Positives About Negatives: A Case Study of an Intermediate Model for Signed Numbers

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In this article, we present a study using an intermediate abstraction as a model for the acquisition of the concept of negative numbers. The intermediate abstraction is a computerized environment based on a detailed epistemological analysis of negative numbers. Four children participated in activities with the intermediate abstraction during eleven 30-min training sessions. This article outlines the development of the children’s representations of negative numbers during the experiment. We analyzed how students used their representations as problem models in transfer tasks with several different referents. The results obtained in the experiment support the use of certain environments for the acquisition of higher level mathematical concepts that cannot be learned informally.

1. INTRODUCTION

Resnick and Greeno (Greeno, 1991; Resnick, 1992; Resnick & Greeno, 1990) articulated a theory which posits that important segments of mathematical knowledge have origins in everyday experience with quantities of physical material and talk about that material. They hypothesized that children start with patterns of cognitive activity in which they compare and reason about changes, combinations, and decompositions of amounts of physical material without exact quantification. For example, children reason that adding more blocks to a pile increases the total...
amount of blocks in a pile, or they recognize that moving children from one car to another at the beginning of a class trip does not change the total number of children going on the trip. This reasoning has been referred to as qualitative (see Forbus, 1985), or protoquantitative (Resnick & Greeno, 1990). Greeno (1991) referred to this type of reasoning as reasoning with mental models or constructing a representation with objects that correspond to the objects in a situation. A crucial feature of this model is that operations on the mental objects are similar to operations on the objects that may be in the situation (Holland, Holyoak, Nisbett, & Thagard, 1986; Johnson-Laird, 1983). Using protoquantitative schemata and the ability to reason with mental models, young unschooled children produce schemata such as cardinality of sets, quantified changes in sets, and quantified part–whole relations between sets.

Although children master these protoquantitative forms of reasoning, they also learn to quantify sets of objects by applying the rules of counting. At first, counting knowledge and protoquantitative reasoning remain separate. As children start to apply counting knowledge to protoquantitative reasoning situations, however, number names come to refer to specific counted or measured amounts of material. Resnick and Greeno (1990) referred to this as the quantitative stage of mathematical thinking. With extended practice, numbers begin to take on a life of their own (e.g., mental objects can be compared, increased and decreased, composed and decomposed). Later, children begin to treat the relations between numbers as objects to be reasoned about.

Resnick (1991) gave a detailed account of how a certain class of mathematical principles—including commutativity, associativity, additive inverse, and equivalence class—are derived directly from early protoquantitative reasoning and children’s growing understanding of how numbers can be used to quantify sets. This route to mathematical understanding, however, seems limited to those concepts that can be directly mapped to physical material. Positive numbers used to quantify extensive amounts of material (e.g., number of items in a set, length in feet or meters, and volume) can be learned in this way. Concepts such as negative numbers, however, may be much harder to understand in this way because negative amounts do not appear in the everyday physical world. In addition, concepts that depend on the relations between two measured amounts (e.g., rates [mph] or density [molecules per unit volume]) would be hard to learn from everyday experience alone because these quantities cannot be directly mapped to observable physical materials.

Thus, it appears that the development of higher level mathematical concepts is not likely to occur informally. This is because the situations required for their development are not part of most children’s everyday lives (Davidov & Markova, 1983; Vygotsky, 1978). As Scribner and Cole (1981) pointed out, helping students develop forms of reasoning that do not occur in everyday life may be the primary raison d’être of institutionalized schooling. However, it is recognized that purely formal methods of instruction, which do not make an effort to understand an individual student’s everyday experiences, also do not understand the majority of students.
In an effort to bridge everyday mathematics with formal mathematics, many mathematics educators (e.g., Dienes, 1963; Freudenthal, 1983; Skemp, 1981) proposed heavy reliance on manipulatives and graphical displays that allowed children to discover relations among quantities and operations on them. The general idea was to introduce formal notations only as a record of already understood relations. Ohlsson (1987) found several reasons to doubt the pedagogical efficiency of graphical displays if the focus was only on the learning of procedures (see also Bell, Costello, & Kuchemann, 1986; Ohlsson, Bee, & Zeller, 1989; Resnick & Omanson, 1987). If a display is strictly isomorphic to the knowledge to be learned, there may be no reason to believe that manipulating the display will be easier than manipulating the formal symbols. However, if the display is not isomorphic, the problem of transfer emerges. Ohlsson concluded that illustrations or manipulatives are helpful only if the activities in which learners are involved focus on mathematical entities rather than on the activity of computing. Because protoquantitative and quantitative reasoning are learned by developing a language that describes actions on real-world objects, the primary function of displays is to provide referential semantics (i.e., a language to describe the objects of the display and their behavior).

Nesher (1989) treated the same problem from a slightly different perspective. She designated displays, manipulatives, actions on these objects, and activities with them as learning systems (LSs). Nesher also elaborated the requirements for an LS designed to facilitate students’ acquisition of mental models, which would be appropriate for formal domains such as number. At the heart of an LS is an exemplification component, a system of manipulable objects and actions that can be mapped one to one onto the elements of the concept to be learned. It is a formal system in its own right, although limited in scope and expressed via physical objects and actions rather than in formal symbols. At the same time, the objects and actions in the exemplification system must make sense to learners. This means that their behavior must be interpretable and predictable by learners on the basis of their prior experience. This is what Greeno (1991), drawing on the language of Gibson (1966), described as affordances of a situation (i.e., the immediately perceivable properties that make a situation meaningful and navigable). Learning happens when learners are able to explain the behavior of the objects of the LS in their own words and map the exemplification objects and actions to formal notations. Nesher assumed that when working with an LS, learners would: (a) manipulate the objects of the exemplification component, (b) develop an exemplification language (i.e., a systematic way of talking about the behavior of the objects in particular configurations and under certain manipulations), (c) come to use a mathematical language gradually (i.e., language about numbers, operators, and general rules that apply to them rather than descriptions of specific displays), and (d) eventually apply the mathematical language to situations and configurations they have never seen.

Educators attempting to construct LSs confront a dilemma because most high-level concepts cannot be illustrated by models relying on everyday experience: Physical displays will not enable the learner to develop referential semantics or use a mathematical language applicable to all situations. Computer systems offer a way to break out of this dilemma. Using appropriate epistemological analysis, it is
possible to create graphical computer models that capture learners' informal knowledge about the physical world without all the real-world constraints. It is possible to create a hypothetical world that contains recognizable but fictional entities that behave according to highly structured rules. This approach was used by White (in press), who constructed intermediate causal models for concepts in physics. White claimed that intermediate causal models served as "conceptual eyeglasses" that unpack the causal mechanism implicit in abstractions such as $F = ma$. These are readily mappable to a variety of real-world contexts because their objects and operators are generic and causal. She detailed the properties of successful intermediate abstractions and traced the development of students' mental models as they worked through an LS about electrical circuits. Similarly, Roschelle (1992) analyzed collaborative learning with an intermediate model in Newtonian mechanics and pointed out that it comprises the display of deep structures to be acquired and a metaphoric language (to be compared to affordances) drawn from the behavior of the system. Similar approaches have been incorporated into computerized teaching models developed by Snir, Smith, and Grosslight (1988) for density and by Ohlsson (1987) for rational numbers. Later in this article we describe such an approach for one of the earliest taught mathematical concepts that cannot be learned from everyday physical experience: negative numbers and operations on them. The first step in the construction of a referential semantics (or a intermediate abstraction) for negative numbers is an epistemological analysis of this concept.

2. THE NATURE OF KNOWLEDGE ABOUT NEGATIVE NUMBERS

The history of negative numbers shows an interesting tension between everyday and formal needs. The concept has its origin in the Far East: The first mathematicians who manipulated negative numbers were from China and India (see Kline, 1972). The 7th century Hindu mathematician Brahmagupta introduced negative numbers to represent debts. (In such situations, positive numbers represented assets.) He also formulated the rules for the four basic operations on negative numbers. However, Hindu mathematicians did not unreservedly accept negative numbers as objects in their own right. For example, the 12th century mathematician Bhaskara, when giving 50 and −5 as two solutions for a problem, wrote, "The second value is in this case not to be taken, for it is inadequate; people do not approve of negative solutions." In another example, Bhaskara, despite his refusal to accept negative solutions, accepted a negative value found for a segment of straight line (an entity that is positive by definition) and associated it with a change of direction (Sesiano, 1985). In practice, negatives were taken to denote debts or segments with a change of orientation, although their existence as mathematical objects was denied.

In Europe, negative numbers arose around the year 1500 as necessary intermediate tools in the manipulation and solution of equations. In his book on algebra,
The Great Art, Cardan called negative numbers “debitum.” Positive numbers were referred to as “true solutions” to equations, whereas negative numbers were defined as “fictitious solutions.” Although Cardan fully accepted negative numbers in mathematical operations and was prepared to treat them as normal for the sake of computations and calculation, he did not take the modern view that negative numbers can represent nonfictitious phenomena. Descartes used negative numbers to give a complete representation of geometrical curves. Such curves are truncated if one is restrained to positive values on the Cartesian plane. Descartes used this extension to utilize geometric methods in solving algebraic problems. Because geometric solutions are independent of coordinates, he was forced to use negatives, although he considered negative coordinates to be meaningless (see Kline, 1972). It was only in the 19th century that negative numbers emerged as directed magnitudes (e.g., in the domain of electricity) and that the set of integers was axiomatically defined in such a way as to give negatives a symmetric status to that of positives.

This brief consideration of the historical development of negative numbers shows that negatives and some operations on them were grounded very early on in everyday situations (debts, direction) and were also required by professional mathematicians to form a coherent system for algebraic and geometric manipulations. However, the extension of positive numbers to a new class that includes negative numbers resisted these two needs until the 19th century. This suggests a substantial epistemological obstacle (cf. Bachelard, 1980) to the construction of negative numbers as true mental objects. The problem for instruction is to overcome this epistemological obstacle and to help students construct a coherent model for negative numbers. This model should be capable of closing the formal system and should at the same time allow reasoning about physical or scientific objects that have negative values. It should permit construction of negative numbers by extending one’s prior understanding of positive numbers (i.e., arithmetic). Understanding arithmetic involves many aspects: the senses of arithmetic concepts (e.g., different senses of number include “number as position” and “number as action”), arithmetic language, procedures, principles, and laws. To understand negatives, each of these aspects must be extended. For example, the educational literature recognized different senses to the ‘+’ sign (Carpenter, Moser, & Romberg, 1982; Nesher & Katriel, 1977; Vergnaud, 1982): static (combine), dynamic (change), or comparing. Therefore, each of these senses needed to be extended to negatives. Overall, we were able to define four types of extension (see Freudenthal, 1983, for a related analysis).

1. Extension of the number to refer to a set of (imaginary) quantities identical in magnitude but opposite in state (i.e., direction or value) to ordinary physical

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1 This term can be compared with surdum (Latin for deaf), which designated irrational numbers, and to imaginary, which still designates complex numbers. Thus, negative, irrational, and complex numbers were used by mathematicians for certain practical reasons long before they were accepted as entities in their own right.
quantities. The mathematical term for the structure containing both positive and negative numbers is a system of *directed magnitudes*.

2. Extension of the domain of application for binary operations to include operations on pairs of positives or negatives and on mixed negative-positive pairs.

3. Extension of unary operations to include changes (i.e., addition and subtraction) on both positive and negative numbers.

4. Extension of the relation of order to a continuous number line that includes both negatives and positives: for example, \((-3) < (-2) < (0) < (+1)\).

Further clarifications of these different senses of extension are presented later. These are referred to as R1, R2, R3, and R4 and are described in general terms that follow (see specific rules in Appendix A):

R1: Directed magnitudes are objects that have a magnitude and two possible states (two directions, two possible colors, and charges). They can be represented by signed numbers of the form \(+a or -a\), where \(a\) is the magnitude of the object, and \((+/-)\) indicates one of the two states.\(^2\) The sense of directed magnitude confers to negative numbers a status of mathematical object.

R2: Binary addition and subtraction can be extended to the set of directed magnitudes. In other words, it is possible to combine or separate directed magnitudes. Such actions lead to numerous rules (prototypic examples of these rules are given in the right-hand column of Appendix A).

