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Assortment-based cooperation between two make-to-stock firms

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Cooperation can potentially improve competitiveness and profitability of firms with limited resources and production capacities. This article presents a continuous-time Markov chain model to study an assortment-based cooperation between two independent firms with limited capacity. An assortment-based cooperation is an agreement to combine the product assortments of two firms and offer the combined assortment to each firm’s customers. Both centralized and decentralized cooperations are studied. In a centralized cooperation, firms jointly make replenishment decisions, whereas in the decentralized case, firms operate under independent base stock policies and manage product exchanges through a discount-based contract where each firm supplies its own product to the other firm at a discounted price and at an agreed-upon fill rate. Under this scheme, assortment-based cooperation also mandates each firm to effectively ration their inventories since they have to deal with two different demand streams. The discount-based contract yields the results of the centralized operation by using specific values of the contract parameters. It is shown that assortment-based cooperation is always beneficial for two symmetrical firms in both centralized and decentralized cooperation. Numerical experiments reveal that assortment-based cooperation is not always beneficial if the firms are not symmetrical.

[Supplementary materials are available for this article. Go to the publisher’s online edition of IIE Transactions for Proofs of all Propositions.]

Keywords: Cooperation, customer choice, assortment pooling

1. Introduction

Small- and Medium-sized Enterprises (SMEs) constitute an important segment of the world economy. It is well known that the vast majority of enterprises in developed economies fall into the SME category. In Europe, SMEs effectively contain 99% of all firms and account for roughly 50% of employment (Doole and Lowe, 2008). Similarly, the private sector is almost entirely composed of SMEs in emerging and developing economies. According to a report by the Organisation for Economic Co-operation and Development (OECD, 2004), 90% of all enterprises in emerging economies can be classified as SMEs. Today, increasing global competition is forcing SMEs to rethink their existing strategies and explore new ways to improve their competitive ability. Competition among rival firms is traditionally viewed as a win–lose situation. However, cooperation can potentially improve competitiveness and profitability of firms with limited resources and production capacities in a win–win manner. By coordinating procurement, production, logistics, and marketing activities through cooperation, firms can benefit from possible cost savings, demand pooling, capacity utilization improvements, and flexibility enhancements. Although there are numerous cases of thriving cooperations (see Johnston and Lawrence (1988), Park (1997), Kumar et al. (1998), Mezgár et al. (2000), Lin (2004), and Ergun et al. (2007) for examples in various industries), there is also evidence indicating that cooperation does not always benefit participating firms and some might indeed be doomed to fail (Park and Russo, 1996; Schmitz, 1999; Park and Ungson, 2001). Therefore, identifying key factors for successful and sustainable cooperation is an important research issue with significant practical implications.

In this article, we focus on a specific form of cooperation referred to as assortment-based cooperation. Assortment-based cooperation is essentially an agreement among a group of firms to combine their product assortments and offer this combined assortment to their own customers. This particular type of cooperation is a common practice in the textile and apparel industry in Turkey (dominated by approximately 40,000 SMEs) which is considered as the backbone of the Turkish economy (according to the

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World Trade Organization statistics for 2009, Turkey’s total textile exports are valued at 8 billion U.S. dollars, which ranks it seventh in the world with a share of 3.7%). International textile fairs and trade shows constitute the foremost medium for Turkish textile firms to collect orders in this fiercely competitive sector. In order to achieve greater visibility by participating in a larger number of such fairs, it is customary for a single firm to represent a group of firms at a fair that might be prohibitively expensive for some SMEs to attend (typical participation fees around $500 per square meter). This particular firm exhibits the catalogue and product samples of all cooperating firms at its own stand and in return takes a commission from each collected order (for a product of another firm). The furniture industry in Turkey (consisting of more than 10,000 SMEs with total exports exceeding $1.3 billion in 2008) also has examples of assortment-based cooperation. A group of furniture firms typically agrees to show all of the models produced by the members of a cooperative at their own showrooms. Although firms maintain their independent production facilities, they benefit from greater exposure of their products to a larger customer base.

