Research Article

Convergence Theorem for a Family of Generalized Asymptotically Nonexpansive Semigroup in Banach Spaces

Bashir Ali¹ and G. C. Ugwunnadi²

¹ Department of Mathematical Sciences, Bayero University, P. M. B. 3011 Kano, Nigeria
² Department of Mathematics, University of Nigeria, Nsukka, Nigeria

Correspondence should be addressed to Bashir Ali, bashiralik@yahoo.com

Received 15 March 2012; Revised 8 June 2012; Accepted 8 June 2012

1. Introduction

Let \( E \) be a real reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm. Let \( J = \{ T(t) : t \geq 0 \} \) be a family of uniformly asymptotically regular generalised asymptotically nonexpansive semigroup of \( E \), with functions \( u, v : [0, \infty) \to [0, \infty) \). Let \( F := F(J) = \cap_{t \geq 0} F(T(t)) \neq \emptyset \) and \( f : K \to K \) be a weakly contractive map. For some positive real numbers \( \lambda \) and \( \delta \), satisfying \( \delta + \lambda > 1 \), let \( G : E \to E \) be a \( \delta \)-strongly accretive and \( \lambda \)-strictly pseudocontractive map. Let \( \{ t_n \} \) be an increasing sequence in \( [0, \infty) \) with \( \lim_{n \to \infty} t_n = \infty \), and let \( \{ \alpha_n \} \) and \( \{ \beta_n \} \) be sequences in \( (0, 1) \) satisfying some conditions. Strong convergence of a viscosity iterative sequence to common fixed points of the family \( J \) of uniformly asymptotically regular asymptotically nonexpansive semigroup, which also solves the variational inequality \( \langle (G - \gamma f)p, j(p - x) \rangle \leq 0 \), for all \( x \in F \), is proved in a framework of a real Banach space.

1. Introduction

Let \( E \) be a real Banach space. We denote by \( J \) the normalized duality map from \( E \) to \( 2^{E^*} \) (\( E^* \) is the dual space of \( E \)), and it is defined by

\[
J(x) = \left\{ f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2 \right\}.
\] (1.1)

A mapping \( T : E \to E \) is said to be contractive if \( \|Tx - Ty\| \leq \alpha \|x - y\| \), for \( x, y \in E \), and some constant \( \alpha \in [0, 1) \). It is said to be weakly contractive if there exists a nondecreasing function \( \varphi : [0, \infty) \to [0, \infty) \) satisfying \( \varphi(t) = 0 \) if and only if \( t = 0 \) and \( \|Tx - Ty\| \leq \|x - y\| - \varphi(\|x - y\|) \), for all \( x, y \in E \). It is known that the class of weakly contractive maps
contain properly the class of contractive ones, see [1, 2]. A mapping $T : E \to E$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in E$ and asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ and $\|T^n x - T^n y\| \leq k_n \|x - y\|$, for all $x, y \in E$. We denote by $F(T) = \{x \in K : Tx = x\}$ the set of fixed points of a map $T$.

A mapping $T : E \to E$ is said to be total asymptotically nonexpansive (see [3]) if there exist nonnegative real sequences $\{u_n\}$ and $\{v_n\}$, with $u_n \to 0$ and $v_n \to 0$ as $n \to \infty$ and strictly increasing and continuous functions $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(0) = 0$ such that

$$\|T^n x - T^n y\| \leq \|x - y\| + u_n \psi(\|x - y\|) + v_n,$$

for all $x, y \in K$. \hspace{1cm} (1.2)

**Remark 1.1.** If $\psi(\lambda) = \lambda$, the total asymptotically nonexpansive mapping coincides with generalized asymptotically nonexpansive mapping. In addition, for all $n \in \mathbb{N}$, if $v_n = 0$, then generalized asymptotically nonexpansive mapping coincides with asymptotically nonexpansive mapping; if $u_n = 0$, $v_n = \max\{0, p_n\}$ where $p_n := \sup_{x,y \in K} (\|T_n x - T_n y\| - \|x - y\|)$, then generalized asymptotically nonexpansive mapping coincide with asymptotically nonexpansive mapping in the intermediate sense; if $u_n = 0$, and $v_n = 0$ then we obtain from (1.2) the class of nonexpansive mapping.

A one-parameter family of generalized asymptotically nonexpansive semigroup is a family $\mathcal{J} = \{T(t) : t \geq 0\}$ of self-mapping of $E$ such that

(i) $T(0) x = x$ for $x \in E$,
(ii) $T(s + t) x = T(t) T(s) x$ for all $t, s \geq 0$ and $x \in E$,
(iii) $\lim_{t \to 0} T(t) x = x$ for $x \in E$,
(iv) there exist functions $u, v : [0, \infty) \to [0, \infty)$ such that $u(t) \to 0, v(t) \to 0$ as $t \to \infty$, and

$$\|T(t) x - T(t) y\| \leq (1 + u(t)) \|x - y\| + v(t) \quad \forall x, y \in E.$$

We will denote by $F$ the common fixed-point set of $\mathcal{J}$, that is,

$$F := \text{Fix}(\mathcal{J}) = \{x \in E : T(t) x = x, t \geq 0\} = \bigcap_{t \geq 0} \text{Fix}(T(t)). \hspace{1cm} (1.4)$$