Here addition and subtraction are referred to as binary operations because they have two inputs and one output.

R3: Addition can be interpreted as an unary operator. That is, addition changes the value of a given directed magnitude. The input to an unary operation is a single directed magnitude, and the output is a new magnitude that was augmented by the operator addition. For example, the addition \(+3\) transforms \(+4\) into \(+7\), and \(-2\) into \(+1\); in this respect, the sense of addition is that of an action. With this new definition of addition, it is possible to reformulate the previous rules by replacing \(+/-b\) (directed magnitude) by \(+/-b\) (action). For example, the Rule r2,4 can be modified to:

\[-a + (-b) = -(a + b) \text{ (Rule r3,4).}\]

This rule would be interpreted as follows: If the number \(-a\) is augmented by \((-b)\), the result will be the number \(-(a + b)\).

\(^2\)The notation chosen here was not used in our LS; it is used here to distinguish between the different senses of negatives. \('+a\' designates a directed magnitude; the \('+\' in an expression such as \('+a + -b\)' designates a combination of two directed magnitudes; and \('+(-\' in an expression such as \(+a + (-b)\) designates a change operated on the directed magnitude \(+a\).
Subtraction can also be interpreted as an unary operator: The action \(-(a)\) is the inverse of the action \(+ (a)\). These modified rules will be referred to as \(r_{3,5}; r_{3,6}; r_{3,7}\) and \(r_{3,8}\). For example, \(r_{3,7}\) would be:

\[+a -(-b) = +(a + b)\]

Such a rule could express that the operation \(- -\) is identical to the operation \(+ +\) because the result of these two actions on any number would be identical. Similarly, other rules would express that the operation \(−+ \) is identical to the operation \(+−\).

R4: Unary addition defined on all directed magnitudes induces a relation of order: A magnitude \(x\) will be called larger than a magnitude \(y\) (i.e., \(x > y\)) if there exists an (unary) addition \(+a\) (with \(a\) positive) that transforms \(y\) in \(x\) (i.e., such that \(x = y + a\)); \(a\) expresses the difference between the two magnitudes (see Appendix A for the rules induced).

In summary, \(R_1\) expresses the sense of negatives as a new sort of quantity. \(R_2\) extends binary addition and subtraction to negatives; it expresses the senses of combination or partition. \(R_3\) details how numbers vary when an addition or a subtraction is operated on them; the sense of positives and negatives here is that of actions. \(R_4\) expresses a relation of order; positives and negatives are positions on a full number line.

3. INSTRUCTIONAL REPRESENTATIONS AND THEIR LIMITS

The preceding provides the terminology for an analysis of instructional representations that have been used to teach negative numbers. A limited number of exemplifications have been used by mathematics educators. Each captures some of the important aspects of the four senses discussed here, but none is able to comfortably encompass them all. Moreover, the knowledge acquired with these models is often tied to particular representations, limiting their applicability in novel situations.

Debts

Negative numbers can be represented as owed quantities. These quantities may be considered fictitious, but money or marbles owed to another can also be thought of as concrete objects that will change hands at some point in the future. A system of debts and assets seems to be particularly natural for modeling \(R_3\), which are rules that involve addition and subtraction as actions (adding on and
taking away). The naturalness of a system of debts and assets becomes somewhat strained when it is used as a base for teaching other rules. R1, rules based on two types of quantities, may not be clearly mapped for all children: Owing or having an object is not a property of that object but rather the result of an action (giving or receiving). Therefore, the two states owing and having are not intrinsic properties of the objects and cannot well exemplify two different states of a quantity. The epistemological obstacles connected with using negatives to deal with debts (although not considering them true numbers) indicates that the link of debts to mathematical objects is not obvious, although even young children appear able to use negative numbers to describe debts (see Peled, Mukhopadhyay, & Resnick, 1988). R2, binary operations, may also not be easily mapped. Again, children may not be able to think of the combination of two opposite quantities if they lack the understanding of two distinct types of quantities. R4 may be easily understood within a model of owing. Children may understand negative positions and a metric within a scale ranging from having a lot to owing a lot. Moving within such an ordering may also be simple. For example, “If Anne owes five dollars, and Theresa has twelve dollars, how much richer is Theresa?”

Davis (1967) developed an unusual story line based on debts to convey a more complete model of negatives. The story involved a postman who delivered checks and bills to addresses at random. An obsessive Mrs. Housewife was also part of the story. Each day, after the mail was delivered, Mrs. Housewife figured out exactly how much money she now had or owed. R1 is naturally conveyed here by means of two different kinds of quantities (i.e., bills and checks). R2 binary addition may be exemplified when Mrs. Housewife takes her mail and adds it up; likewise, R3 may be shown as her running balance that changed by each new check or bill. R4, an ordering and a metric, may also be easily grasped, as previously discussed under a model of debts. This postman model has a clever way of dealing with the difficult problem of making subtraction of negatives understandable. The postman eventually comes back to take away the wrong mail, thereby removing an erroneous bill. Therefore, Mrs. Housewife subtracts her earlier subtraction. This model may prove difficult because the story relies on people who act in quite odd (not to mention sex-stereotyped) ways. Some children would be distracted by the fact that the real world does not work in such a “Kafka-esque” way.

Cancellation Models (Debts With a More Physical Representation)

Freudenthal (1983) referred to a possible model of teaching negative numbers that involved a system of checkers. According to this model, white and black checkers may be used to represent two different types of quantities (to instantiate the R1
rules). Binary addition may be represented by a simple rule whereby checkers of different colors annihilate each other. However, subtracting negatives is a bit more complex because pairs of checkers need to be formed artificially so that the new amount can be generated. For example, the value of the expression $3 - (-4)$ may be exemplified by using a group of three black checkers to represent the positive quantity and then by forming four pairs, each consisting of a black and a white checker (as a first step in instantiating the negative quantity to be taken away). Finally, the four white artificially generated checkers are removed. $R_3$ may be represented by adding or removing some checkers from an initial amount. As with the formal system (discussed later), the level of rules to be memorized becomes quite taxing in dealing with difficult problems. The relation of order ($R_4$) induced by this model is that of two distinct number lines without a natural link between them.

Elevators (or Distances)

These are models mappable to the full number line with actions allowing passage from one point to another. In a system using elevators, numbers may be used to represent both positions (i.e., the third floor) and actions (i.e., going up three floors) but not truly to represent quantities. Therefore, the $R_1$ rules may be more difficult to grasp. Floors below ground are a different type of position (in reference to a standard) but not necessarily a different type of quantity. Binary operations are meaningless (i.e., two positions cannot be combined). However, actions ($R_3$ rules) going up versus down would be easily distinguished. By definition, this model is adequate for a full number line sense of negatives ($R_4$).

The microworld "Integers" (Thompson & Dreyfus, 1988) is an example of another elevator model. By using "Logo-like" programming, children can plan simple or complex moves of a "turtle" on a number line. The system, therefore, exemplifies $R_3$ rules very well (especially rules of the kind $a - (-b) = a + b$) and $R_4$ but not $R_1$ or $R_2$. In their test of the model, Thompson and Dreyfus reported that children had difficulty learning formal rules.

To use an elevator model in which numbers represent quantities, Janvier (1985) described a test of a model of negative numbers involving hot air balloons developed by Luth (1967). Sand bags and helium balloons may be attached to the hot air balloons to move them lower and higher, respectively. $R_1$, two sorts of quantities, appears to be well represented in this model by the two kinds of baggage. $R_2$, binary addition and subtraction, seems quite difficult to imagine. Binary addition seems to involve somehow adding two different hot air balloons. $R_3$, unary actions, appears simple in this case, children readily grasp the effect of one sandbag or one helium balloon on a hot air balloon. $R_4$, an ordering and a metric, also seems very hard to imagine in this model. Children may have to imagine a series of many balloons, each at a different level, to grasp such rules. Moreover, like Davis’s postman, this model seems artificial and therefore difficult to use in novel situations.
Time
A model using time, with a scale from B.C. to A.D., may also be used in an effort to convey information about negatives. Such a model may have similar limitations in instantiating the relevant set of rules as the models previously discussed. R1 may be difficult to understand for the same reasons as the elevator model. R2 is not possible at all: Dates cannot be added together or subtracted. Most of the rules of R3 may be demonstrated by adding and subtracting durations to and from dates. R4 may also be naturally understood in terms of date and duration. However, time is a difficult concept for most young children; it is not easily manipulated in everyday experience.

Temperature
A model using temperature to convey rules about negative numbers has particular advantages and disadvantages. On the one hand, children may be familiar with the terminology “degrees below zero.” This may be an advantage for conveying senses R1 and R4. Children may naturally think of negative degrees as a different type of quantity and have a mental image, such as a thermometer, to represent an ordering and a metric. On the other hand, temperatures are intensive measurements, and therefore, R2 or binary operations do not make sense. For example, combining some amount of liquid at $-15^\circ$ with the same amount of liquid at $+20^\circ$ does not yield $+5^\circ$ liquid. R4, ordering and metric rules, incur similar problems: The $0$ on a thermometer does not refer to a lack of degrees; $+10^\circ$ and $-10^\circ$ do not clearly refer to temperatures equally distant from $0^\circ$. Therefore, activities with temperatures as referents may become extremely formal in nature.

Formal Models
A formal model of teaching negatives relies on the manipulation of symbols. Minus signs are used to distinguish a new and different type of quantity. Rules are taught that children must learn and memorize because they cannot directly check them. For example, they cannot verify that subtracting a negative is an equivalent operation to adding. Children are expected to learn rules that they cannot yet fully instantiate or grasp. All rules previously discussed (R1, R2, R3, and R4) are supported within this model, but formal proofs are required to truly understand them, which provides a test of faith for children. New math programs have attempted to justify such rules. A typical example of an activity using such an approach is the following:

If we designate $(-a)$ to be the unique solution of the equation $a + x = 0$, then we can prove from that premise that $(-a) + (-b) = -(a + b)$:

<table>
<thead>
<tr>
<th>Premises</th>
<th>Inferences</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a + (-a) = 0$ and $b + (-b) = 0$,</td>
<td>$a + (-a) + b + (-b) = 0$.</td>
</tr>
</tbody>
</table>
The addition is commutative, 
\[ a + b + (-a) + (-b) = 0. \]
\[ (-a) + (-b) \text{ is a solution of} \]
\[ a + b + x = 0 \]
The solution is unique and defined as 
\[ -(a + b), \]
\[ -(a + b) = (-a) + (-b) \text{ holds.} \]

Typically, such proofs uncover properties of negatives that are used to preserve coherence in algebraic manipulations. However, the nature of the new numbers may not be meaningful to students but merely a new syntax. Proofs are often seen as the teacher’s invention (Léonard & Sackur-Grisvard, 1987).

A Learning System for Negative Numbers

Except for the formal model, the traditional models for teaching negatives are not complete; none of them affords the discovery of all the senses and rules concerning negatives. Table 1 outlines the rules that each of these models exemplify. The models have another drawback: Even if students manage to solve tasks when reasoning about sand bags or helium balloons, the knowledge may remain bound to that context. The models just discussed are incomplete, require tools that have a very abstract degree of sophistication, or rely on a context that makes it difficult to apply negative number knowledge to other situations.

