily checked that

$$\left(1 + (n-1) \left(a - \frac{1-a}{n-1}\right)^2 \right)^{-1} \left( (a - \frac{1-a}{n-1}) s_2 + \frac{n(1-a)}{n-1} \right)^2 \geq \frac{s_2^2}{n-1}$$
if and only if
\[
\frac{(an - 1)^2}{(n - 1)^2} s_2^2 + 2n \left( \frac{an - 1}{n - 1} \right) \left( \frac{1 - a}{n - 1} \right) s_2 + \frac{n^2(1 - a)^2}{(n - 1)^2} \geq \frac{s_2^2}{n - 1} \left( 1 + \frac{1 - a}{n - 1} \right)^2
\]
if and only if
\[
s_2^2 - \frac{2n(na - 1)(1 - a)}{n - 1} s_2 - \frac{n^2(1 - a)^2}{n - 1} \leq 0.
\]
The two roots of the left-hand side are given by
\[
s_2^{(1)} = \frac{n(na - 1)(1 - a)}{n - 1} - \sqrt{\left( \frac{n(na - 1)(1 - a)}{n - 1} \right)^2 + \frac{n^2(1 - a)^2}{n - 1}},
\]
which is obviously negative and
\[
s_2^{(2)} = \frac{n(na - 1)(1 - a)}{n - 1} + \sqrt{\left( \frac{n(na - 1)(1 - a)}{n - 1} \right)^2 + \frac{n^2(1 - a)^2}{n - 1}},
\]
which is positive. With these information, the proof can be easily completed. □

As mentioned earlier, in this case we also obtain a new family of inequalities that sharpens inequality (2) for a union of some classes of nonnegative matrices. To illustrate this, we take another example which appears to be of independent interest. As earlier it deals with the case \( a = 0 \) in the preceding theorem and sharpens inequality (2) for those symmetric doubly stochastic matrices \( A \) for which \( TrA^2 \leq \frac{n(\sqrt{\pi - 1})}{n - 1} \).

**Example 2.5** Let \( A \) be an \( n \times n \) symmetric doubly stochastic matrices matrix with \( \sigma(A) = (1, \lambda_2, \cdots, \lambda_n) \) and with \( 1 \leq Tr(A^2) \leq \frac{n(\sqrt{\pi - 1})}{n - 1} \). Then
\[
(n - 1)s_4 \geq \frac{(n - s_2)^2}{n} \geq s_2^2.
\]

3. **Condition (3) is not sufficient**

First we start with the following result for which the proof can be found in [9].

**Lemma 3.1:** Let \( n \) be odd and \( \sigma = (1, 0, \cdots, 0, -1) \in \mathbb{R}^n \). Then \( \sigma \) can not be the spectrum of an \( n \times n \) symmetric doubly stochastic matrix.

The next result helps establishing that condition (3) is not sufficient for the case \( n = 3 \).

**Lemma 3.2:** Let \( 0 \leq a \leq 1 \). Then for any positive integer \( k \),
\[
\bullet \ 2 \left( 1 + 2 \left( \frac{3a - 1}{2} \right)^{\frac{2k}{n - 1}} \right)^{2k - 1} \geq \left( \frac{3a + 1}{2} \right)^{2k}.
\]
\[
\bullet \ 2 \left( 1 + 2 \left( \frac{3a - 1}{2} \right)^{\frac{2k}{n - 1}} \right)^{2k - 1} \geq \left( \frac{3}{2}(1 - a) \right)^{2k}.
\]

**Proof:** If we let \( z = 2 \left( 1 + 2 \left( \frac{3a - 1}{2} \right)^{\frac{2k}{n - 1}} \right)^{2k - 1} - \left( \frac{3a + 1}{2} \right)^{2k} \), then obviously we need
to prove that \( z \geq 0 \). In addition, \( z \geq 0 \) if and only if

\[
y = \left( 1 + 2 \left( \frac{3a - 1}{2} \right)^{\frac{k}{2k-1}} \right) - \frac{1}{2^{\frac{k}{2k-1}}} \left( \frac{3a + 1}{2} \right)^{\frac{k}{2k-1}} \geq 0.
\]