The family $\mathcal{J} = \{T(t) : t \geq 0\}$ is said to be asymptotically regular if

$$\lim_{s \to \infty} \|T(s + t) x - T(s) x\| = 0, \hspace{1cm} (1.5)$$

for all $t \in [0, \infty)$ and $x \in E$. It is said to be uniformly asymptotically regular if, for any $t \geq 0$ and for any bounded subset $C$ of $E$,

$$\lim_{s \to \infty} \sup_{x \in C} \|T(s + t) x - T(s) x\| = 0. \hspace{1cm} (1.6)$$
For some positive real numbers $\delta$ and $\lambda$, a mapping $G : E \to E$ is said to be $\delta$-strongly accretive if for any $x, y \in E$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Gx - Gy, j(x - y) \rangle \geq \delta \|x - y\|^2,$$

and it is called $\lambda$-strictly pseudocontractive if

$$\langle Gx - Gy, j(x - y) \rangle \leq \|x - y\|^2 - \lambda \|(I - G)x - (I - G)y\|^2.$$  

Let $E$ be a real Banach space, and let $\delta, \lambda$, and $\tau$ be positive real numbers satisfying $\delta + \lambda > 1$ and $\tau \in (0, 1)$. Let $G : E \to E$ be a $\delta$-strongly accretive and $\lambda$-strictly pseudocontractive, then the following holds, see [4], for $x, y \in E$:

$$\|(I - G)x - (I - G)y\| \leq \left(1 - \sqrt{\frac{1 - \delta}{\lambda}}\right)\|x - y\|,$$

$$\|(I - \tau G)x - (I - \tau G)y\| \leq 1 - \tau \left(1 - \sqrt{\frac{1 - \delta}{\lambda}}\right)\|x - y\|,$$

that is, $(I - G)$ and $(I - \tau G)$ are contractive mappings.

Let $C$ be a nonempty closed-convex subset of $E$ and $T : E \to E$ a map. Then, a variational inequality problem with respect to $C$ and $T$ is found to be $x^* \in C$ such that

$$\langle Tx^*, j(y - x^*) \rangle \geq 0, \quad \forall y \in C, \; j(y - x^*) \in J(y - x^*).$$

Recently, convergence theorems for fixed points of nonexpansive mappings, common fixed points of family of nonexpansive mappings, nonexpansive semigroup, and their generalisation have been studied by numerous authors (see, e.g., [5–21]).

Acedo and Suzuki [22], recently, proved the strong convergence of the Browder’s implicit scheme, $x_0, u \in C$,

$$x_n = \alpha_n u + (1 - \alpha_n)T(t_n)x_n, \quad n \geq 0,$$  

to a common fixed point of a uniformly asymptotically regular family $\{T(t) : t \geq 0\}$ of nonexpansive semigroup in the framework of a real Hilbert space.

Li et al. [23] proved strong convergence theorems for implicit viscosity schemes for common fixed points of family of generalized asymptotically nonexpansive semigroups in Banach spaces.

Let $S$ be a semigroup and $B(S)$ the subspace of all bounded real-valued functions defined on $S$ with supremum norm. For each $s \in S$, the left translator operator $l(s)$ on $B(S)$ is defined by $(l(s)f)(t) = f(st)$ for each $t \in S$ and $f \in B(S)$. Let $X$ be a subspace of $B(S)$ containing $1$, and let $X^*$ be its topological dual. An element $\mu$ of $X^*$ is said to be a mean on $X$ if $\|\mu\| = \mu(1) = 1$. Let $X$ be $l_1$ invariant, that is, $l_s(X) \subseteq X$ for each $s \in S$. A mean $\mu$ on $X$ is said to be left invariant if $\mu(l_sf) = \mu(f)$ for each $s \in S$ and $f \in X$.
Recently, Saeidi and Naseri [24] studied the problem of approximating common fixed point of a family of nonexpansive semigroup and solution of some variational inequality problem in a real Hilbert space. They proved the following theorem.

**Theorem 1.2** (Saeidi and Naseri [24]). Let $\mathcal{H} = \{T(t) : t \in S\}$ be a nonexpansive semigroup in a real Hilbert space $H$ such that $F(\mathcal{H}) \neq \emptyset$. Let $X$ be a left invariant subspace of $B(S)$ such that $1 \in X$, and the function $t \rightarrow \langle T(t)x, y \rangle$ is an element of $X$ for each $x, y \in H$. Let $f : E \to E$ be a contraction with constant $\alpha$, and let $G : H \to H$ be strongly positive map with constant $\overline{\gamma} > 0$. Let $\{\mu_n\}$ be a left regular sequence of means on $X$, and let $\{\alpha_n\}$ be a sequence in $(0, 1)$ such that $\lim_{n \to \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Let $\gamma \in (0, \overline{\gamma}/\alpha)$, and let $\{x_n\}$ be a sequence generated by $x_0 \in H$,

$$x_{n+1} = (I - \alpha_n G)T(\mu_n)x_n + \alpha_n \gamma f(x_n), \quad n \geq 0.$$  

(1.12)

Then, $\{x_n\}$ converges strongly to a common fixed point of the family $\mathcal{H}$ which is the unique solution of the variational inequality $\langle (G - \gamma f)x^*, j(x - x^*) \rangle \geq 0$ for all $x \in F(\mathcal{H})$. Equivalently one has $P_{F(\mathcal{H})}(I - G + \gamma f)x^* = x^*$.

More recently, as commented by Goilkaranesh and Naseri [25], Piri and Vaezi [4] gave a minor variation of Theorem 1.2 as follows.