In this article, we present an intermediate model about negatives that embodies the senses of the operations on them: negatives as directed magnitudes (R₁), the combination and partition of directed magnitudes (R₂), negatives and positives as actions (R₃), and the comparison of directed magnitudes (R₄). We propose the study of the development of the negative numbers concept for students manipulating the objects of the intermediate abstraction. The guidance provided to the students is minimal; learners

<table>
<thead>
<tr>
<th>Models of Negatives</th>
<th>Directed Magnitudes (R₁)</th>
<th>Binary Operations (R₂)</th>
<th>Unary Operations (R₃)</th>
<th>Order &amp; Metric (R₄)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Debts/assets</td>
<td>No</td>
<td>Not easily for most rules</td>
<td>Mostly yes, no for r₃,₇ and r₃,₈</td>
<td>Yes</td>
</tr>
<tr>
<td>Cancellation</td>
<td>Yes</td>
<td>Not easily for most rules</td>
<td>Mostly yes, no for r₃,₇ and r₃,₈</td>
<td>No</td>
</tr>
<tr>
<td>Elevators</td>
<td>No</td>
<td>Not easily for most rules</td>
<td>Mostly yes, no for r₃,₇ and r₃,₈</td>
<td>Yes</td>
</tr>
<tr>
<td>Time</td>
<td>No</td>
<td>No</td>
<td>Mostly yes, no for r₃,₇ and r₃,₈</td>
<td>Yes (though abstract)</td>
</tr>
<tr>
<td>Temperature</td>
<td>No</td>
<td>No</td>
<td>Mostly yes, no for r₃,₇ and r₃,₈</td>
<td>Yes for order, no for metric</td>
</tr>
</tbody>
</table>

TABLE 1
The Rules Exemplified by Traditional Models of Negatives
are presented problems that constrain their exploration, which hopefully leads them to discover particular properties of the objects and actions that are essential to the concept of negatives.

4. DESCRIPTION OF THE LS

The LS under study is an extension of a computerized environment, Trainworld, which was designed to map positive numbers onto situations involving additive and multiplicative structures (Peled & Resnick, 1987). This system was remodeled and enhanced to build a new system—the Planner. The exemplification component is referred to as Trainworld. The system as a whole is referred to as the Planner, including the full set of activities organized around the objects of Trainworld.

4.1. The Exemplification Component

Trainworld is a computerized environment that graphically represents numbers and operations by means of a system of trains and machines that operate dynamically on trains. Trainworld runs on Macintosh microcomputers.

*Trains as representations of numbers (R1).* We represent numbers by means of trains with different lengths. Each train has a “smokestack” on which the train’s length is written (see Figure 1). Trains that represent positive quantities are gray; trains that represent negative quantities are white; and “smokestacks” are oriented to the left, which alludes obliquely to the respective positions of negative and positive numbers on the number line. Trains are generated by using a build machine. A student clicks on the machine with a mouse, drags it to the right or to the left, and releases it when the desired size train is reached. Therefore, trains are magnitudes with two possible states (white and gray), and they exemplify R1 (i.e., the existence of directed magnitudes). The simple action of creating trains did not refer explicitly to any process that students were familiar with. However, it made perfect sense to students. In fact, students easily made sense of all the machines of the intermediate abstraction: The system’s behavior afforded

![FIGURE 1 Creating trains: An exemplification of directed magnitudes (R1).](image-url)
FIGURE 2 Gluing trains: An exemplification of binary addition (R2).
the various senses of extending the number system to include negatives. For example, in the intermediate abstraction, two trains of opposite signs annihilate or “eat” each other (according to children’s explanations) when they are put together (see the description of the gluing machine that follows).

The cutting and gluing machines as representations of binary addition and subtraction (R2). The cutting and gluing machines embody binary addition and subtraction of two directed magnitudes (R2). They allow trains to be combined or partitioned. The gluing machine takes two input trains and glues them together to form an output train, which is their sum. For example, a +3 train and a +2 train enter the machine from its two sides, and a 5 train drops out of the machine onto the bottom track (see Figure 2a).

Similarly, in Figure 2b, a −3 and a −2 train glued together produce a −5 train (Rule r2,4). When gluing gray and white trains (Rule r2,3), the operation is done dynamically by progressive neutralization (i.e., the trains “eat each other” or shrink toward the center line [Figure 2c]). To use the cutting machine, a train is put onto the bottom track. A fixed segment that represents a knife allows children to decide how they want to cut the train. For example, Figure 3 shows how a 9 train is cut into a +3 and a +6 train (Rule r2,5). Similarly, a −9 train may be cut in a −3 and a −6 train (Rule r2,6). The name cutting machine refers to only a simple description of its functioning. A more accurate designation would have been “Partition machine.” However, the term cutting is certainly more accessible and more appealing to children. An interesting (and puzzling) situation arises when the knife is placed outside of the train. In this case, the cut produces two trains, one positive and one negative by means of the progressive expansion of two opposite trains around the knife, until one of them reaches the input train. For example, Figure 4 illustrates the equality: 6 = 9 + (−3) induced by the Rule r2,5. For the rules involving subtraction the exemplification is indirect: The system does not show explicitly that −9 − (−3) = −6 or 6 − (−3) = 9 but rather that −9 = (−3) + (−6) and 6 = 9 + (−3). However, subtraction is embedded in a single machine that has cutting as its sole action. This contrasts with postman-like or cancellation models in which subtracting negatives first requires the creation of a bill and a

FIGURE 3 Cutting trains: An exemplification of binary subtraction (R2—here r2,5).
FIGURE 4 Cutting "outside": An exemplification of $t_2$. 
FIGURE 5  Loading and unloading: An exemplification of unary addition ($R_3$).
FIGURE 6  Exemplifying the composition of unary operators (R₃—here r₃,9).
check of the same amount or the creation of the same number of checkers of
different colors. Subtracting positives and negatives within those models requires
different actions. The load and unload machines as representations of unary
addition and subtraction (R3). The load and unload machines embody unary
addition and subtraction as actions on directed quantities (Rules R3). For example,
Figure 5 displays a +6 train entering a [−3] unload machine and three units being
dynamically unloaded from the train. The output is a +3 train (Rule r3,3). The figure
also shows a −3 train loaded by a [+2] load machine. A chain of these machines
can be constructed. For example, Figure 6 shows a compound consisting of a 3
Unload and a 1 load machine. It unloads 3 units and then loads 1 unit to any train
entering the chain. The chain functions exactly as a [−2] unload machine. Conse-
quently, unary operators are represented as objects even if they do not operate on
trains. Chaining and equivalence confer on these machines the possibility of
constructing operations on operators.

The compare machine as a representation of distance between numbers
on the number line (R4). The compare machine allows students to order
trains on a number line and measure their relative length. It takes as its input two
trains that are placed on two parallel number lines. A segment that represents the
difference in position between the two trains is then generated. For example, when
a +9 train and a +5 train are placed together in the compare machine (Figure 7),
a segment of Length 4 is generated (Rule r4,4). This segment is movable and can
be placed on a number line to be measured. For trains of opposite signs, the

![Figure 7](image-url)

FIGURE 7 Exemplifying an order and a metric between trains (R4—here r4,1 and r4,4)
comparison generates the sum of the absolute values of the trains. For example, entering a \(-2\) train and a \(+3\) train into a compare machine generates a 2-unit long and a 3-unit long segment that then create a 5-unit long segment. Figure 8 exemplifies Rule \(r_{4,6}\).

4.2. Acting on the Exemplification: The Set of Activities

Introducing the objects of the exemplification component does not fully explain how knowledge about negative numbers is acquired. Further conditions designed to lead students to an explicit formulation of rules about negatives follow:

- All the activities were problem-solving oriented.
- The set of activities was built in terms of learning units that constrained and guided learners’ exploration. We hoped that students’ relatively free exploration would lead them to discover particular properties of the objects and actions in the intermediate abstraction that are essential to the concept of negative numbers. One of the most important constraints we used in the intermediate abstraction was to require students to plan problems and predict solutions. By these means, students were invited both to explain the functioning of the machines and to express in their own words the relations existing between the objects of the system.

The design of the activities and the constraints imposed on the activities are described in the remainder of this section.
The tracks: A necessary ingredient for problem solving. By definition, problem solving is goal oriented. In the realm of numbers, the goal is often to fit a value or to find a final value by solving equations or executing operations on numbers. A tool that permits a fit between quantities is then necessary. The tool constructed for this purpose was a track. The goal is then to make a train that fits on a track. Constraints are imposed in the form of given trains (directed magnitudes) and machines (operators or actions). This may then force students to explore relations between trains and operators. For example, if a +6 train is entered into a [-3] unload machine, the output—a +3 train—will fit a +3 track, which expresses the equality $6 - 3 = 3$ (see Figure 9). If students try to place a +3 train in a +4 track, the +4 track expels the +3 train and emits a "boing" sound to demonstrate a lack of fit.

FIGURE 9 Solving problems by matching trains and tracks.

FIGURE 10 Example A: An activity for the acquisition of $R_3$ rules.
Activities were often designed around a set of trains, machines, and tracks; these can be taken to express equations with unknown values. For instance, see Example A in Figure 10. The problem can be formulated as, “Choose a load machine and an unload machine in order to fit the trains on the tracks.” The students were asked to choose the values of the machines in order that a minimal number of steps would be required to fit the trains on the tracks. This request avoided the troublesome choice of a [+1] load machine and a [-1] unload machine (this choice is correct, but does not allow students to generate very interesting explanations or rules).

Example A is an activity that concentrates on R3 rules (unary operators) because in this example the learner has to find operators such that \(-5 + (\pm ?) = -2\), or \(4 = 8 + (+?) + (-?)\). Example A is an opportunity for students to study the realm of integers as actions on directed magnitudes (R3). We do not require that a particular rule must be formulated or learned from this activity. The constraints here are intended to allow children to freely explore the consequences of particular situations. Possible constraints in Example A are: (a) the choice of machines has to work for all of the trains, (b) the number of steps has to be minimal, and (c) the choice of trains and tracks has been made by the experimenter in order that the shortest solution will imply iterations and compounds of machines.

Example B displays a simple situation in which all the trains, tracks, and

\[\text{FIGURE 11} \quad \text{Example B: An activity for the acquisition of R}_2 \text{ rules.}\]

A good choice here would be a [+3] load machine and a [-2] unload machine. The trains will match the tracks by drawing on the following equalities: \(-5 = -2 + (+3); 6 = -3 + (+3) + (+3) + (+3); 8 = 4 + (+3) + (-2).\)
machines are completely defined (see Figure 11). The cutting/gluing machine is used to fit the trains to the tracks. Here, students are also asked to use a minimal number of steps, which gives them the opportunity to formulate how the cutting/gluing machine acts when operating on numbers of opposite signs. Example B was designed to study the addition of directed magnitudes (R2), although it was not intended to teach a specific rule. For a few activities, the links to specific rules are explicit: these are activities involving the compare machine (see Example C in Figure 12). The problem here is to find a train, which when compared with the -5 train, will generate a segment 6 units long. The two answers (a -11 train and a +1 train) are directly linked to the rules that express the idea of a metric on the number line (r4,5 and r4,6).

The planning component: From the exemplification component to the mathematical relations. The most salient constraint we imposed on some activities was to force students to plan out their actions in solving equations. The planning component demands from the students more than knowledge of how to manipulate the objects of the exemplification component. Using the exemplification component does not necessarily lead students to reflect on actions, to formulate expectations, or to elaborate language used to describe the system. Our belief was that the students' language would become more formal when they used the planning component. We thought they would concentrate on general properties of machines for two reasons: (a) The trains are replaced by numbers; and (b) the machines used are similar to those used in Trainworld, but they appear in reduced size and their functioning is not visible.

In Example D for instance, the students are presented with a strip (see Figure 13). The strip is a shorthand way to ask the question: What is the unload machine you need to use on a -3 train to get a result that will fit on a -8 track? Letters A

![Figure 12](image-url) Example C: An activity for the acquisition of R4 rules.
and B under the machines designate the two stages of the problem. The letter A appearing in the track means that the result of the first operation has to fit that track. The strip functions like a mini computer program: First, the student has to predict the solution; then enter it in place of the question mark; and finally run the program in Trainworld with the full size objects. For example, if the student guesses that the solution is an \([-11]\) unload machine, the computer automatically displays a \(-3\) train, which enters into a \([-11]\) unload machine (see Figure 14). The output—a \(-14\)
train—fails to fit the −8 track. This attempt is displayed in a table. Students can run as many attempts as they wish until the correct result is attained. All the results are displayed in the table. It is crucial to notice that in the strip the property of unload machines to “add onto white trains” is invisible, whereas in the exemplification component, it was salient.

The problem shown in Figure 15 (Example E) is to find the train that, when glued to a −12 train, will fit a −5 track. Although this question can be related to actions, the problem with the Planner is fundamentally different than when posed with “real trains.” When using real trains, the phrase “white and gray trains eat each other when glued” is actually demonstrated directly.