We model the salient characteristics of a firm with limited production capacity as a make-to-stock system with a single server. We assume that each firm produces a different product as its assortment (although we can show the validity of our results for multi-product assortments, we focus on a single-product model for the clarity of our presentation) and uses a base stock policy for inventory replenishment. If the first choice of an arriving customer is the same as the product that is produced by the firm, or if the customer is willing to substitute her first choice with the firm’s product, an order is received. A received order, on the other hand, is fulfilled only if there is a product available in inventory and otherwise is lost when inventory is empty. In essence, only a portion of the customer demand is translated into actual sales while the remainder is lost due to either the mismatch between the producer’s assortment and the customer’s choice or due to stockout. At this point, assortment-based cooperation acts as a means for the firm to increase the likelihood of converting demand to sales by extending its assortment. Accordingly, when a customer inquires for her first choice product, the order can potentially be fulfilled when this product is either available in the assortment of the firm itself or the assortment of another firm in the cooperative or the customer is ready to substitute with a product in the combined assortment. We consider two different schemes in which cooperation can be established—centralized and decentralized. Centralized cooperation requires both firms to make their inventory replenishment decisions jointly to maximize their aggregate expected profit, whereas under decentralized cooperation each firm makes its replenishment decision independently to maximize its individual expected profit. In our decentralized model, we assume that the two firms manage all product exchanges through a discount-based contract in which each firm supplies its own product to the other firm at a discounted price and at an agreed-upon fill rate. In other words, if a firm sells the product of the other firm to its own customer, it keeps part of the other firm’s profit as a commission. Under the discount-based contract, a decentralized-assortment-based cooperation also might require each firm to effectively ration their inventories since they have to deal with two different demand streams after cooperation. We show that the discount-based contract yields the benefits of the centralized system by selecting appropriate values for the contract; i.e., the discount rates and the fill rates.

We measure the net effect of cooperation on the profitability of the two firms as the difference between their total expected profit rate when they form an assortment-based cooperation and their total expected profit rate when they operate independently. In addition, to capture the impact of capacitated production, we propose two different service levels as performance measures. The firm-based service level is defined as the probability that a customer visiting a particular firm ends up purchasing a product (any product), and the customer-based service level is defined as the probability that a customer whose first choice is a particular product ends up purchasing the product (without having to go through any substitutions). Cooperation clearly alters the demand traffic intensity that each firm faces. Depending on intricate substitution effects, governed by a wide range of problem parameters, this can be either beneficial or detrimental for the firm. The traffic intensity also impacts the base stock levels, average inventory levels, and order fill rates, making the evaluation of potential benefits from an assortment-based cooperation a challenging task. We present an analytical model to show a number of structural properties of a firm’s traffic intensity, average inventory, expected profit rate, and service levels. We also prove that assortment-based cooperation is always beneficial if the two firms are symmetric (have identical parameters). Moreover, symmetrical firms and their customers enjoy better service levels under cooperation. Firms should understand how key problem parameters affect the outcome of an assortment-based cooperation to decide whether or not to cooperate with another firm and to choose a fitting firm with which to cooperate. Since such a study is analytically intractable, we rely on our extensive numerical experiments to generate insights.

There is extensive literature on cooperation focusing on joining forces at different stages of a supply chain. Cooperation in procurement entails firms to essentially combine their replenishment orders from a single supplier. The obvious benefit of joint ordering is to take advantage of better prices (through quantity-based discounts) and sharing of fixed ordering costs. Gerchak and Gupta (1991), Meza et al. (2004), and Dror and Hartman (2007) study economic order quantity models to show that by placing their orders simultaneously, firms can reduce their total cost compared with the total cost in which they all order separately, mainly because of the lower total number of orders. Heuvel et al. (2007) and Guardiola et al. (2009) extend this work to discrete-time economic lot sizing models with finite time
horizon and time-varying demand. Müller et al. (2002) and Hartman and Dror (2005) consider single-period newsvendor-type models in which different firms combine their orders for a single product with a single supplier. In the newsvendor setting, Slikker et al. (2005) study a cooperation scheme in which firms jointly place their orders with a supplier but in addition allow transshipment of inventories among firms in the cooperation, essentially pooling their inventories. Using stochastic models, Diks and De Kok (1996), Rudi et al. (2001), Axsaeter (2003), Agrawal et al. (2004), Wee and Dada (2005), and Zhao et al. (2006) similarly focus on inventory ordering and balancing implications of transshipment. Benjaafar et al. (2005) investigate the overall value of pooling for a production/inventory system with multiple inventory locations, each receiving its own demand stream and operating under an independent base stock policy. When the inventory is pooled, these individual demand streams are satisfied from a single location. Yu et al. (2006) address the situation in which firms decide whether to operate their own production/inventory facilities or to enter a cooperation by sharing a facility with another firm. On the other hand, cooperation in marketing and sales activities, generally referred to as co-marketing, involves contractual relationships undertaken by firms whose respective products are complements in the market. Through marketing cooperation, firms reduce marketing costs by bundling their products, and at the same time increase customer loyalty by addressing their customers’ needs with offerings from the partner firm. Using empirical data, Bucklin and Sengupta (1993) identify situations in which marketing cooperations are most successful and Swami-nathan and Moorman (2009) explore whether such cooperations indeed add value to the firms. Assortment-based cooperation is a cooperation scheme that spans inventory pooling and co-marketing. Firms improve the utilization of their production capacities through additional demand streams and need to satisfy the demand of the other firm, essentially treating their own stock as pooled inventory. Moreover, assortment-based cooperation involves selling a firm’s own product side by side with another firm’s product, which are substitutes, in contrast with the traditional approach in a co-marketing alliance selling complementary products.