**Theorem 1.3** (Piri and Vaezi [4]). Let $\mathcal{H} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on a real Hilbert space $H$ such that $F(\mathcal{H}) \neq \emptyset$. Let $X$ be a left invariant subspace of $B(S)$ such that $1 \in X$, and the function $t \rightarrow \langle T(t)x, y \rangle$ is an element of $X$ for each $x, y \in H$. Let $f : E \to E$ be a contraction with constant $\alpha$, and let $G : H \to H$ be $\delta$-strongly accretive and $\lambda$-strictly pseudocontractive with $\delta + \lambda > 1$. Let $\{\mu_n\}$ be a left regular sequence of means on $X$, and let $\{\alpha_n\}$ be a sequence in $(0, 1)$ such that $\lim_{n \to \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Let $\{x_n\}$ be a sequence generated by $x_0 \in H$,

$$x_{n+1} = (I - \alpha_n G)T(\mu_n)x_n + \alpha_n \gamma f(x_n), \quad n \geq 0,$$  

(1.13)

where $0 < \gamma < (1 - \sqrt{(1 - \delta/\lambda)})/\alpha$, then, $\{x_n\}$ converges strongly to a common fixed point of the family $F(\mathcal{H})$ which is the unique solution of the variational inequality $\langle (G - \gamma f)x^*, j(x - x^*) \rangle \geq 0$ for all $x \in F(\mathcal{H})$. Equivalently one has $P_{F(\mathcal{H})}(I - G + \gamma f)x^* = x^*$.

Very recently, Ali [26] continued the study of the problem in [4, 24] and proved a strong convergence theorem in a Banach space setting much more general than Hilbert space. He actually proved the following theorem.

**Theorem 1.4** (Ali [26]). Let $E$ be a real Banach space with local uniform Opial’s property whose duality mapping is sequentially continuous. Let $\mathcal{H} = \{T(t) : t \geq 0\}$ be a uniformly asymptotically regular family of asymptotically nonexpansive semigroup of $E$ with function $k : [0, \infty) \to [0, \infty)$ and $F : F(\mathcal{H}) = \cap_{t \geq 0} F(T(t)) \neq \emptyset$. Let $f : E \to E$ be weakly contractive, and let $G : E \to E$ be $\delta$-strongly accretive and $\lambda$-strictly pseudocontractive with $\delta + \lambda > 1$. Let $\eta := (1 - \sqrt{(1 - \delta/\lambda)})/\alpha$ and $\gamma \in (0, \min\{\eta, \delta/2\})$. Let $\{\beta_n\}$ and $\{\alpha_n\}$ be sequences in $(0, 1)$, and let $\{\tau_n\}$ be an increasing sequence in $[0, \infty)$ satisfying the following conditions:

$$\lim_{n \to \infty} \alpha_n = 0, \quad \lim_{n \to \infty} \frac{\alpha_n}{\beta_n} = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad 0 < \lim \inf \beta_n \leq \lim \sup \beta_n < 1.$$  

(1.14)
Define a sequence \( \{ x_n \} \) by \( x_0 \in E \),
\[
x_{n+1} = \beta_n x_n + (1 - \beta_n) y_n,
\]
\[
y_n = (I - \alpha_n G)T(t_n)x_n + \alpha_n \gamma_n f(x_n), \quad n \geq 0. \tag{1.15}
\]

Then, the sequence \( \{ x_n \} \) converges strongly to a common fixed point of the family \( \mathcal{F} \) which solves the variational inequality
\[
\langle (G - \gamma f)q, j(x - q) \rangle \geq 0, \quad \forall x \in F. \tag{1.16}
\]

Remark 1.5. It is well known that all \( L^p \) (\( 1 < p < \infty \)) spaces satisfy Opial’s condition and possess a weakly sequentially continuous duality mapping. However, \( L^p \) (\( 1 < p < \infty \)) spaces and consequently all Sobolev spaces do not satisfy either of the properties.

It is our purpose in this paper to prove a strong convergence theorem for approximating common fixed points of family of uniformly asymptotically regular generalized asymptotically nonexpansive semigroup in a real reflexive and strictly convex Banach space \( E \) with a uniformly Gâteaux differentiable norm. Our theorem is applicable in \( L_p(\ell_p) \) spaces, \( 1 < p < \infty \) (and consequently in Sobolev spaces). Our theorem extends and improves some recent important results. For instance, our theorem presents a convergence of an explicit scheme that extends Theorem 1.4 to a more general setting of Banach spaces that includes \( L^p \) (\( 1 < p < \infty \)) spaces on one hand and for more general class of maps on the other hand.

2. Preliminaries

Let \( S := \{ x \in E : \| x \| = 1 \} \) denote the unit sphere of a real Banach space \( E \). \( E \) is said to have a Gâteaux differentiable norm if the limit
\[
\lim_{t \to 0} \frac{\| x + ty \| - \| x \|}{t}
\]
exists for each \( x, y \in S ; E \) is said to have a uniformly Gâteaux differentiable norm if for each \( y \in S \), the limit is attained uniformly for \( x \in S \). A Banach space \( E \) is said to be strictly convex if \( \| x + y \|/2 < 1 \) for \( x \neq y \) and \( \| x \| = \| y \| = 1 \).

Let \( K \) be a nonempty, closed, convex, and bounded subset of a real Banach space \( E \), and let the diameter of \( K \) be defined by \( d(K) := \sup \{ \| x - y \| : x, y \in K \} \). For each \( x \in K \), let \( r(x, K) := \sup \{ \| x - y \| : y \in K \} \) and \( r(K) := \inf \{ r(x, K) : x \in K \} \) denote the Chebyshev radius of \( K \) relative to itself. The normal structure coefficient \( N(E) \) of \( E \) (introduced in 1980 by Bynum [27], see also Lim [28] and the references contained therein) is defined by \( N(E) := \inf \{ (d(K)/r(K)) : K \text{ is a closed convex and bounded subset of } E \text{ with } d(K) > 0 \} \). A space \( E \) such that \( N(E) > 1 \) is said to have uniform normal structures. It is known that every space with a uniform normal structure is reflexive, and that all uniformly convex and uniformly smooth Banach spaces have uniform normal structure (see, e.g., [29]).