We hoped that the language used by students, particularly in the planning stage, would move from a mere description of the functioning of the exemplification component toward a type of language that could be mapped onto a more formal mathematical language.

In summary, this intermediate abstraction was a workshop in which students could act on objects. The behavior of the objects was self-evident. Children were encouraged to create their own vocabulary and discover their own properties and rules. These activities created a system of constraints that we hoped would lead students to the articulation of mathematical rules and principles.

5. THE EXPERIMENT

We constructed an experiment around the Planner in which the growth of the concept of negative numbers was analyzed. Four 5th-grade students participated as subjects in this study. All the students were from the same innercity parochial school. During all the training sessions, students worked with the Planner in pairs. The experiment consisted of four phases: the pretest, the training sessions, the posttest, and the caboose (see Table 2). The activities were labeled according to the phase to which they belonged (pre, training, post, caboose), the referent for the activity (debt, temp, elev, time, formal [i.e., no referent], or planner), and a number to designate the order. For example, (pre, temp, 1) is the 1st pretest activity with temperatures as the referent (see Table 2). Similarly, (post, formal, 12) is the 12th formal activity in the posttest.
TABLE 2
Description of the Phases of the Study

Pretest
- Existence of negatives
- Ordering of numbers (Pre, formal 1 & 2)
- Solution of equations (Pre, formal 3-10)
- Word problems (Pre, debt/temp/elev, 1 or 2)

Training sessions
- Demonstration and explanation of Trainworld (2 sessions)
- Working with the exemplification component (4 sessions)
  - Negative trains, directed magnitudes
  - Compare machine, relations or order between negatives (Example C, Section 3)
  - (Un)load machines, unary operations on full number line (Example A, Section 3)
  - Cut/glue machines, binary addition and subtraction (Example B, Section 3)
- Secret box (1 session)
- Working with the planning component (4 sessions [Examples D and E, Section 3])

Posttest
- Ordering of numbers (Posta, formal, 1 & 2)
- Solution of equations (Post, formal, 3-28)
- Word problems (= preproblems) (Post, debt/temp/elev, 1 or 2)
- Complex word problems (Post, debt/temp/elev/time, 3 and up)

Caboose session
- Extension the subtraction (cutting outside) (1 session)
- Redoing equations (Post, formal, 18-27)

Sample of tasks
1. (Pre, formal, 3) $4 - 6 = ?$
2. (Pre, temp, 1) If it was 5 degrees yesterday, and the temperature went down 12 degrees overnight, what is the temperature today?
3. (Pre, elev, 1) I got on an elevator and went down three floors. The floor I got out on was the fourth floor underground. What was the floor I got on at?
4. (Post, formal, 12) $?-4=-6$
5. (Post, formal, 21)$^b$ $-3=4+?$
6. (Post, formal, 27)$^b$ $-8=？+2$
7. (Post, temp, 5) Yesterday, it was minus five degrees. The weather forecaster on TV said the temperature would change by eight degrees. If he was right, what is the temperature?
8. (Post, time, 1 & 2) Augustus Caesar was born in 63 B.C. and died in 14 A.D.
   How old was he when he died?
   Julius Caesar was born in 100 B.C. He was assassinated when he was 56 years old. What year was he killed?

$^a$The 10 posttest tasks (post, formal, 1-10) were identical to the pretest tasks (pre, formal, 1-10).
$^b$This task was used twice, first in the posttest and then after the appendix session using the cutting machine with the “knife” outside.
The Pretest

The pretest checked children’s informal and formal knowledge about negatives. The children were tested individually by the experimenters. The second author, a native English speaker, conducted all the interviews. The first author, a nonnative English speaker, took notes and interjected questions as the occasion arose. Interviews were tape recorded and transcribed. The interviewer encouraged the children to answer all questions but tried to maintain an informal atmosphere. Children were asked to answer all questions, even if they were not sure, if their answer sounded silly, and so on. The pretest sections are described next and in Table 2.

Knowledge of the existence of negatives. The children were asked if they had ever heard of negative numbers. They were also encouraged to say anything they could think of that was relevant.

Ordering of numbers (pre, formal, 1–2). Children ordered index cards on which positive and negative numbers were written. They were asked to put the cards in order and point out the smallest and largest numbers.

Solution of equations (pre, formal, 3–10). Children were given simple subtraction problems to solve. Most of them required a negative answer. If a child tried to invert the order of the numbers to avoid a negative result, the experimenter asked, “But is it the same problem then or a different one?”

Solution of story problems (pre, debt/temp/elev, 1 & 2). Children read and solved a set of simple word problems. The referents for these problems were debts, temperature, and elevators.

Pretest tasks were designed to assess the individual children’s starting points (i.e., what their mental models of negatives looked like before any formal training).

The Training Sessions: Manipulating the Objects of the Planner

The two pairs of children were exposed to eleven 30-min training sessions with the Planner. These activities were designed in the hopes of allowing children to acquire a model of negatives that would include most or all of the properties of negatives listed in Section 2. We did not explain how the Planner worked in an explicit way but rather tried to keep the discussion going. Often, we simply told the children to “wait and see” when they had a specific question. We tried to build on the children’s own vocabulary. We hoped that instruction would occur naturally through the children’s discussions. We also relied on our activities to do the teaching, somewhat Socratically, by providing the “right questions” as children built up an understanding of negative numbers. The training sessions consisted of two phases: first, a
series of activities with the exemplification component in which students manipulated the objects of the system; and second, a series of activities with the planning component in which students reasoned about semisymbolic trains and machines without actually seeing the objects function. Note that all sessions with the Planner were tape recorded and transcribed. The phases of the training sessions follow:

Activities With the Exemplification Component (Training, Exemp, 1–21; Session 1 to 7)

Demonstration of the exemplification component (training, exemp, 1–8). The experimenters presented the different machines in the system and showed the children how they work with gray trains. Care was taken not to introduce any vocabulary about negatives. The children were asked to watch what happened when the trains were placed into machines and to describe what they saw.

White trains as directed magnitudes (exemplification of R₁; training, exemp, 9–10). The second phase of activities involved introducing children to negative quantities. We presented white trains to the children as “different kinds of trains.” Children were asked to match both white and gray trains to tracks.

The compare machine: Relations of order among trains (exemplification of R₄; training, exemp, 11–12). The children work with the compare machine to demonstrate relations between negative numbers (see Example C, Section 4).

The cutting/gluing machine: Binary operations on trains (exemplification of R₃; training, exemp, 13–16). The next phase was binary addition and subtraction on negative numbers using the full number line (see Example B, Section 3). The knife was used on the inside of trains only; by this means all rules except r₂,₅ and r₂,₆ may be exemplified.

The load and unload machines: Unary actions on trains (exemplification of R₃; training, exemp, 17–20). We then moved on to using the load and unload machines to exemplify unary addition and subtraction. Activities included problems in which the unknown was a train or the value of a machine.

Secret box (training, exemp, 21). Students then worked with a secret box. In this activity, children guessed the content of a black box by entering trains into it. The aim of these activities was to show the child that various sets
of operators yield the same results as a particular problem we posed. For example, if entering a +3 train into a secret box gave a result of −2, children were encouraged to guess all the different operations and sets of operations that might be inside the box. “Minus 5” was the simplest answer, but the children learned that sets of operations, such as “minus 7 and plus 2” also worked. Secret box activities exemplified r3,9 (i.e., rules induced by combinations of actions).

Activities With the Planning Component (Training, Planner, 22–53; Session 8 to 11)

The first phases of our study allowed students to directly manipulate the objects of the system. The next phase, using the planning component, forced the pairs of children to reason with semisymbolic displays. The Planner activities required children to plan out answers to questions given as strips (see Examples D and E in Section 3). These strips were no longer concrete models but were meant only to remind the children of how machines functioned. Children were shown how they might use plans to find a missing value for a train, a machine, or a track (these activities are isomorphic to solving equations). Children then guessed what they thought might be answers to problems and the different planned solutions were run (as shown in Examples D and E in Section 3). Children developed skill in planning out their answers. They learned to use feedback from an incorrect plan to modify and create new plans. Activities using the planning component were the same type of activities as those solved earlier with the exemplification component.

The Posttest

The posttest was designed to provide a broad range of situations to assess (quite stringently) the structures and mental models that each child developed with the Planner as a formal system. By using many situations, we hoped to uncover the nature as well as the limits of children’s understanding. Parts of the posttest were identical to parts of the pretest (ordering numbers, solving equations, and word problems) and included the same tasks given in the pretest. Additionally, students were asked both more difficult and more varied questions. New tasks included more formal equations and more word problems. Table 2 contains a sample of formal tasks and a temperature problem. We also included word problems involving time to provide a very stringent transfer task to test children’s mental models.

The Caboose: “Cutting Outside” the Trains

A set of activities was used at the very end of the study, after the children participated in the posttest tasks. These activities used the cutting machine with the knife outside of the train to present a demonstration of the subtraction of negatives. Demonstrations included subtraction of a negative from a positive and a positive from a negative
(formulated as Rules r2,6 and r2,7). Finally, children were asked to once again solve 10 formal equations from the posttest (post, formal, 18–27).

6. INDIVIDUAL PERFORMANCES

6.1 Coding Responses Given During the Tests and the Training Sessions

In this section, we describe our classification system for the strategies that children used in solving and explaining pretest and posttest problems and used during the training system. This classification system is the first step to characterize students’ mental model and organization of knowledge during the different phases of the experiment. Examples from protocols are given to illustrate each strategy in Appendix B.

“Nothing” (No). The child considers any negative number to be nothing or equal to zero. In computations, if the child reaches a result that is not positive, he or she identifies it as zero.

“Connections” (Con). This strategy appears to be articulated in two stages. First, the solver generates several possible answers; second, he or she chooses an answer from among the possible set. The possible answers are all obtained by adding or subtracting the given numbers in a task (and sometimes by inverting one of those additions or subtractions). For example, in the equation \( ? - 8 = -5 \), the student may know that the answer is one of the four following numbers: 3 (the value of 8 - 5), 13 (the value of 8 + 5), -3 (the inverse of 3, or 5 - 8), or -13 (the inverse of 13, or -8 - 5). It is interesting to note that the Con strategy does not take into consideration the structure of the problem, only the numbers involved.

Symmetrization (Sym). This strategy generally occurs when all the numbers involved in a task are negative. The child appears to pretend that the given numbers are positive, carries out the computation, and then adds a minus sign to the result.

Inverse (Inv). The student rearranges the given numbers in a different order than that presented in the task. After the computation, the sign of the result is changed. This strategy is similar to the Sym strategy, although different because of the prior rearrangement of numbers. Here, as well as for the Sym strategy, the structure of the task is changed in the process of computation.

Dynamic (Dyn). The computation is apparently done by mentally changing the size of the quantities. Explanations contain descriptions of quantities changing gradually. The most common terms involved in this strategy are: smaller, bigger, this one
will eat that one, and it will shrink. This strategy is not merely qualitative; children still perform arithmetic computations, but computations appear to be based on a qualitative understanding of the size of resulting quantities. For example, the child considers if the resulting quantity will be bigger or smaller, gray or white.

**Taking away (Tk).** This strategy is similar to the Dyn strategy. The solver describes how a quantity changes by mentally removing parts of the quantity. The language used often contains statements such as “I take pieces off” and “I remove it.” The child often refers explicitly to elements of the exemplification component (i.e., trains).

**Nullification (Nul).** This strategy occurs when the two given quantities in an equation are of opposite signs (e.g., “−a” and “b”). This strategy is articulated in three stages. First, the child translates the problem to find an action that gets him or her from −a to b (or from b to −a). Second, the child finds the action that gets him or her from −a to 0 (or from b to 0). Finally, the child finds the action that gets him or her from 0 to b (or from 0 to −a). The value of the composed action, a + b (or −b −a) is equal to the unknown (see Appendix B for examples).

**Counting (Count).** This strategy involves counting the number of units between given numbers (positive or negative) to solve a problem.