Akcay and Tan (2008) examine an assortment-based cooperation among independent firms and show that such a cooperation scheme between symmetric single-product firms is always beneficial, whereas a threshold-type criterion should be satisfied so that assortment-based cooperation is beneficial for asymmetric firms. However, their model inherently assumes that firms do not have any capacity limitations (i.e., they do not have an explicit model for the manufacturing operations of the firms) and the system is not affected by variability in demand and capacity. As a result, they ignore the effects of an assortment-based cooperation on customers due to limited capacity. Since this particular cooperation scheme is essentially intended as a competitive strategy for SMEs, for which capacity limitations are among their most critical shortcomings, capacitated production needs to be an explicit feature of the model. Furthermore, the effects of variability in supply and production must also be studied to understand the potential benefits of assortment-based cooperation. In this study, we extend the model in Akçay and Tan (2008) to capture the effects of capacity limitations on benefits of cooperation in a stochastic setting. When firms have finite production capacities, they need to manage their product inventories carefully to satisfy both their own orders and those from the partner firm. While capturing the dynamics of such a production process, our objective is to identify the situations in which an assortment-based cooperation always benefits firms and to understand the impact of key problem parameters such as utilizations of the production facilities of firms, firm sizes, product market shares, and substitution behavior of customers on the benefit of cooperation. Furthermore, we investigate how assortment-based cooperation changes the service levels, which are of critical importance in a capacitated production environment.

There are three main contributions of our study. First, we present a detailed analytical model of assortment-based cooperation between two firms with limited production capacities. Our main result shows that assortment-based cooperation, which can be implemented using a simple discount-based contract, is effective only when firms with the right characteristics cooperate and the discount-based contract yields the potential benefits of the centralized system by selecting the contract parameters appropriately. In the process, we also introduce customer-based and firm-based service levels as critical performance measures in assessing the trade-offs involved with assortment-based cooperation when firms are capacitated. Second, our study incorporates customer behavior in a traditional make-to-stock production system model. Since the products are substitutable, individual product demands for each firm are linked through consumer choice processes. Finally, we derive general structural results for make-to-stock queues with and without rationing thresholds and lost sales, which can also be used in other problem settings.

2. Problem description

We consider two firms, each producing a different product in its assortment in a competitive market. In the market, there is a great deal of non-price competition, based on product differentiation. Suppose that the products in this particular market are good substitutes for each other, yet they are not perfect substitutes. In other words, they are sufficiently differentiated such that customers have clearly defined preferences. Each customer identifies her first choice based on her own valuations of the products available in the market.

Let us assume that Firm 1 produces Product A and Firm 2 produces Product B. We define $\alpha_A$ and $\alpha_B$ as the fractions
of customers whose first choices are Product A and Product B, respectively. Consequently, \(q_0 = 1 - \alpha_A - \alpha_B\) is the fraction of customers whose first choice is neither Product A nor Product B but some other product in the market. We denote this outside option for customers as Product 0. Suppose potential customers arrive to the market as a Poisson process and let \(\lambda\) be their arrival rate. We define \(q_1\) as the fraction of these customers who visit Firm 1 and \(q_2\) as the fraction of customers who visit Firm 2. In order to distinguish between the firm-based and product-based preferences of the customers, we refer to \(q_1\) and \(q_2\) as the size of Firm 1 and Firm 2 and \(\alpha_A\) and \(\alpha_B\) as the market share of Product A and Product B in the remainder of this article.