Note that for \( 0 \leq a \leq \frac{1}{3} \), the first term in \( y \) is greater than or equal to 1 and the second term is less than or equal to 1. So that in this case \( y \geq 0 \). Now for the case where \( \frac{1}{3} \leq a \leq 1 \), we study the variations of \( y \). The derivative \( y' \) of \( y \) is given by

\[
y' = 2 \left( \frac{2k}{2k - 1} \right) \left( \frac{3a - 1}{2} \right)^{\frac{k}{2k-1}} - \frac{1}{2^{\frac{k}{2k-1}}} \left( \frac{2k}{2k - 1} \right) \left( \frac{3a + 1}{2} \right)^{\frac{k}{2k-1}}.
\]

Note that \( y' = 0 \) if and only if \( 2 \left( \frac{3a - 1}{2} \right)^{\frac{k}{2k-1}} = \frac{1}{2^{\frac{k}{2k-1}}} \left( \frac{3a + 1}{2} \right)^{\frac{k}{2k-1}} \), which is positive for all \( a = \frac{2^{2k+1}}{3(2^{2k+1}-1)} \). Moreover, \( y' < 0 \) for \( \frac{1}{3} \leq a < \frac{2^{2k+1}}{3(2^{2k+1}-1)} \), and \( y' > 0 \) for \( \frac{2^{2k+1}}{3(2^{2k+1}-1)} < a \leq 1 \). In addition, we have the following:

- For \( a = \frac{1}{3} \), we see that \( y = 1 - \frac{1}{2^{\frac{k}{2k-1}}} \), which is positive for all \( k \geq 1 \).

- For \( a = \frac{2^{2k+1}}{3(2^{2k+1}-1)} \), we have \( y = 1 + 2 \left( \frac{3a - 1}{2} \right)^{\frac{k}{2k-1}} - \frac{1}{2^{\frac{k}{2k-1}}} \left( \frac{2k}{2k - 1} \right) \left( \frac{3a + 1}{2} \right)^{\frac{k}{2k-1}} \), which in turn implies that \( y = 1 + \frac{2k}{2k-1} - \frac{1}{2^{\frac{k}{2k-1}}} \left( \frac{2k}{2k - 1} \right) = 1 - \frac{2}{2^{1/2k}} \), which is also positive for all \( k \geq 1 \).

- For \( a = 1 \), we have \( y = 3 - 2 = 1 \), which is clearly positive.

From the sign of \( y' \) we conclude that the curve of \( y \) goes down from the point \((\frac{1}{3}, 1 - \frac{1}{2^{\frac{k}{2k-1}}})\) to the point \((\frac{2^{2k+1}}{3(2^{2k+1}-1)}, 1 - \frac{2}{2^{1/2k}})\) and then up to the point \((1, 1)\). So that \( y \) is positive for \( \frac{1}{3} \leq a \leq 1 \), and the proof of the first part is complete.

For the second part, we employ a similar strategy. Now if we let \( z = 2 \left( 1 + 2 \left( \frac{3a - 1}{2} \right)^{\frac{k}{2k-1}} - \left( \frac{3}{2} (1 - a) \right)^{\frac{k}{2k-1}} \right) \), then \( z \geq 0 \) if and only if

\[
y = \left( 1 + 2 \left( \frac{3a - 1}{2} \right)^{\frac{k}{2k-1}} \right) - \frac{1}{2^{\frac{k}{2k-1}}} \left( \frac{3}{2} (1 - a) \right)^{\frac{k}{2k-1}} \geq 0.
\]

However, contrary to the first case, for \( \frac{1}{3} \leq a \leq 1 \), the first term in \( y \) is greater than or equal to 1 and the second term is less than or equal to 1. So that in this case \( y \geq 0 \). Now for \( 0 \leq a \leq \frac{1}{3} \), we study the variations of \( y \). As before, the derivative is given by

\[
y' = 2 \left( \frac{2k}{2k - 1} \right) \left( \frac{3a - 1}{2} \right)^{\frac{k}{2k-1}} - \frac{1}{2^{\frac{k}{2k-1}}} \left( \frac{2k}{2k - 1} \right) \left( \frac{3}{2} (1 - a) \right)^{\frac{k}{2k-1}}.
\]