**Line (L).** The student appears to use an ordinal relation among trains, quantities, or numbers. The quantities in the task may be thought of as positions or actions that enable one to attain a certain position. The computations are then apparently conducted according to the semantic meaning of the actions. The terms used in this strategy are: goes down, goes up, it will be colder or hotter, higher, before or after, and owing or having more or less.

**Trainworld (Train).** During the posttest, students often used language that referred to objects in the intermediate abstraction. In many cases, it was possible to code the explanation according to one of the previously mentioned strategies (e.g., see the previous usages of Take away in Appendix B). However, often the only explanation provided was of the sort “I thought about Trainworld;” or “I thought about a Gluing machine.” In those cases we coded the explanation as a Train strategy.

**Hybrid strategies.** Students often used a combination of two strategies to solve a problem. For instance, in the task “−5 = ? + (−8)” (post, formal, 22), Ash uses a mixture of the Con and the Dyn strategies (see Appendix B).

In conclusion, student explanations were coded into 10 strategies: No(thing), Con(nections), Sym(metrization), Inv(erce), Dyn(amic), Tk (Take Away), Nul(lification), Count(ing), L(ine), and Train(world). One of the strategies, No, was always wrong. In the case of hybrid strategies, a double code was used. The strategies were coded independently by the first two authors. Disagreements were
resolved through discussion with the third author. The 10 strategies can be organized into three classes. Figure 16 represents these classes in an iconic way.

One class, *Actions*, includes strategies that imply an action or a change of quantity. This class includes Dyn, Tk, and Nul. Children speak about movement, growth, shrinking, and so on when they use those strategies.

The second class, *Ordinality*, includes strategies that link integers (and trains) into some sort of an ordered set. Children using these strategies appear to take into account the positions of numbers to make predictions and judgments. These strategies include L, Count, and Nul, which belong to two strategy classes.

The third class of strategies is *Rules*. These refer to relations among operations on quantities, numbers, or trains. These strategies are Con, Sym, and Inv.

*No* is in a class by itself. This is a black hole strategy that does not allow any other explanation to take place: Any computation reduces negatives to being equal to zero. *Train* is a very general strategy; it does not express how the child specifically used trains, machines, or their relations to solve problems.

6.2. Individual Performances

This section is an attempt to characterize the students’ model about negatives at each stage of the experiment. Table 3 shows the performances of the four
children on the pretest compared to their performances for the same tasks on the posttest and to their overall performance on the posttest. The data show that the performances of the four children improved substantially both for the formal equations and the word problems, specifically if we take into account that during the treatment, the Planner was used as a formal system without relation to any referent. In the pretest, the four children correctly solved only 47% of the formal tasks (on average). In the posttest, three out of the four children answered all of the pretest items correctly. As mentioned in Section 5, the new questions in the posttest were far more difficult than those in the pretest. Therefore, the overall score of 76% correct reflects only part of the improvement on the posttest. Performance on the elevator problems shows similar gains. The improvement is more accentuated for temperature problems; correct performance was very low on the pretest (19%) and very high in the posttest (88% retest, 75% overall). The performances on the debt/asset problems are almost perfect in both the pretest and posttest.

Table 4 shows the performances on the most difficult formal tasks (of the sort $\pm a = \pm b$ and $\pm a = \pm + ?$). These problems were presented twice, once in the posttest and again after a session using the cutting machine with the knife outside (referred to as the caboose phase of the study). Again, improvement is seen in children's performances. Exposure to the objects of the intermediate abstraction for only

<table>
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<th>Elev</th>
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TABLE 4
Performances on the Difficult Formal Equations Before and After the Cutting Outside Tasks (Caboose Session)

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<td>Si</td>
<td>5/10</td>
<td>10/10</td>
</tr>
<tr>
<td>Mi</td>
<td>7/10</td>
<td>10/10</td>
</tr>
</tbody>
</table>

TABLE 5
Strategies Used During the Pretest and the Posttest

<table>
<thead>
<tr>
<th>Classes of Strategies</th>
<th>Distribution (%)</th>
<th>Specific Strategies</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ash</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pre</td>
<td>Actions</td>
<td>76</td>
</tr>
<tr>
<td>Pre</td>
<td>Nothing</td>
<td>24</td>
</tr>
<tr>
<td>Post</td>
<td>Actions</td>
<td>42</td>
</tr>
<tr>
<td>Post</td>
<td>Ordinality</td>
<td>31</td>
</tr>
<tr>
<td>Post</td>
<td>Rules</td>
<td>27</td>
</tr>
<tr>
<td>Sh</td>
<td>Rules</td>
<td>100</td>
</tr>
<tr>
<td>Post</td>
<td>Actions</td>
<td>9</td>
</tr>
<tr>
<td>Post</td>
<td>Rules</td>
<td>89</td>
</tr>
<tr>
<td>Mi</td>
<td>Pre</td>
<td>Actions</td>
</tr>
<tr>
<td>Mi</td>
<td>Post</td>
<td>Actions</td>
</tr>
<tr>
<td>Mi</td>
<td>Ordinality</td>
<td>19</td>
</tr>
<tr>
<td>Mi</td>
<td>Train</td>
<td>50</td>
</tr>
<tr>
<td>Si</td>
<td>Pre</td>
<td>Actions</td>
</tr>
<tr>
<td>Si</td>
<td>Nothing</td>
<td>56</td>
</tr>
<tr>
<td>Si</td>
<td>Post</td>
<td>Actions</td>
</tr>
<tr>
<td>Si</td>
<td>Ordinality</td>
<td>10</td>
</tr>
<tr>
<td>Si</td>
<td>Rules</td>
<td>22</td>
</tr>
<tr>
<td>Si</td>
<td>Nothing</td>
<td>5</td>
</tr>
<tr>
<td>Si</td>
<td>Train</td>
<td>25</td>
</tr>
</tbody>
</table>


one additional session (the caboose session) seems sufficient to help students apply their knowledge as they work through difficult formal tasks.

Table 5 shows a comparison between the strategies used in the pretest and those used in the posttest. The strategies are organized into the classes: Actions, Ordinality, Rules, Nothing, and Trainworld. For example, Ash used
the Action strategies on 76% and the No(thing) strategy on 24% of the pretest tasks. For each category and each child, the specific strategies are shown. For example, the only Action strategy used by Ash in the pretest was Tk, although in the posttest, she used Tk(Take away), Dyn(amic) and Nul(lification). Taken together, Table 3 and Table 5 indicate that: (a) the children had little knowledge of negatives in the pretest; (b) their performances on the posttest were very good—difficult formal tasks were carried out very easily, and story problems, which students did not solve during the training sessions, were solved successfully; and (c) the explanations provided during the posttest drew on a large number of strategies. In the rest of this section, these points are further demonstrated as we attempt to describe our students’ mental models of negatives in the pretest and the posttest.

Knowledge Change Between the Pretest and the Posttest

The most salient point that emerges from Table 5 is the growth and diversification of strategies used correctly between the pretest and the posttest: The only correct strategy used in the pretest was Tk. No was used frequently by Ash (24%) and Si (56%; see the two examples of the No strategy in Section 6.1). Use of this strategy indicates that these children had no basic knowledge about negatives. They also did not succeed on word problems, except for debt/asset problems in which all operations referred directly to amounts of money, without invoking any negative quantity, per se. Results in the posttest show that each student used several classes of strategies. Actions and Ordinality are used by Ash, Mi, and Si frequently; Rules are used by Ash (41%) and by Si (12%). Train(world) is used intensively by Mi (50% on formal tasks, 25% on story problems). Moreover, they diversified the number of strategies used within each of the classes. Note in Table 5 the large number of specific strategies used by three of the four children in the posttest.

Figure 17 attempts to show schematically and qualitatively the pretest and posttest mental models of the four students. The distribution of strategies is represented for formal tasks and story problems separately. The icons used in this figure are based on the classes of strategies displayed in Figure 16. The correctness is indicated on top of each of the strategy icons. This is the proportion of tasks that each subject solved correctly, irrespective of the strategy used. For example, Sh always uses Rules but succeeds in applying them to stories in the pretest on only 50% of the tasks.

The development of strategies through the training sessions is traced individually and analyzed in Section 7. Although each child developed knowledge about negatives and showed diversification of strategies, they each did so in a highly individual way. Also, the pretest and posttest phases of the study were administered individually; therefore, we report on each child’s performance separately in this section. For this purpose, the results shown in Figure 17 and Table 5 are commented on and further detailed through excerpts of children’s protocol. These protocol also serve as evidence of the uniform application of many of these strategies in both story problems and formal tasks.

The case of Ash. Among the four students, Ash showed the biggest change
between the pretest and the posttest. In the pretest, her knowledge about negatives was very weak. In the posttest, Ash succeeded not only in solving the large majority of the formal and story problems but also in using a very large number of strategies. Some of the strategies are very sophisticated, like Nullification or Con(nection) coupled with Dyn(amic; see Appendix B for examples of protocol). Except for
temperature questions, her performance on story problems was flawless (see Table 3). A very interesting point is that she used the same Actions and Ordinality explanations for both formal and story problems. For instance, the Nul strategy is used for formal tasks (see the two examples in 6.1) and for tasks about time and temperature as shown next.

For the task (Post, Time 1); Augustus Caesar was born in 63 B.C. and died in 14 A.D. How old was he when he died?

Ash: Sixty-three plus fourteen.
E: OK. Why would that be a good thing to do?
Ash: Because I was to start to say what I did. The other problems like make the first number sixty-three and that was zero and since he died after, on the fourteenth, he did, that would be sixty-three and fourteen cause ... it's like the other problems with minus sixty-three and regular fourteen. So, I just did it the same way. Made the first number sixty-three and that was zero and then added on fourteen more because this is fourteen.

For (post, temp, 2): If it was minus six degrees yesterday, and it is five degrees today, what happened overnight?

Ash: What's the difference between the two temperatures? ...Eleven. I made six the first number again and that was zero, then I added five more on.

Ash even compares a debt/assets problem to a B.C.–A.D. time problem:

Ash(post, debt, 2)
Ash: I have to do thirty minus twelve.
E: Ok. Write that down.
Ash: And that's because if he had thirty all together...and thirty altogether he owed, he didn't have any money left. So I knew it had to be the rest of the money he has left from the twelve dollars. That's how you do the other one. When before Christ, like B.C. ....
E: The Julius Caesar one?
Ash: Yes, 'cause the rest of the years, the years between one hundred is how like the year that he got killed!

The case of Sh. Sh's performances on the pretest tasks were very good. He reported that he, "learned about negatives with my Dad." He was very skillful and answered quickly on the formal portion of the pretest (80% correct performance). He appeared to do his computations by applying rules, but he did not give explanations about his computations. In contrast to this skilled performance, Sh's performances were rather mixed on the story problems. For elevator and temperature problems, it was only after the experimenter coaxed him and reformulated the problem somewhat that he succeeded in solving some of the tasks. For example, when asked the task (pre, temp, 1), Sh seemed very hesitant to apply the procedures he had used on formal equations:
Sh: If it was 5 degrees yesterday, and the temperature went down 12 degrees overnight, what is the temperature today? You subtract. I don’t know how to write that problem.

E: Ok. Let’s start out with five yesterday. And then it goes down twelve. How do you think you do that?

Sh: Uh…five minus twelve. Yeah five minus twelve.

E: Can you tell me now the temperature today?

Sh: Five minus twelve...(pause). This is a minus seven. Right five minus…(pause)

Sh provided few explanations on the posttest. However, three points can be noted.

1. For the formal part of the posttest, his performance on tasks that demand direct computation (i.e., $a \pm b = ?$) was flawless. He also performed well on new tasks of the kind $a \pm b$ and $a + ? = \pm b$. For the most difficult tasks, Sh answered only 6 of the 10 questions correctly (see Table 4). Although such questions can be mapped onto situations encountered in sessions with the intermediate abstraction, the cutting machine with the knife outside seems a particularly appropriate model for these problems. All children correctly answered all of these questions after the single caboose session.

2. His performance on the story problems is outstanding (in contrast with his performance on the pretest).

3. A very notable failure is seen in time problems. In (post, time, 1) Sh fails to find the answer because he has difficulties in matching numbers and quantities to this situation:

Sh: How long will it take until you can get to...how long does it take until you can get to A.D.? From 63 B.C.?