When a customer visits a firm, she attempts to purchase her favorite product. Empirical studies in the marketing literature suggest that a customer demonstrates a variety of reactions if her first choice is not available. She can switch to another product, buy the missing item in a competing store, defer the purchase to a next shopping occasion, or drop the purchase altogether (Campo et al., 2003). These studies also reveal that item switching (substitution) is the predominant reaction, whereas store switching and cancelling/deferring the purchase are less often observed (yet remain important issues as they may entail serious negative consequences for the manufacturer and/or retailer). Therefore, we assume that a customer might order an alternative product if her first choice is not produced by the firm. We refer to this substitution scheme as assortment-based substitution. Let \(\theta\) be the probability that a customer is willing to purchase her second choice if her favorite product is not produced by the firm she visits. Consequently, she does not make a purchase and leaves the firm with probability \(1 - \theta\). A study on the retailing industry shows that on average 40% of customers purchase a substitute product when they are faced with such a situation (Gruen et al., 2002). We define \(\delta_{ik}\) as the probability of a customer substituting product \(k\) for product \(i\). Note that, with this definition, it is also possible for consumers whose favorite product is 0 to substitute Product A or Product B. When the substitute product is also unavailable, consumers decide whether to substitute again or alternatively not to purchase. However, for analytical tractability, a common assumption in the assortment planning literature is that only a single round of substitution takes place. If the second choice product is also not produced by the firm, then the consumer decides not to purchase. Smith and Agrawal (2000) and Kok and Fisher (2007) mention that the dynamics of multiple rounds of the substitution procedure can be approximated with a single-round substitution model by selecting a larger value for the substitution probability \(\theta\). There are various ways of expressing \(\delta_{ik}\) in order to reflect different probabilistic substitution mechanisms (see Smith and Agrawal (2000) and Kok et al. (2009) for a selection of these schemes). In this article, we adopt a proportional substitution scheme in which the rate of substitution of a product is proportional to its expected demand. Hence, we express \(\delta_{ik}\) as \(\theta\alpha_k/(1 - \alpha_k)\).

We model the production facility of each firm as a make-to-stock system with a single server. The production times are exponentially distributed mean \(1/\mu_1\) in Firm 1 and mean \(1/\mu_2\) in Firm 2. Accordingly, Firm 1 and Firm 2 use independent base stock policies to manage their inventories and set their own base stock levels. Production in each firm continues as long as the finished goods inventory level is below the base stock level and stops when the finished goods inventory reaches the base stock level. A received order is fulfilled only if the requested product is available in the inventory. Otherwise, the customer cancels the order to try again later. In essence, we consider a lost sales system for each firm. Furthermore, we assume that customers do not change their orders when they are faced with a stock-out situation. Although, we do not explicitly model stock-out-based substitution in this article, it is captured implicitly through Poisson arrival of customers that also includes customers who come back again after leaving the system due to not finding the product they want to buy in stock in their prior visits.

Let \(c_A\) and \(c_B\) be unit production costs of Product A and Product B and \(h_A\) and \(h_B\) be their corresponding unit inventory holding costs. The selling price of Product A is \(s_A\), whereas the selling price of Product B is \(s_B\). We denote the per unit profits of Product A and Product B as \(p_A = s_A - c_A\) and \(p_B = s_B - c_B\), respectively.

As a final note, we should mention that our model can be generalized to the case where each firm has multiple products in its assortment and each product has its own dedicated production process. We pursue the single-product model in this article and present our results accordingly, in order to avoid messy notation and to keep our analysis more tractable for the reader.

3. Independent firms without cooperation

In this section, we study the case in which Firm 1 and Firm 2 do not cooperate with each other and operate independently. Next, we derive the expected profit rate of Firm 1 as a function of its base stock level and its traffic intensity and then discuss performance measures and their structural properties. Derivations regarding Firm 2 are exactly the same as Firm 1 and the results can be obtained by changing the indices, and hence they are omitted for brevity.