Let \( E \) be a real Banach space with uniformly Gâteaux differentiable norm, then the normalized duality mapping \( J : E \to 2^E \), defined by (1.1), is single valued and uniformly continuous from the norm topology of \( E \) to the weak* topology of \( E^* \) on each bounded subset of \( E \), see, for example [30].
Definition 2.1. Let $\mu$ be a continuous linear functional on $l^\infty$, and let $(a_0, a_1, \ldots) \in l^\infty$. We write $\mu_n(a_n)$ instead of $\mu(a_0, a_1, \ldots)$. The function $\mu$ is called a Banach limit when $\mu$ satisfies $||\mu|| = \mu_n(1) = 1$ and $\mu_n(a_{n+1}) = \mu_n(a_n)$ for each $(a_0, a_1, \ldots) \in l^\infty$.

For a Banach limit $\mu$, it is known that $\liminf_{n \to \infty} a_n \leq \mu_n(a_n) \leq \limsup_{n \to \infty} a_n$ for every $a = (a_0, a_1, \ldots) \in l^\infty$. So if $a = (a_0, a_1, \ldots) \in l^\infty$ and $a_n - b_n \to 0$ as $n \to \infty$, we have $\mu_n(a_n) = \mu_n(b_n)$.

We will make use of the following well-known result.

Lemma 2.2. Let $E$ be a real-normed linear space. Then, the following inequality holds:

$$
||x + y||^2 \leq ||x||^2 + 2(y, j(x + y)) \quad \forall x, y \in E, j(x + y) \in J(x + y).
$$

(2.2)

In the sequel, we shall also make use of the following lemmas.

Lemma 2.3 (Suzuki [31]). Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a real Banach space $E$, and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf \beta_n \leq \limsup \beta_n < 1$. Suppose that $x_{n+1} = \beta_n y_n + (1 - \beta_n) x_n$ for all integer $n \geq 1$ and $\limsup_{n \to \infty} (||y_{n+1} - y_n|| - ||x_{n+1} - x_n||) \leq 0$. Then, $\lim_{n \to \infty} ||y_n - x_n|| = 0$.

Lemma 2.4 (Shioji and Takahashi [32]). Let $(a_0, a_1, a_2, \ldots) \in l^\infty$ be such that $\mu_n a_n \leq 0$ for all Banach limits $\mu$. If $\limsup_{n \to \infty} (a_{n+1} - a_n) \leq 0$, then $\liminf_{n \to \infty} a_n \leq 0$.

Lemma 2.5 (Xu [33]). Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:

$$
a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \sigma_n + \gamma_n, \quad n \geq 0,
$$

(2.3)

where (i) $\{\alpha_n\} \subset [0, 1]$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ (ii) $\limsup_{n \to \infty} \sigma_n \leq 0$ (iii) $\gamma_n \geq 0$ and $(n \geq 0)$, $\sum_{n=0}^{\infty} \gamma_n < \infty$. Then, $a_n \to 0$ as $n \to \infty$.

3. Main Results

Theorem 3.1. Let $E$ be a real reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm, and let $\mathcal{F} = \{T(t) : t \geq 0\}$ be uniformly asymptotically regular family of generalized asymptotically nonexpansive semigroup of $E$, with functions $u, v : [0, \infty) \to [0, \infty)$ and $F := F(\mathcal{F}) = \cap_{t \geq 0} F(T(t)) \neq \emptyset$. Let $f : E \to E$ be weakly contractive, and let $G : E \to E$ be $\delta$-strongly accretive and $\lambda$-strictly pseudocontractive with $\delta + \lambda > 1$. Let $\eta := (1 - \sqrt{(1 - \delta)/\lambda})$ and $\gamma \in (0, \min\{\delta, \eta/2\})$. Let $\{\beta_n\}$ and $\{a_n\}$ be sequences in $(0, 1)$ and $\{t_n\}$ an increasing sequence in $[0, \infty)$ satisfying the following conditions:

$$
\lim_{n \to \infty} a_n = 0, \quad \lim_{n \to \infty} \frac{u(t_n)}{a_n} = 0, \quad \lim_{n \to \infty} \frac{v(t_n)}{a_n} = 0, \quad \sum_{n=1}^{\infty} a_n = \infty,
$$

(3.1)

$$
0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1, \quad \lim_{n \to \infty} t_n = \infty.
$$
Define a sequence \( \{x_n\} \) by \( x_0 \in E \),

\[
x_{n+1} = \beta_n x_n + (1 - \beta_n) y_n,
\]

\[
y_n = (I - \alpha_n G)T(t_n)x_n + \alpha_n \gamma f(x_n), \quad n \geq 0.
\]

Then, the sequence \( \{x_n\} \) converges strongly to a common fixed point of the family \( \mathcal{F} \) which solves the variational inequality

\[
\langle (G - \gamma f)q, (x - q) \rangle \geq 0, \quad \forall x \in F.
\]

**Proof.** We start by showing that solution of the variational inequality (3.3) in \( F \) is at most one. Assume that \( q, p \in F \) are solutions of the variational inequality (3.3), then

\[
\langle (G - \gamma f)p, j(q - p) \rangle \geq 0, \quad \langle (G - \gamma f)q, j(p - q) \rangle \geq 0.
\]

Adding these two inequalities, we get

\[
\langle (G - \gamma f)p - (G - \gamma f)q, j(p - q) \rangle \leq 0.
\]

Therefore,

\[
0 \geq \langle (G - \gamma f)p - (G - \gamma f)q, j(p - q) \rangle = \langle G(p) - G(q), j(p - q) \rangle - \gamma (f(p) - f(q), j(p - q))
\]

\[
\geq \delta \|p - q\|^2 - \gamma \|f(p) - f(q)\| \|p - q\|
\]

\[
\geq \delta \|p - q\|^2 + \gamma \psi(\|p - q\|) \|p - q\| - \gamma \|p - q\|^2
\]

\[
= (\delta - \gamma) \|p - q\|^2 + \gamma \psi(\|p - q\|) \|p - q\|.
\]

Since \( \delta > \gamma \), we obtain that \( p = q \), and so the solution is unique in \( F \).