E: What do you think? What would you think about it?

Sh: No. I mean how much will 63 B.C. be able…[pause] how many B.C. do it have to before it can get to A.D.?

Sh: I was saying that 63. I thought that 63 would go into a hundred, I think. Cause…nobody told me what 63 B.C. ...[pause]. Sixty-two years before Christ was born?

The case of Mi. During the pretest, although Mi is able to correctly solve 70% of the formal tasks, she does so by translating all of them into taking away or owing situations (see Table 5). For example, when asked to order positive and negative numbers, she succeeded by conceptualizing the negatives as “amounts taken away”: “$(-3)$ is smaller than $(-2)$,…because minus 3 is like you take away 3” (see also examples for take away strategy in Appendix A). This point of view is also expressed as she solves the problem “$-3 + 2 =$?” (pre, formal, 6):

Mi: You should, uh, take away 3 and add a 2…
Mi seemed very confused on both temperature and elevator problems during the pretest. This may reflect that Mi’s initial conception of negatives is biased toward an understanding of negatives as quantities. Negative quantities (like debts) are one way of modeling how negatives work, but this representation is problematic for tasks involving negatives as positions (e.g., points on a thermometer).

Similarly to Ash, Mi used many strategies and applied them uniformly to several settings in the posttest (see Figure 17). For most of the formal tasks, she said that she solved the problem by “thinking about trainworld.” Mi also mentioned a specific Trainworld machine or function at times. For example, to solve the problem “? -3 = -,” she said, “If you have a zero train and you take away 2 plus [sic] a minus three. You will take away three.” This appears to be a novel use of Tk. It is interesting to note that for word problems, Mi did not refer to Trainworld at all. The concept of historical time, B.C. versus A.D., presented a difficult challenge for all of the children in this study and was the only task on which Mi failed. However, Mi was able to understand the concept when the initial question was modified with small numbers:

E: If a child was born in 3 B.C. and died in 4 A.D., how old was he?
Mi: Seven, because he was 4. And 3 years before Christ, and 4 years after Christ.

The case of Si. In the pretest, Si was unaware of the existence of negatives. (He used the No strategy in 56% of cases; his overall performance was the lowest of the four children.) Si also performed rather poorly on the posttest, although substantial progress was made, especially in regard to performance on formal problems (70% correct vs. 20% correct). Si used several strategies (see Figure 17). No was used for the first three posttest formal tasks, but then Si began to use language borrowed from Trainworld and subsequently succeeded in most of the formal tasks. His favorite strategy was Tk, which he usually talked about in terms of “taking pieces off.” He often appeared to be referring to gluing a positive and a negative quantity. Figure 17 shows that these strategies were used equally in both formal and story problems.

In conclusion, the results in the posttest suggest that students could map knowledge of negatives acquired while they worked with the intermediate abstraction both to formal tasks and word problems of the posttest. In the next section, we show that the Ordinality, Action, and Rules strategies were developed and articulated by all the children during the training sessions. We also attempted to examine the language adopted by the students as they worked with the exemplification component.
7. THE DEVELOPMENT OF KNOWLEDGE DURING THE TRAINING SESSIONS

The two pairs, Ash-Sh and Mi-Si, worked on identical activities during eleven 30-min training sessions. The sessions were tape recorded. The language that children used to communicate with each other about the intermediate abstraction and the problems they solved were analyzed according to two different perspectives: degree of reference to the exemplification component and the strategies used.

7.1 The Tools for the Analysis of the Sessions

*Measuring the Degree of Reference to the Exemplification Component*

This perspective is used by most researchers using computer microworlds and tutoring models (e.g., Roschelle & Behrend, in press). Roughly, this is a measure of the extent to which children's language is independent of the computer model. We analyzed the vocabulary that children used without considering the mathematical structures referred to. Two levels of analysis were considered:

Exemplification language describes the behavior of the system in terms of the objects of the system and their manipulation. The vocabulary at this level contains terms like *trains, put, glue,* and *here.* Such descriptions can be predictive. An example of an exemplification language explanation was given by Si:

Activity 16: Trains (−9, 5, −12, 4), cutting/gluing machine, Tracks (−2, 7, 3)5

Si: You can take this. If you take two them the same, put this back,...right here...and go fourteen like, and put it together, you'll come out with seven. In multiple seven. Then you take nine, and you ain't got no more left.

At this level, explanations can also refer to properties of objects in the system that the children have inferred. Children refer to the unload machine, for example, as "always adding on the white trains." Sometimes they refer in more general terms to causal explanations of the functioning of the intermediate abstraction.

Mathematical language contains statements and rules about numbers and operations and does not refer to the intermediate abstraction at all. Children using this type of language appear to purposely choose to speak in general terms. An example is provided by Sh:


---

5In this section, most activities are described by a triplet of trains, machines, and tracks (as in Section 3). In Activity 16, the students were given two trains (5 and 14) and two white trains (−9 and −12), a cutting/gluing machine, and three tracks (−2, 7, −3). The task is to use the machine to fit the trains to the tracks.
Sh: All you got do is when it’s the minus number up against a positive...all you got do is you have just minus the negative number against that, then you come up with the answer.

E: Wait. What do you mean?
Sh: Like if it was a negative nine up against uh...positive eighteen, I’d come up with a regular nine, cause all I do is let minus go.

Classes of Strategies

The second way to analyze the protocols was to classify them according to the three strategy classes—Ordinality, Actions, and Rules.

Coding a Session

The transcript of Activity 18 for the pair Ash–Sh reads as follows:

Activity 18: Trains(−4, −6, −13)/machine(unload[−?])/Tracks(−8, −10, −12)

1 Ash: Oh, I know the answer already. We could do two. Two...no.
2 Sh: No, we could do four, because four plus four would equal eight. And four plus six is ten.
3 Ash: There’s no minus, Sh.
4 E: Do we got a problem here to do?
5 Ash: Yes.
6 E: Ok, so you guys were thinking, put in a two in that unload machine, or a four? Which one?
8 Sh: How about six?
9 Ash: No. Because thirteen is...a problem.
10 E: What’s the problem with the thirteen?
11 Ash: It’s all...it’s more than eleven. And if plant two on that one, that’ll 12 be too high. Like if we put more on it then it will just get higher.
12 E: Ok, ok. Do you understand what Ash just said, Sh?
13 Sh: Yes. Uh...what she said is that wouldn’t cut up if we add on. It’s already high.
14 Ash: It’s more than all of them.
15 E: This is an unload machine. It adds or it...it takes off?
16 Sh: Can it go in to one of those white trains? Can we make this into a gray train?
17 E: No.
18 Ash: Cause thirteen is higher than all these numbers. And this one adds on, so say if we
19 wanted three here. If we put three on...it’ll be fifteen. And three more will be eighteen. It
20 will just keep getting higher. And it won’t fit on that.
22 E: Mm-hmm. Ok. So what does this machine on white trains?
23 Ash: It takes off in that it...
24 Sh: No, no, the only time it takes off is when it's a positive. And when it's a negative it...makes
25 that positive number be negative number. Like this is thirteen, and it'll go to fifteen.

Comments.

1. In Line 2, Sh gives the rule “4 + 4 = 8” even though the trains are white and the machine is an unload machine. This excerpt is thus classified as Rules (the specific strategy is Sym).

2. In Lines 3 and 9, Ash appears to interpret the problem at a high level. She says that -13 is “too high” and that by using the unload machine, which functions by adding onto white trains, the -13 is not attainable (“there is no minus”). This excerpt is classified as Action strategy using mathematical language. Lines 12–12, 16, and 19–21 elaborate the same idea in different ways: Up to Line 15, Ash expresses a relation of order; in Lines 19–21 she uses an Action argument (“this one adds on”).

3. The rules Sh expresses in Lines 24–25 are classified in the Action class (Tk strategy) with mathematical language.

In summary, our coding indicated that Rules, Ordinality, and Action strategies were all used with mathematical language by the pair Ash–Sh during Activity 18.

7.2 The Development of the Two Pairs

The order of activities used with the Planner reflects our intentions as experimenters. First, we familiarized the students with the trains and machines using only positives and let them play with the exemplification component in hopes that they would build up a way to talk about the intermediate abstraction model (i.e., acquire an exemplification language). Next, with the planning component, we asked students to predict the behavior of the objects in the system. Finally, as the objects in the planning component do not actually function, we see if students can reason through problems without relying on manipulations.

The sequence of activities was also directed to encourage the learning of several specific senses of negative numbers: R1 (directed quantities) and R4 (relation of order and metric) were exemplified by the build and compare machine; R2 (binary operations) were demonstrated with the cutting/gluing machine; R3 (unary operations) may be shown with the load/unload machines.

Figure 18 shows the number of correct and incorrect strategies used during the different phases of the study. A strategy was coded as being used correctly during a phase if it was correctly used at least twice (similarly, a strategy was coded as incorrect if misused twice during a phase). Thus, a strategy could be used and misused during the same phase. Figure 18 shows the two pairs separately. Overall, the two pairs both
learned many strategies successfully. The number of strategies used correctly increased during the exemplification and planning phases. The number of strategies misused dropped to zero for the two pairs during the exemplification component. This number then increased abruptly during the planning phase, and dropped again (to zero for Mi, to one for Ash, and to two for Si) during the posttest.

Two additional points are indicated in Figures 19 and 20. Figure 19 shows the percentage of Hybrid strategies used. Hybrid strategies were particularly used by Ash–Sh during the planning phase. We believe that during the first activities of this phase (when students first used the new semisymbolic representations), the pairs developed Rules strategies by combining already known strategies with the new demands of the more symbolic representation. Figure 20 shows how the language of the pairs developed during the training sessions in relation to machines. For example, during the black box activities, 100% of the language used by Ash–Sh was mathematical, whereas only 33% of the language used by the pair Mi–Si was classified as such. Overall, the pair Ash–Sh used mostly mathematical language, whereas the pair Mi–Si used little of it. A detailed description of the development of the two pairs follows.

The Pair Ash–Sh

Some salient features of this pair are shown in Figures 18, 19, and 20.

- The pair seldom used exemplification language.
- Exposure to the planning component appeared to cause an abrupt change in both the language and class of strategies used. This pair’s language had been mathematical, but during the first planning component activities it switched to exemplification language. After using the Planner for awhile, the children returned to mathematical language. The number of strategies used also multiplies in the planning phase. Evidence for these points follows:
1. Even during the first sessions, it was difficult to find this group using vocabulary that described the objects of the system or their functioning. An example is the following excerpt from the first activity after the experimenters demonstrated how white and gray trains behaved when glued together:

Activity 15: Trains (-5, -8, 10, -2, 6)/ gluing machine/Tracks (-2, 5).

1  Ash:  I know what we could do for five, we could do ten minus...
2   Sh:  Five.
3  Ash:  The min...the five...the minus five.... For ten
4     I mean for five. And for two we could just use this train.
In Line 1, Ash appears to use the gluing machine as an operator between numbers without referring to the machine itself. In Line 9, Sh seems close to expressing an explicit rule. Similarly, in Activity 18 (previously reported) the pair also used a high level of mathematical language, although this activity was the very first time they had seen the unload machine acting on negatives. This use of mathematical language is seen up until the introduction of the planning component.

As early as Activity 18, Ash and Sh expressed general rules (Lines 19–21, and 24–25 of that transcript). In the later sessions with the exemplification and planning components, trains and machines were rarely referred to in explanations. Rather, the pair appeared to use the objects of the system as a general framework in which to carry out computations.

2. During the first activities with the Planner (Activities 24–27), Ash–Sh made incorrect predictions and were surprised when the strip was run. From Activity 28 onward, the children discussed their predictions; tried to make sense of them; and appeared to generate new strategies, often hybrid in nature (see Figure 19).

Activity 28: Strip (?→unload[−8]→−5)

1 Ash: I’m not sure, but I know it’s a minus three or regular three cause eight minus three is uh...I mean eight minus five is three and...(pause)...Cause eight minus five...five minus eight. Eight minus five is three. Five plus three is eight. So, I know it has to be a minus three or regular three.