Suppose Firm 1 operates as a make-to-stock queue with traffic intensity \(\hat{\lambda}_1\) and maximizes its own individual profit by setting its base stock level \(S_1\). We denote the effective demand rate at the production facility of Firm 1 for Product A, including all substitutions to Product A, as \(\hat{\lambda}_A = \hat{\lambda}_1(\alpha_A + \alpha_0\delta_{0A} + \alpha_B\delta_{BA})\). Consequently, the traffic intensity of Firm 1, the ratio of the effective demand rate to the production rate at Firm 1, is given by \(\hat{\rho}_1 = \hat{\lambda}_A/\mu_1\) (the flow of customer orders is illustrated in Fig. 1). We define the expected profit rate of a firm as the difference
between the expected revenue generated through sales and the expected inventory holding cost. Orders received from customers whose first choice is Product A and from customers who decide to substitute their first choice with Product A only if at least one product is available in the inventory of Firm 1. Let us define the fill rate for Product A, \( \hat{f}_A(\hat{S}_1, \hat{\rho}_1) \), as the stationary probability that the inventory of Firm 1 is not empty. Then, the expected profit rate of Firm 1 can be written as \( \hat{\pi}_1 = \lambda A \hat{S}_1 \hat{\rho}_1 - h A I_A(\hat{S}_1, \hat{\rho}_1) \), where \( I_A(\hat{S}_1, \hat{\rho}_1) \) is the expected inventory of Product A.

Since we assume that customers arrive as a Poisson process and production times are exponentially distributed, we can model the production-inventory system of Firm 1 as a Continuous-Time Markov Chain (CTMC). Note that the dynamics of customer arrival and service ensures that the process is ergodic. We can determine the stationary probability distribution of the inventory level and service level of Product A by solving the balance equations of the CTMC associated with the production process of Firm 1. These results are based on the analysis of the underlying M/M/1/N queueing model (see Buzacott and Shantikumar (1993)).

Subsequently, the average inventory level of Product A and the fill rate at Firm 1 can be obtained by using this stationary distribution. Let \( \hat{X}_A(t) \) be the inventory level of Product A at Firm 1 at time \( t \). We compute the average Product A inventory at Firm 1 when the system is started in steady-state as

\[
\hat{I}_A(\hat{S}_1, \hat{\rho}_1) = E[\hat{X}_A(t)] = \sum_{x=0}^{\hat{S}_1} x P(\hat{X}_A(t) = x) = \frac{\hat{S}_1 + \hat{\rho}_1 \hat{S}_1^{\hat{\rho}_1+1}}{1 - \hat{\rho}_1} - \hat{\rho}_1. \tag{1}
\]

where \( P(\hat{X}_A(t) = x) \) is the stationary probability that the inventory level \( \hat{X}_A(t) \) is equal to \( x \), for \( 0 \leq x \leq \hat{S}_1 \). Similarly, the fill rate of Product A can be written as

\[
\hat{f}_A(\hat{S}_1, \hat{\rho}_1) = P(\hat{X}_A(t) > 0) = \frac{1 - \hat{\rho}_1^{\hat{S}_1 + 1}}{1 - \hat{\rho}_1}. \tag{2}
\]

In order to measure the performance of the system (aside from expected profit rates) before and after two firms engage in an assortment-based cooperation, we propose two different service levels. The firm-based service level is defined as the probability that a customer visiting a particular firm ends up purchasing a product (any product), and the customer-based service level is defined as the probability that a customer whose first choice is a particular product ends up purchasing the product at any firm without having to go through any substitutions.

Let \( \hat{\phi}_1 \) and \( \hat{\phi}_2 \) denote the firm-based service levels for Firm 1 and Firm 2 when they operate independently. Based on our definition of the firm-based service level, we can express \( \hat{\phi}_1 \) and \( \hat{\phi}_2 \) as

\[
\hat{\phi}_1(\hat{S}_1, \hat{\rho}_1) = (\alpha_A + \alpha_B \delta_{BA} + \alpha_0 \delta_{0A}) \hat{f}_A(\hat{S}_1, \hat{\rho}_1), \tag{3}
\]

\[
\hat{\phi}_2(\hat{S}_2, \hat{\rho}_2) = (\alpha_B + \alpha_B \delta_{AB} + \alpha_0 \delta_{0B}) \hat{f}_B(\hat{S}_2, \hat{\rho}_2). \tag{4}
\]