Now, let \( p \in F \), since \((1 - \alpha_n \eta)(u(t_n)/\alpha_n) \to 0 \) and \((1 - \alpha_n \eta)(\nu(t_n)/\alpha_n) \to 0 \) as \( n \to \infty \), then there exists \( n_0 \in \mathbb{N} \) such that \( (1 - \alpha_n \eta)(u(t_n)/\alpha_n) < (\eta - \gamma)/2 \) and \((1 - \alpha_n \eta)(\nu(t_n)/\alpha_n) < (\eta - \gamma)/2 \) for all \( n \geq n_0 \). Hence, for \( n \geq n_0 \), we have the following:

\[
\|y_n - p\| \leq \|(I - \alpha_n G)(T(t_n)x_n - p)\| + \alpha_n \|\gamma f(x_n) - G(p)\|
\]

\[
\leq (1 - \alpha_n \eta) \|[1 + u(t_n)]\|x_n - p\| + \nu(t_n)\| + \alpha_n \|\gamma f(x_n) - f(p)\| + \alpha_n \|\gamma f(p) - G(p)\|
\]

\[
\leq [1 - \alpha_n (\eta - \gamma) + (1 - \alpha_n \eta)u(t_n)]\|x_n - p\| + (1 - \alpha_n \eta)\nu(t_n) + \alpha_n \|\gamma f(p) - G(p)\|,
\]

(3.7)
so that

\[
\|x_{n+1} - p\| \leq \beta_n \|x_n - p\| + (1 - \beta_n) \|y_n - p\|
\]

\[
\leq \left[ \beta_n + (1 - \beta_n) \left[ 1 - \alpha_n (\eta - \gamma) + (1 - \alpha_n \eta) u(t_n) \right] \right] \|x_n - p\|
\]

\[
+ (1 - \alpha_n \eta) (1 - \beta_n) v(t_n) + \alpha_n (1 - \beta_n) \|y f(p) - G(p)\|
\]

\[
\leq \left[ 1 - \alpha_n (1 - \beta_n) \left( (\eta - \gamma) - (1 - \alpha_n \eta) \frac{u(t_n)}{\alpha_n} \right) \right] \|x_n - p\|
\]

\[
+ \alpha_n (1 - \beta_n) \left[ \|y f(p) - G(p)\| + (1 - \alpha_n \eta) \frac{v(t_n)}{\alpha_n} \right]
\]

\[
\leq \left[ 1 - \alpha_n (1 - \beta_n) \left( (\eta - \gamma) - (1 - \alpha_n \eta) \frac{u(t_n)}{\alpha_n} \right) \right] \|x_n - p\|
\]

\[
+ \alpha_n (1 - \beta_n) \left( (\eta - \gamma) - (1 - \alpha_n \eta) \frac{u(t_n)}{\alpha_n} \right)
\]

\[
\times \frac{2 \|y f(p) - G(p)\| + (1 - \alpha_n \eta) \left( \frac{v(t_n)}{\alpha_n} \right)}{\eta - \gamma}
\]

\[
\leq \max \left\{ \|x_n - p\|, \frac{2 \|y f(p) - G(p)\|}{\eta - \gamma} + 1 \right\}.
\]

By induction, we have

\[
\|x_n - p\| \leq \max \left\{ \|x_0 - p\|, \frac{2 \|y f(p) - G(p)\|}{\eta - \gamma} + 1 \right\}, \forall n \geq 0.
\] (3.9)

Thus, \( \{x_n\} \) is bounded and so are \( \{T(t_n)x_n\}, \{GT(t_n)x_n\}, \{y_n\}, \) and \( \{f(x_n)\} \).

Observe that

\[
y_{n+1} - y_n = ((I - \alpha_{n+1} G)T(t_{n+1})x_{n+1} - (I - \alpha_{n+1} G)T(t_{n+1})x_n)
\]

\[
+ ((I - \alpha_{n+1} G)T(t_{n+1})x_n - (I - \alpha_n G)T(t_{n+1})x_n)
\]

\[
+ ((I - \alpha_n G)T(t_{n+1})x_n - (I - \alpha_n G)T(t_n)x_n)
\]

\[
+ (\alpha_{n+1} y f(x_{n+1}) - \alpha_{n+1} y f(x_n)) + (\alpha_{n+1} y f(x_n) - \alpha_n y f(x_n)),
\] (3.10)
so that

\[
\left\| y_{n+1} - y_n \right\| \leq (1 - \alpha_{n+1} \eta) (1 + u(t_{n+1})) \left\| x_{n+1} - x_n \right\| + (1 - \alpha_{n+1} \eta) v(t_{n+1}) + |\alpha_n - \alpha_{n+1}| \left\| GT(t_{n+1}) x_n \right\| + (1 - \alpha_n \eta) \sup_{z \in \{x_n\}} \left\| T(s + t_n) z - T(t_n) z \right\|
\]

and by Lemma 2.3,

\[
\left\| x_{n+1} - x_n \right\| \leq \left\| x_{n+1} - x_n \right\| + |\alpha_n - \alpha_{n+1}| \left\| f(x_n) \right\| + (1 - \alpha_n \eta) \sup_{z \in \{x_n\}} \left\| T(s + t_n) z - T(t_n) z \right\|
\]

which implies that

\[
\lim_{n \to \infty} \sup \left( \left\| y_{n+1} - y_n \right\| - \left\| x_{n+1} - x_n \right\| \right) \leq 0.
\]

and by Lemma 2.3,

\[
\lim_{n \to \infty} \left\| y_n - x_n \right\| = 0.
\]