2 E: What are you counting out?

3 Ash: I was trying to minus...since we were minusing eight, I was doing two, one, zero.

4 E: Which train you can put here?

5 Ash: You can unload the three. (Ash means the solution is a +3 train, unloaded by the [−8] machine.)

6 Sh: Minus three. (Sh enters a −3 train, in contrast with the +3 train correct solution.)

7 Ash: I didn’t get three.

8 E: What did you get?

9 Ash: I did minus two, minus one, minus zero equal zero. And then one, two, three, four, five.
E: Ok, but...if you put the –3 train, what will happen with the...with a minus.

Ash: Minus eleven. It...it gets higher in the minuses.

Comments.

At Line 1, Ash expresses the result as being one of two possibilities. This appears to gradually lead toward an articulation of the connection strategy. Ash generally uses this strategy in hybrid form. At Line 11, she appears to use a new technique possibly linked to the Nul strategy.

The Rules class of strategies emerges during sessions with the Planner. An interesting example is given subsequently for Activity 39. This includes the emergence of the Sym strategy by Sh (Lines 7–8) and a hybrid of Con and Dyn by Ash (Lines 3, 4, and 6):

Activity 39: (build[?], unload[–3], track[–5])

1 Ash: Oh, we unload three?
2 E: Mm-hmm. That’s an unload.
3 A: Ok. Unload. So that will be a higher minus number. Well I would...minus three...I
4 would say a minus eight. No, wait, it’ll get higher. So...
5 E: You want to try minus eight?
6 Ash: A minus...a regular...No...
7 Sh: Ash...I got an idea. Say...just think this is a regular three and think this is a regular five
8 train. What would you add on to the three to get to this five. Just a regular two.

The last two excerpts appear to show a pattern. When using the semisymbolic representation of the Planner, the subjects seem to exhibit two parallel processes: (a) They try to mentally figure out how the machines and trains behave (see Activity 39, Lines 3–4, and Activity 28, Line 11; and (b) they try to combine the numbers in the strips by carrying out operations with them (see Activity 39, Lines 7–8, and Activity 28, Lines 1–3). Running the strips, both mentally and on the computer, may help the subjects to build up good Rules strategies (and may help them avoid generating faulty rules).

The Pair Mi–Si

Some salient features about this pair are shown in Figures 18, 19, and 20:

1. The pair generally speaks in terms of the Trainworld model (exemplification language). They rarely use mathematical language. This is in marked contrast to the patterns of language used by the other pair.
2. This pair uses the Action strategies very frequently. Even when they are in the planning section of the study, they often predict actions that will occur when the plan is run. The Action strategies are very successfully used by this pair.

Examples of how this pair successfully solves the problems posed to them follows, along with an attempt to explain the processes and representations involved.

In Activity 17, the comments made by Mi–Si demonstrate the type of language and the strategy most frequently used by this pair.

Activity 17: Trains (−2, −4, −5)/machines (unload[−2])/ Tracks (−8, −10, −12)

1 Si: Look, I have to take this and put it, that will be nine. Take...put minus two and then two is a nine.
2 (Si points at the −5 white train and explains how he obtains −9 by using the unload machine twice.)
3 (pause) Take two out. Look, if you put this out right there, and put that right there it will take...and I keep on adding to it until it get to four. And eight...
4 Mi: You can’t get five. Three...if you keep...if you add on two in front of the nine you going to eleven.
5 Si: Unless you add, keeping adding nine...add nine...look, don’t, don’t use that yet. Just have this one go to ten. And you keep on adding two on nine, two more...Oh, we can’t do that. There this one can fit in both of them, in way.
6 Mi: Because the six, now you have two more times because it tell you add it on. If you add it on three more time it comes to twelve.
7 E: Ok, so which on you want to do it in. You want to make it ten or twelve?
8 Si: Twelve. Wait, if you add two more onto this it will be nine, twelve into nine is...eleven. You put eleven use the one. Because this one wouldn’t fit in. And none of them because uh, you take this and put it in there you’ll have...nine. And you will be short a bumper. You add two more to nine and it’s eleven. And uh,...you’ll be uh,...eleven, yeah, then it will total, uh, thirteen.
9 E: Ok. So we can’t make that last one work. How this unload machine works with the white trains?
10 Mi: Well, on the gray train it takes them off and on the white trains it add them on.
In Lines 1–4, and 12–15, Si clearly uses exemplification language. He has difficulty in predicting that the −5 train cannot fit on any of the tracks (after being operated on by the unload machine). It is only in Line 15 that he appears to realize this impossibility by reasoning with objects of the system. Mi is more successful in her predictions. She also provides causal explanations about the functioning of the system, “you add on two in front of the nine” in Line 5; “on gray trains, it takes them off” in Line 17.

Planning component activities demonstrate more strongly the differences between Mi–Si and Ash–Sh. Ash–Sh in this phase progressively used more rules expressed in mathematical language; Mi–Si continue to use Action strategies and exemplification language, even to the very last activities:

Activity 40: (Strip(?→unload[4]→Track[2 8–3]))

1 Si: Four…. If you take a regular four, it will give…it will subtract four off. No, uh…seven.
2 E: Seven. Ok, we will run it.
3 Mi: Which one…what’s seven?
4 Si: Regular…Oh! Oh! I meant minus seven.
5 Mi: If you take a five…I know another way. I know one that works. If you take the negative five train.
6 And put it in the unload train, it will become a one train.
7 Si: Oh, it adds on.
8 E: Wait. Mi, I’m sorry. Can you explain that again. Start with five…
9 Mi: A regular became…And then you take four off. So it’s a regular one train. And then you take four more off, and it will be minus three. That’s what I would do.
10 E: See you said five put in there, and then the minus…So how you reach the solution?
11 Mi: Cause I did…I add a regular one and then take four off the two and got a three train.

Activity 40 demonstrates this pair’s failure to use Rules: In Lines 1 and 4, Si fails to use the numbers in the task to come up with a solution. Mi proposes +5 (Line 5), perhaps by using $4 - 3 = 1$ and $5 = 4 + 1$ (Line 14–15). Mi then refers to the different actions that may be performed on a train to make it become a negative train. She speaks in terms of steps or stages of actions and in terms of causes and effects on a train. When she is asked how she solved the problem, she reiterates the causes and effects that the machines would have on the train (Line 12).

There seem to be some social and interaction factors affecting the type of conversations that Mi and Si had. Si was the subject with the poorest performance on the pretest tasks; he seemed to have rather weak math skills. He was, however, very excited to be working on a computer task. Therefore, he may have preferred to construe the entire experiment as a computer game rather than as a math project. This might be a factor in his descriptions of the Planner; such descriptions were
very closely linked to the game and not to generalizations about numbers. Part of Mi's language seems to be directed at Si in an effort to help him understand what is going on. We reported that Mi was a good subject on both the pretest and the posttest. She also appears to be a very careful thinker and did not like to guess what was going on in the Planner. Rather, she would often pause for a minute, apparently to try to figure things out. Siegler (1988) referred to a group of students who performed well on math tasks, yet took a long time to solve problems as perfectionists. Mi seems to qualify as a perfectionist on our task. Therefore, it may be quite sensible that this pair nearly always talked about the exemplification component and causal relations. On the strategy side, actions are very often used. These strategies were perhaps the most game-like, and they interested Si the most. But other types of strategies were also successfully used by this pair.

7.3. An Interpretation of the Possible Effects of the Training Sessions on Children's Problem-Solving Processes in Formal Tasks and Story Problems

The performances on the formal tasks and the story problems indicate that transfer of knowledge occurred. We propose an interpretation of this successful transfer in the light of the data gathered during the training sessions and from theoretical findings in discourse analysis (Kintsch, 1986). Kintsch recognized three general levels of analysis for the process of solving word problems. First, the text base (i.e., a mental representation of the text) was formed during the process of comprehension. This was built up of propositions and expresses the semantic content of the text. Second, a situation model was formed; this was a mental representation of the situation described in the text. In constructing that situation model, children choose an arithmetical structure to solve the problem. Nathan, Kintsch, and Young (in press) refined this stage by defining a problem model (i.e., the subprocess of choosing formal relations to match the problem). We argue that during the posttest phase of the study, the children were able to construct correct problem models by mapping the posttest situation to a representation of a planner situation or by using a mental model derived from the intermediate abstraction.

Each of the tasks asked in the tests and the training sessions had a given referent. The possible referents were formal, time, elevator, temperature, debts, train, and planner (the last two for the training sessions); we adopt the generic notation T(ref) for each of the tasks, ref being one of these seven referents. With that notation, we argue that the process of solving tasks is sketched as follows:

\[ T(\text{ref}) \rightarrow (T'(\text{intermediate abstraction or mental model derived from it})) \rightarrow \text{solution} \]

In other words, when given a problem T(ref), the student mapped it to an internal representation (some form of the intermediate abstraction or a mental model derived from the intermediate abstraction). Mapping the situation to the
intermediate abstraction directly and not reasoning with a mental model means that the student may be bound to the objects of the exemplification component. Choosing $T'$ identical to $T$ means having a correct problem model. Finally, the unknown is obtained. The central portion of the solution (between previously shown curled brackets) may be skipped when the child is familiar with the referent. We also argue that students' familiarity with the referent of a problem figures strongly in whether they successfully build an appropriate situation model. If the mapping hypothesis is right, language and strategies used during the training sessions by the two pairs are broadly present during the posttest. Moreover, students having difficulties understanding actions of a particular referent fail to map a situation described in a particular task to a situation taken from the Planner or derived from it. The mapping hypothesis is examined separately here for each of the students.

**Evidence for a Mapping (Situation→Intermediate Abstraction)**

**The case of Ash.** The strongest evidence of a good mapping between formal tasks or word problems and a mental model of negatives (derived from working with the intermediate abstraction) was the way in which children, especially Ash, applied the strategies learned during the training sessions to the posttest. Ordinality, Action, and Rules strategies, which where extensively used by Ash in the posttest (see Figure 17), were developed during the training sessions. For example, Ash generated the Nul strategy in Activity 28 for the first time (see Section 7.2) and then used it in numerous occasions during the posttest (as detailed in Section 6.2).

**The case of Sh.** Unlike the other children, Sh came into this study with considerable knowledge about negative numbers. He had a set of rules for adding and subtracting negatives that his father had taught him. He performed very well on the pretest, particularly on formal problem (perhaps because these were the types of problems he had experience with). He had more difficulty on story problems, he appeared unable to map his rule-based knowledge to different situations (as in the example from the posttest that follows). His performance improved on the posttest for both formal and story problems, which indicates that he did learn more about negatives from the study and that he did develop a variety of strategies during the training sessions. However, his explanations on the posttest (when he gave them) were still very rule based. His difficulties, when they arose, appeared to stem from a lack of an ability to map his rules to the situation at hand. The excerpts of Sh's protocol for a time problem indicated a lack of familiarity with the nature of the quantities involved. For example, Sh asked, “How many B.C. do it have [sic] to before it can get to A.D.?” This lack of familiarity hinders the formation of a correct map between the situation and the intermediate abstraction.

**The case of Mi.** An interesting difficulty arose early on in Mi's formal posttest. The first six subtraction problems were presented, and Mi performed
poorly. She solved \(-3 + 2 = -5\), \(3 + (-7) = -10\), and \(-3 - 2 = -1\), but Mi did not seem confident or happy with her answers. Then, as the experimenter asked her to explain her reasoning on the sixth of these problems, she asked, "Is this related to the computer?" After she was told that she could think about the computer sessions if it helped, she performed very well and corrected her earlier errors. For some of the problems, she explained which machines and trains she had used to come up with her answers. This is some evidence that Mi needed to be linked to the intermediate abstraction to solve the formal problems.