In addition, \( y' = 0 \) if and only if \( 2 \left( \frac{3a - 1}{2} \right)^{\frac{k}{2k-1}} = -\frac{1}{2^{\frac{k}{2k-1}}} \left( \frac{3}{2} (1 - a) \right)^{\frac{k}{2k-1}} \), if and only if \( a = \frac{2^{2k}-3}{3(2^{2k}-1)} \). Moreover, \( y' < 0 \) for \( 0 \leq a < \frac{2^{2k}-3}{3(2^{2k}-1)} \), and \( y' > 0 \) for \( \frac{2^{2k}-3}{3(2^{2k}-1)} < a \leq \frac{1}{3} \). Also, we have the following:

- For \( a = 0 \), we see that \( y = 1 + \frac{1}{2^{\frac{k}{2k-1}}} = \frac{1}{2^{\frac{k}{2k-1}}} (3)^{\frac{2k}{2k-1}} = 1 + \frac{1}{2^{\frac{k}{2k-1}}} (1 - \left( \frac{3}{2} \right)^{\frac{2k}{2k-1}}) \), which is positive for all \( k \geq 1 \), since this last term increases as \( k \) increases and
its minimum is attained at $k = 1$ and is equal to $1 + \frac{1}{2}(1 - \frac{9}{4})$, which is obviously positive.

- For $a = \frac{2^{2k-3}}{3(2^{2k-1})}$, we have

$$y = 1 + 2 \left(\frac{2^{2k-3}}{2^{2k-1}} - 1\right) - \frac{1}{2^{2k-1}} \left(3 - \frac{2^{2k-3}}{2^{2k-1}}\right) \left(\frac{2^{2k}}{2^{2k-1}}\right),$$

which in turn implies that $y = 1 + 2 \left(\frac{1}{2^{2k-1}}\right) - \frac{1}{2^{2k-1}} \left(\frac{2^{2k}}{2^{2k-1}}\right) = 1 + \frac{2 - 2^{2k+1}}{(2^{2k-1})2^{2k-1}} = 1 - 2 \frac{2^{2k-1}}{(2^{2k-1})2^{2k-1}} = 1 - \frac{2}{2^{2k-1}}$, which is also positive for all $k \geq 1$.

- For $a = \frac{1}{3}$, we have $y = 1 - \frac{1}{2^{2k-1}}$, which is always positive for all $k \geq 1$.

From the sign of $y'$ and similar to what was mentioned earlier in the first case, the curve of $y$ goes down from the point $\left(0, 1 + \frac{1}{2^{2k-1}} \left(1 - \frac{3}{2} \frac{2^{2k}}{2^{2k-1}}\right)\right)$ to the point $\left(\frac{2^{2k-3}}{3(2^{2k-1})}, 1 - \frac{2}{2^{2k-1}}\right)$ and then up to the point $\left(\frac{1}{3}, 1 - \frac{1}{2^{2k-1}}\right)$. So that $y$ is positive for $0 \leq a \leq \frac{1}{3}$, and the proof of the second part is complete.

**Theorem 3.3:** Condition (3) is not sufficient for the case $n = 3$.

**Proof:** Let $\sigma = (1, 0, -1)$ then clearly for all positive integers $k$ and $m$, we have $s_{2km} = 2$ and

$$s_m = \begin{cases} 2 & \text{for } m \text{ even} \\ 0 & \text{for } m \text{ odd} \end{cases}.$$

Now for this $\sigma$ and for $n = 3$, condition (3) is satisfied by virtue of the preceding lemma. However by Lemma 3.1, $\sigma$ can not be the spectrum of a $3 \times 3$ symmetric doubly stochastic matrix. We conclude that the necessary condition (3) is not sufficient.

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**References**


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