Thus,

\[
\left\| x_{n+1} - x_n \right\| = (1 - \beta_n) \left\| y_n - x_n \right\| \to 0 \quad \text{as} \quad n \to \infty.
\]

Next, we show that \( \lim_{n \to \infty} \left\| y_n - T(t) y_n \right\| = 0 \), for all \( t \geq 0 \). Since

\[
\left\| x_n - T(t_n) x_n \right\| \leq \left\| x_n - x_{n+1} \right\| + \left\| x_{n+1} - T(t_n) x_n \right\|
\]

\[
\leq \left\| x_n - x_{n+1} \right\| + \beta_n \left\| x_n - T(t_n) x_n \right\| + (1 - \beta_n) \left\| y_n - T(t_n) x_n \right\|,
\]

\[
\leq \left\| x_n - x_{n+1} \right\| + \beta_n \left\| x_n - T(t_n) x_n \right\| + (1 - \beta_n) \left\| y_n - T(t_n) x_n \right\|,
\]

\[
\leq \left\| x_n - x_{n+1} \right\| + \beta_n \left\| x_n - T(t_n) x_n \right\| + (1 - \beta_n) \left\| y_n - T(t_n) x_n \right\|.
\]
we have
\[(1 - \beta_n)\|x_n - T(t_n)x_n\| \leq \|x_n - x_{n+1}\| + (1 - \beta_n)\|y_n - T(t_n)x_n\| \tag{3.17}\]
\[= \|x_n - x_{n+1}\| + \alpha_n(1 - \beta_n)\|y_f(x_n) - GT(t_n)x_n\|.
\]

From \(\alpha_n \to 0\) as \(n \to \infty\) and (3.15), we obtain
\[\lim_{n \to \infty} \|x_n - T(t_n)x_n\| = 0. \tag{3.18}\]

Also,
\[\|y_n - T(t_n)y_n\| \leq \|y_n - x_n\| + \|x_n - T(t_n)x_n\| + \|T(t_n)x_n - T(t_n)y_n\| \tag{3.19}\]
\[\leq (2 + u(t_n))\|y_n - x_n\| + \alpha_n + \|x_n - T(t_n)x_n\| \to 0 \quad \text{as} \quad n \to \infty.
\]

Since \(\lim_{n \to \infty} t_n = \infty\) and \(\{T(t) : t \geq 0\}\) is uniformly asymptotically regular,
\[\lim_{n \to \infty} \|T(t)T(t_n)x_n - T(t_n)x_n\| = \lim_{n \to \infty} \sup_{x \in C} \|T(t)T(t_n)x - T(t_n)x\| = 0,
\]
\[\lim_{n \to \infty} \|T(t)T(t_n)y_n - T(t_n)y_n\| = \lim_{n \to \infty} \sup_{y \in C} \|T(t)T(t_n)y - T(t_n)y\| = 0,
\]
where \(C\) is any bounded subset of \(E\) containing \(\{x_n\}\). Since \(\{T(t)\}\) is continuous, we get that
\[\|y_n - T(t)y_n\| \leq \|y_n - T(t_n)y_n\| + \|T(t_n)y_n - T(t)(T(t_n)y_n)\| \tag{3.21}\]
\[+ \|T(t)(T(t_n)y_n) - T(t)y_n\|.
\]

This implies that
\[\lim_{n \to \infty} \|y_n - T(t)y_n\| = 0, \quad \forall t \geq 0. \tag{3.22}\]

Next, we show that
\[\lim_{n \to \infty} \sup_{t \geq 0} \langle (\gamma f - G)p, j(y_n - p) \rangle \leq 0. \tag{3.23}\]

Define a map \(\phi : E \to \mathbb{R}\) by
\[\phi(y) := \mu_n\|y_n - y\|^2, \quad \forall y \in E. \tag{3.24}\]

Then, \(\phi(y) \to \infty\) as \(\|y\| \to \infty\), \(\phi\) is continuous and convex, so as \(E\) is reflexive, there exists \(q \in E\) such that \(\phi(q) = \min_{u \in E} \phi(u)\). Hence, the set
\[K^* := \left\{ y \in E : \phi(y) = \min_{u \in E} \phi(u) \right\} \neq \emptyset. \tag{3.25}\]
Since $\lim_{t \to \infty} \|y_n - T(t)y_n\| = 0$, $\lim_{t \to \infty} u(t) = 0$, $\lim_{t \to \infty} v(t) = 0$, and $\phi$ is continuous for all $z \in K^*$, we have

$$
\phi\left(\lim_{t \to \infty} T(t)z\right) = \lim_{t \to \infty} \phi(T(t)z) = \lim_{t \to \infty} \mu_n \|y_n - T(t)z\|^2 \\
\leq \lim_{t \to \infty} \mu_n ((1 + u(t))\|y_n - z\| + (v(t)))^2 = \mu_n \|y_n - z\|^2 = \phi(z).
$$

(3.26)

Hence, $\lim_{t \to \infty} T(t)z \in K^*$. Let $p \in F$. Since $K^*$ is a closed-convex set, there exists a unique $q \in K^*$ such that

$$
\|p - q\| = \min_{x \in K^*} \|p - x\|.
$$

(3.27)