For the word problems, she did not directly use the objects of the intermediate abstraction but apparently did use a mental model of negatives that she had constructed from her experience with the intermediate abstraction. For instance, \((\text{post, temp, 5})\), a temperature problem, (see Mi's protocol as an example of the L strategy in 6.1) revealed what may be Mi's interestingly configured idiosyncratic thermometer. She appeared to have a V-shaped thermometer in mind. Negative 13 is up from \(-5\) because it is colder. But she operates well within this configuration. She does not appear to have a discontinuous thermometer in mind, but rather the negative portion is tilted upward. The formation of such a representation is understandable, given the way one creates negative and positive trains in Trainworld. To create a negative train, one drags the mouse to the left; to create a positive train, one drags to the right. The farther you drag, the greater the absolute value of the train. On the thermometer, Mi appears to use height on either side of the V to determine cold or heat. Mi's successful solution to the altered B.C.–A.D. problem (presented in 6.2 with smaller numerical values) can be understood in the same manner: Given any word problem, Mi first maps it onto her understanding of the intermediate abstraction actions and quantities. There are then familiar intermediate abstraction situations that she can draw on, and she solves the novel problem with her mental model of the intermediate abstraction. For formal problems, she appears to "remain in the Planner" because there is no need to reinterpret the results. Using a mental model derived from the Planner can sometimes have interesting consequences. For example, Mi used both money and trainworld to explain the different ways that she could answer the "circle the largest number" questions. She explained that the answers would be different, depending on whether you were thinking of trains or money. Indeed, a \(-6\) train is longer than a \(-4\) train. But if you only owe 6 dollars, you have less money (in a sense) than if you owe 4 dollars. We believe that it is the role of further instruction to address these outcomes (in this case, by dealing with the concept of absolute value).

The case of Si. As during the training sessions, Si generally used an exemplification-bound language during the posttest. A large difference between Mi and Si is that when each attempts to create a link between a referent and the system, Si often does not form any situation model at all and consequently does not choose a problem model well matched to the task. An example is Si working through the task \(-3 + ? = -1\) (Si answers +4):
Si: The way I thought about this. Minus three plus minus four is seven white. But if you uh, use a regular four, four is bigger so it will go inside that and it will take three pieces off...

Si here uses a connection strategy: He knows that the solution is \(-3 -1\) or "something like that."\(^6\) He tries \(-4\) and finds that it is incorrect (he gets a \(-7\)). He then inverts the attempt into \(+4\). The way he checks this attempt is in fact correct, but he fails to recognize that he needs to obtain \(-1\) and not \(+1\). Such behavior occurs relatively frequently: Si chooses an incorrect planner situation, reasons correctly within the intermediate abstraction, and fails to recognize the mismatch between the planner solution and the solution to the task at hand. Similarly, his take away strategy example in Section 6.1 shows that to find the difference between the temperatures \(-6\) and \(+5\), he "takes five pieces off the \(-6\)" and obtains \(-1\). Si may fail to see temperatures as extensive quantities, and thus he chooses the wrong model to solve the task. The same phenomenon also occurs for time problems, not only for Si but also for Sh.

In summary, the analysis of the posttest showed consistency with the strategies used during the training sessions for each of the students. The transfer tasks in the posttest indicate that children continued to use the same strategies and types of language that they had used in the training sessions. Ash and Sh appeared to reason problems out by creating a problem model derived from their work with the intermediate abstraction. However, they quickly stopped referring to the objects of the system. Si and Mi overtly used objects of the intermediate abstraction to solve tasks with different referents. Mi–Si used language bound to the exemplification component that they worked with during the training sessions. When given word problems, Mi stopped using the exemplification language and was able to reason correctly with various referents. Therefore, her reliance on the model during the training sessions appears to be more a learning-style choice than an incapability to infer general properties from the intermediate abstraction. Si was often unsuccessful in mapping his intermediate abstraction knowledge to new situations. He often chose an incorrect intermediate abstraction situation to model tasks. His weak arithmetic skills may also have interfered with his development of negative number knowledge.

8. CONCLUSIONS

The mapping hypothesis is confirmed: When solving problems about negatives, three out of the four students created problem models derived from the intermediate abstraction and mapped them correctly to the situations described in their tasks. The weakest student (Si) tried (often unsuccessfully) to directly map situations in word problems to objects of the planner. In Section 3, we saw that each of the machines of

\(^6\)There are actually four possible solutions using this strategy: \(-3 - 1 = -4\), \(-3 + 1 = -2\), \(3 - 1 = 2\) and \(3 + 1 = 4\).
the intermediate abstraction embodied a specific sense of negatives. Consequently, each student's successful completion of the set of activities tended to indicate that the different senses of negatives were learned during the training sessions and that students were able to map their knowledge to various referents by creating correct problem models.

The results of the empirical study conducted with the Planner raise questions of what may be necessary features to include in designing models to guide learning of mathematical concepts that cannot be learned informally. Characteristics that we hypothesize to be important in guiding students to successfully work with these models are: completeness, linkability, neutrality, and self-evidence.

Completeness

The system exemplifies all the senses of the concept to be acquired. It allows children to develop a language that can embody all of the aspects of this concept and subsequently, to refer to all its possible referents. In Ohlsson's terminology (1987), it was a semantic field. Researchers must conduct a careful epistemological analysis of the knowledge area to be taught. For negative numbers, we needed to express the senses of negatives and of the operations on them and then to embody them in the objects of the intermediate abstraction.

Linkability

This is a feature borrowed from White (in press); she writes, "Intermediate models provide the foundation for a coherent knowledge structure by linking different levels of abstraction and different model structures." For the Planner, the sequence of tasks was designed to guide the children through increasingly higher levels of abstraction. The first tasks used the exemplification component, then the planning component was introduced, and finally, the caboose tasks used the cutting machine with the knife outside to come up with the most difficult tasks.

Neutrality

The representations used should not allude to any particular referent. In the Planner, although numbers were represented by trains and machines, the representation was neutral in the sense that the model was not bound to a specific situation (in contrast with early models of negative numbers discussed in Section 2). Bassock and Holyoak (in press) showed that generic representations facilitate the transfer of models to multiple contexts.

Self-Evidence

The objects of the system and their behavior should make sense. For example, when working with the objects of the intermediate abstraction (even from the very first activities), the students successfully manipulated objects and predicted and explained their behavior. Directed magnitudes and some operations with them may be grounded,
we believe, in experiences children have with principles of symmetry. The principle of symmetry is embodied in various forms (e.g., objects having two identical sides, such as the human body) and in various actions (e.g., folding and unfolding, going forwards and backwards). These different experiences with symmetry may be anchors for children's understanding of the behavior of trains and machines. The features listed previously, we believe, contribute to foster the acquisition of knowledge that cannot be learned informally, although children manipulate objects of an intermediate model and talk about it. Such an approach enhances flexible expertise and fosters the development of generally applicable mental models.

ACKNOWLEDGMENT

Preparation of this article was supported by the National Science Foundation Grant No. MDR-8850703 for the project entitled “Intelligent Microworlds for Scaffolding Number Concept Development.”

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### APPENDIX A

#### The Senses of Negatives and Operations Upon Them

**R1.** Negatives as directed magnitudes.

**R2.** Addition and subtraction as binary operations on directed magnitudes (combining and partitioning directed magnitudes):

<table>
<thead>
<tr>
<th>Rules</th>
<th>Conditions</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r_{2,1} ): (+a + +b = +(a + b))</td>
<td>( +3 + +2 = +5 )</td>
<td></td>
</tr>
<tr>
<td>( r_{2,2} ): ( -a + +b = -(a - b) )</td>
<td>when ( b &lt; a ) ( -3 + +2 = -1 )</td>
<td></td>
</tr>
<tr>
<td>( -a + +b = +(b - a) )</td>
<td>when ( b &gt; a ) ( -2 + +3 = +1 )</td>
<td></td>
</tr>
<tr>
<td>( r_{2,3} ): (+a + -b = +(a - b))</td>
<td>when ( b &lt; a ) ( +3 + -2 = +1 )</td>
<td></td>
</tr>
<tr>
<td>(+a + -b = -(b - a))</td>
<td>when ( b &gt; a ) ( +2 + -3 = -1 )</td>
<td></td>
</tr>
<tr>
<td>( r_{2,4} ): (-a + -b = -(a + b))</td>
<td>( -3 + -2 = -5 )</td>
<td></td>
</tr>
<tr>
<td>( r_{2,5} ): (+a - +b = +(a - b))</td>
<td>when ( b &lt; a ) ( +3 - +2 = +1 )</td>
<td></td>
</tr>
<tr>
<td>(+a - +b = -(b - a))</td>
<td>when ( b &gt; a ) ( +2 - +3 = -1 )</td>
<td></td>
</tr>
<tr>
<td>( r_{2,6} ): (-a - +b = -(a + b))</td>
<td>( -3 - +2 = -5 )</td>
<td></td>
</tr>
<tr>
<td>( r_{2,7} ): (+a - -b = +(a + b))</td>
<td>( +3 - -2 = +5 )</td>
<td></td>
</tr>
<tr>
<td>( r_{2,8} ): (-a - -b = -(a - b))</td>
<td>when ( b &lt; a ) ( -3 - -2 = -1 )</td>
<td></td>
</tr>
<tr>
<td>(-a - -b = +(b - a))</td>
<td>when ( b &gt; a ) ( -2 - -3 = +1 )</td>
<td></td>
</tr>
</tbody>
</table>

**R3.** Addition and subtraction as unary operator (actions changing directed magnitudes).

With this new definition of addition, it is possible to reformulate the above rules by replacing \(+ / - b\) (directed magnitude) by \(+ / - b\) (action). For example, the rule \( r_{2,4} \) can be modified to:

\[ r_{3,4}: -a + (-b) = -(a + b) \]

**R4.** Directed magnitudes as comparable entities (relation of order):

\( r_{4,1} \): if \( a < b \) then \( +a < +b \)

\( r_{4,2} \): if \( a < b \) then \( -b < -a \)

\( r_{4,3} \): \(-a < +b\)

The relation of order induces a metric:

\( r_{4,4} \): if \( a < b \) then \(+b = +a + (b - a)\) \((b - a)\) is the difference

\( r_{4,5} \): if \( a < b \) then \(-b = -a - (b - a)\) \((b - a)\) is the difference

\( r_{4,6} \): \(+b = -a + (a + b)\) \((a + b)\) is the difference
The Strategies Used During the Experiment

Nothing (No)

Subject Si for ordering the numbers (-2, -3, 0, -1) (Pre, formal, 1)
   Si: Well . . . they're all zeros.
   E: They're all zeros. What do you mean.
   Si: It's minus in front of it.
   E: So that means it's zero.
   Si: Minus one is zero.
   E: Ok. So you can't put these in order?
   Si: Not really.

Si for the problem: It was five degrees yesterday, and the temperature went down twelve degrees overnight. What is the temperature today? (Pre, temp, 1)
   Si: Zero degrees. Because if it was five degrees and it went down twelve degrees, there wouldn't be no more degrees.

Connections (Con)

Si for ? + 5 = 1 (Post, formal, 9), correct answer -4.
   Si: Four . . . Minus four and five. If you put a regular four it will be nine. So since that is the subtract sign it would subtract that and to that and there would be one.

Ash for Activity 28: Strip (? - unload[-8]--- 5)
   Ash: I'm not sure, but I know it's a minus three or regular three . . . Cause eight minus five . . . five minus eight. Eight minus five is three. Five plus three is eight. So, I knew that . . . I know it has to be a minus three or regular three.

Symmetrization (Sym)

Ash for -3 - 2 = ? (Post, formal, 8), correct answer -5.
   Ash: I knew that both of them were minuses, I mean negatives. I imagined that they were both just regular three's and two's. A regular three and a regular two as three plus two is a five.

Inverse (Inv)

Ash for 4 - 6 = ? (Post, formal, 3), correct answer -2.
   Ash: Oh, um . . . first I was, I did six. I put the problem backward and that was six minus four. And that was a regular two, but since the problem was the other way, I did a minus two.

Dynamic (Dyn)

Ash for -3 + 2 = ? (Post =, formal, 6), correct answer -1 [Ash writes the correct answer].
   Ash: I knew that if I added on a regular two then the minus would get smaller.

Taking away (Tk)

Mi for 5 + ? = -4 (Post, formal, 15), correct answer - 9
   Mi: Uh, a nine. Five plus . . . a minus nine.
   E: Ok. How did you think of that?
   Mi: With trainworld . . . if you have a regular five train and you take nine off it goes to a minus four.