Since $p = \lim_{t \to \infty} T(t)p$ and $\lim_{t \to \infty} T(t)q \in K^*$,

$$
\left\| p - \lim_{t \to \infty} T(t)q \right\| = \left\| \lim_{t \to \infty} T(t)p - \lim_{t \to \infty} T(t)q \right\| \\
= \lim_{t \to \infty} \left\| T(t)p - T(t)q \right\| \\
\leq \lim_{t \to \infty} ((1 + u(t))\|p - q\| + v(t)) \\
\leq \|p - q\|.
$$

(3.28)

Therefore, $\lim_{t \to \infty} T(t)q = q$. Since $T(s + h)x = T(s)T(h)x$ for all $x \in E$ and $s \geq 0$, we have

$$
q = \lim_{t \to \infty} T(t)q = \lim_{t \to \infty} T(s + t)q = \lim_{t \to \infty} T(s)T(t)q \\
= T(s) \lim_{t \to \infty} T(t)q = T(s)q.
$$

(3.29)

Therefore, $q \in F$ and so $K^* \cap F \neq \emptyset$.

Let $p \in K^* \cap F(T)$ and $\tau \in (0, 1)$. Then, it follows that $\phi(p) \leq \phi(p - \tau(G - \gamma f)p)$, and using Lemma 2.2, we obtain that

$$
\|y_n - p + \tau(G - \gamma f)p\|^2 \leq \|y_n - p\|^2 + 2\tau (\langle G - \gamma f)p, j(y_n - p + \tau(G - \gamma f)p)\),
$$

(3.30)

which implies that

$$
\mu_n (\langle jf - G) p, j(y_n - p + \tau(G - \gamma f)p)\rangle) \leq 0.
$$

(3.31)
Moreover,

\[
\mu_n(\langle (\gamma f - G)p, j(y_n - p) \rangle) = \mu_n(\langle (\gamma f - G)p, j(y_n - p) - j(y_n - p + \tau(G - \gamma f)p) \rangle) \\
+ \mu_n(\langle (\gamma f - G)p, j(y_n - p + \tau(G - \gamma f)p) \rangle) \\
\leq \mu_n(\langle (\gamma f - G)p, j(y_n - p) - j(y_n - p + \tau(G - \gamma f)p) \rangle).
\]

(3.32)

Since \( j \) is norm-to-weak* uniformly continuous on bounded subsets of \( E \), we have that

\[
\mu_n(\langle (\gamma f - G)p, j(y_n - p) \rangle) \leq 0.
\]

(3.33)

Observe that from (3.14) and (3.15), we have

\[
\lim_{n \to \infty} \|y_{n+1} - y_n\| = 0.
\]

(3.34)

This implies that

\[
\limsup_{n \to \infty} [\langle (\gamma f - G)p, j(y_n - p) \rangle - \langle (\gamma f - G)p, j(y_{n+1} - p) \rangle] \leq 0,
\]

(3.35)

and so we obtain by Lemma 2.4 that

\[
\limsup_{n \to \infty} \langle (\gamma f - G)p, j(y_n - p) \rangle \leq 0.
\]

(3.36)

Finally, we show that \( x_n \to p \) as \( n \to \infty \). Since \( \lim_{n \to \infty} (u(t_n)/\alpha_n) = 0 \), if we denote by \( \sigma(t_n) \) the value \( 2u(t_n) + u(t_n)^2 \), then we clearly have \( \lim_{n \to \infty} (\sigma(t_n)/\alpha_n) = 0 \). Let \( N_0 \in \mathbb{N} \) be large enough such that \( (1 - \alpha_n\eta)(\sigma(t_n)/\alpha_n) < (\eta - 2\gamma)/2 \), for all \( n \geq N_0 \), and let \( M \) be
a positive real number such that $\|x_n - p\| \leq M$ for all $n \geq 0$. Then, using the recursion formula (3.2) and for $n \geq N_0$, we have

$$
\|y_n - p\|^2 = \|a_n(\gamma f(x_n) - G(p)) + (I - a_n G)(T(t_n)x_n - p)\|^2
$$

$$
\leq (1 - \alpha_n \eta) \|T(t_n)x_n - p\|^2 + 2 \alpha_n \gamma f(x_n) - G(p), j(y_n - p)\rangle
$$

$$
\leq (1 - \alpha_n \eta) [1 + u(t_n)] \|x_n - p\|^2 + v(t_n)\|^2
$$

$$
+ 2 \alpha_n \gamma f(x_n) - G(p), j(y_n - p)\rangle - 2 \alpha_n \gamma \|y_n - p\| \psi(\|x_n - p\|)
$$

$$
+ 2 \alpha_n \gamma \|y_n - x_n\| \|x_n - p\|
$$

$$
\leq [(1 - \alpha_n \eta)(1 + \alpha(t_n)) + 2 \alpha_n \gamma] \|x_n - p\|^2
$$

$$
+ \alpha_n [2(\gamma f(p) - G(p), j(y_n - p)) + 2(1 - \alpha_n \eta)(1 + u(t_n)) \frac{v(t_n)}{\alpha_n} \|x_n - p\|^2
$$

$$
+ (1 - \alpha_n \eta) \frac{v(t_n)^2}{\alpha_n} + 2 \gamma \|y_n - x_n\| \|x_n - p\|]
$$

$$
= [1 - \alpha_n \left(\eta - 2 \gamma - (1 - \alpha_n \eta) \frac{\sigma_n}{\alpha_n}\right)] \|x_n - p\|^2
$$

$$
+ \alpha_n [2(\gamma f(p) - G(p), j(y_n - p)) + 2(1 - \alpha_n \eta)(1 + u(t_n)) \frac{v(t_n)}{\alpha_n} \|x_n - p\|^2
$$

$$
+ (1 - \alpha_n \eta) \frac{v(t_n)^2}{\alpha_n} + 2 \gamma \|y_n - x_n\| \|x_n - p\|]
$$

so that

$$
\|x_{n+1} - p\|^2 \leq \beta_n \|x_n - p\|^2 + \alpha_n \|y_n - p\|^2
$$

$$
\leq \left(\beta_n + \alpha_n \left[(1 - \beta_n) \left[1 - \alpha_n \left(\eta - 2 \gamma - (1 - \alpha_n \eta) \frac{\sigma_n}{\alpha_n}\right)\right]\right]\right) \|x_n - p\|^2
$$

$$
+ \alpha_n \|y_n - p\| \|x_n - p\| + \frac{v(t_n)}{\alpha_n} \|x_n - p\|^2 + (1 - \alpha_n \eta) \frac{v(t_n)^2}{\alpha_n} + 2 \gamma \|y_n - x_n\| \|x_n - p\|
$$
Let Corollary 3.2. strongly to a common fixed point of the family \( J \)
where \( A \)
converges strongly to a common fixed point of the family \( J \).

Let /parenleftmath\( 3.3 \)\( /parenrightmath
\( J \)
\( F, f, G, \delta, \lambda, \eta, \gamma, \beta_n, \alpha_n, t_n \)
be as in Theorem 3.1. Then, the sequence \( \{ x_n \} \)

\[
\begin{align*}
\leq & \left[ 1 - \alpha_n(1 - \beta_n) \left( (\eta - 2\gamma) - (1 - \alpha_n \eta) \frac{\alpha_n}{\alpha_n} \right) \right] \| x_n - p \|^2 \\
& + \alpha_n(1 - \beta_n) \left[ 2\gamma f(p) - G(p), j(y_n - p) + 2(1 - \alpha_n \eta)(1 + u(t_n)) \frac{\nu(t_n)}{\alpha_n} M^2 \\
& + (1 - \alpha_n \eta) \frac{\nu(t_n)^2}{\alpha_n} + 2\gamma \| y_n - x_n \| M \right]
\end{align*}
\]

\[
= \left[ 1 - \alpha_n(1 - \beta_n) \left( (\eta - 2\gamma) - (1 - \alpha_n \eta) \frac{\alpha_n}{\alpha_n} \right) \right] \| x_n - p \|^2 \\
+ \alpha_n(1 - \beta_n) \left( (\eta - 2\gamma) - (1 - \alpha_n \eta) \frac{\alpha_n}{\alpha_n} \right) \\
\times \left[ 2\gamma f(p) - G(p), j(y_n - p) + 2(1 - \alpha_n \eta)(1 + u(t_n)) \frac{\nu(t_n)}{\alpha_n} M^2 + \mathcal{A}_n \right],
\]

(3.38)

where \( \mathcal{A}_n \) denotes \( (1 - \alpha_n \eta) \frac{\nu(t_n)^2}{\alpha_n} + 2\gamma \| y_n - x_n \| M \).

Observe that \( \sum_{n=1}^{\infty} \alpha_n(1 - \beta_n)((\eta - 2\gamma) - (1 - \alpha_n \eta)(\sigma_n/\alpha_n)) = \infty \) and

\[
\limsup_{n \to \infty} \left( 2\gamma f(p) - G(p), j(y_n - p) + 2(1 - \alpha_n \eta)(1 + u(t_n)) \frac{\nu(t_n)}{\alpha_n} M^2 + \mathcal{A}_n \right) \leq 0.
\]

(3.39)

Applying Lemma 2.5, we obtain \( \| x_n - p \| \to 0 \) as \( n \to \infty \). This completes the proof. \( \square \)

The following corollaries follow from Theorem 3.1.

**Corollary 3.2.** Let \( E \) be a real uniformly convex and uniformly smooth Banach space, \( \mathcal{J} = \{ T(t) : t \geq 0 \} \), and let \( F, f, G, \delta, \lambda, \eta, \gamma, \beta_n, \alpha_n, \{ t_n \} \) and \( \{ x_n \} \) be as in Theorem 3.1. Then, the sequence \( \{ x_n \} \)
converges strongly to a common fixed point of the family \( \mathcal{J} \) which solves the variational inequality (3.3).

**Corollary 3.3.** Let \( E = H \) be a real Hilbert space, and let \( \mathcal{J} = \{ T(t) : t \geq 0 \} \),
\( F, f, G, \delta, \lambda, \eta, \gamma, \beta_n, \alpha_n, \{ t_n \} \) and \( \{ x_n \} \) be as in Theorem 3.1. Then, the sequence \( \{ x_n \} \)
converges strongly to a common fixed point of the family \( \mathcal{J} \) which solves the variational inequality

\[
\langle (G - \gamma f)q, x - q \rangle \geq 0, \quad \forall x \in F.
\]

(3.40)

**Corollary 3.4.** Let \( \mathcal{J} = \{ T(t) : t \geq 0 \} \) be a family of nonexpansive semigroup of a real reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm \( E \), and let \( F, f, G, \delta, \lambda, \eta, \gamma, \beta_n, \alpha_n, \{ t_n \} \) and \( \{ x_n \} \) be as in Theorem 3.1. Then, the sequence \( \{ x_n \} \)
converges strongly to a common fixed point of the family \( \mathcal{J} \) which solves the variational inequality (3.3).
Acknowledgment

The authors thank the anonymous referees for useful comments and observations, that helped in improving the presentation of this paper.

References


Submit your manuscripts at http://www.hindawi.com