On the other hand, let \( \hat{\gamma}_A(\hat{S}_1, \hat{\rho}_1) \) and \( \hat{\gamma}_B(\hat{S}_2, \hat{\rho}_2) \) denote the customer-based service levels for Product A and Product B when the two firms operate independently. Then, \( \hat{\gamma}_A(\hat{S}_1, \hat{\rho}_1) = (q_1/(q_1 + q_2)) \hat{f}_A(\hat{S}_1, \hat{\rho}_1) \) and \( \hat{\gamma}_B(\hat{S}_2, \hat{\rho}_2) = (q_2/(q_1 + q_2)) \hat{f}_B(\hat{S}_2, \hat{\rho}_2) \), respectively.

In order to compare the performance of the firms when they operate independently with the case when they form an assortment-based cooperation, we investigate the impact of traffic intensity and the base stock level on the firm-based service level, expected profit, and service levels. We provide the proofs of the propositions in the Online Appendix to the article.

Our first proposition shows that Firm 1 can potentially increase the fill rate by increasing its base stock level. Note that the fill rate at Firm 1 is bounded from above by \( \min(1, 1/\hat{\rho}_1) \) (as \( \hat{S} \to \infty \), we have \( \hat{f}_A(\hat{S}_1, \hat{\rho}_1) \to 1/\hat{\rho}_1 \) if \( \hat{\rho}_1 > 1 \) and \( \hat{f}_A(\hat{S}_1, \hat{\rho}_1) \to 1 \) if \( \hat{\rho}_1 < 1 \)). Furthermore, increasing the base stock level has diminishing marginal returns on the fill rate. Proposition 1 also states that, as the traffic intensity of Firm 1 increases, the fill rate decreases, as expected.

**Proposition 1.** \( \hat{f}_A(\hat{S}_1, \hat{\rho}_1) \) is nondecreasing and concave in \( \hat{S}_1 \) and nonincreasing in \( \hat{\rho}_1 \).

Based on Proposition 1, we can claim that Firm 1 can also boost its firm-based service level, \( \hat{\phi}_1(\hat{S}_1, \hat{\rho}_1) \), and the customer-based service level of Product A, \( \hat{\gamma}_A(\hat{S}_1, \hat{\rho}_1) \), by setting a higher base stock level \( \hat{S}_1 \). However, there is only so much that Firm 1 can do to improve the service levels by adjusting its base stock level, as \( \hat{\phi}_1(\hat{S}_1, \hat{\rho}_1) \) is bounded from above by \( (\mu_1/\lambda q_1) \min(1, 1/\hat{\rho}_1) \), and \( \hat{\gamma}_A(\hat{S}_1, \hat{\rho}_1) \) is bounded from above by \( (q_1/(q_1 + q_2)) \min(1, 1/\hat{\rho}_1) \). As a direct consequence of increasing the base stock level of Product A,
the expected inventory level at Firm 1 also increases. On the other hand, for a given base stock level \( \hat{S}_1 \), the average inventory level of Product A at Firm 1 goes down when Firm 1 faces a larger traffic intensity.

**Proposition 2.** \( \hat{I}_A(\hat{S}_1, \hat{\rho}_1) \) is nondecreasing in \( \hat{S}_1 \) and nonincreasing in \( \hat{\rho}_1 \).

Using Propositions 1 and 2, we can show that under a higher traffic intensity, Firm 1 improves its expected profit at any given base stock level. The next proposition states this particular result.

**Proposition 3.** \( \hat{\Pi}_1(\hat{S}_1, \hat{\rho}_1) \) is nondecreasing in \( \hat{\rho}_1 \) for given \( \hat{S}_1 \) and \( \mu_1 \).

We can directly extend the above results to Firm 2 and Product B as well and show that \( \hat{f}_B(\hat{S}_2, \hat{\rho}_2) \) is nondecreasing and concave in \( \rho_2 \) and nonincreasing in \( \hat{\rho}_2 \), \( \hat{I}_B(\hat{S}_2, \hat{\rho}_2) \) is nondecreasing in \( \hat{S}_2 \) and nonincreasing in \( \hat{\rho}_2 \), and \( \hat{\Pi}_2(\hat{S}_2, \hat{\rho}_2) \) is nondecreasing in \( \hat{\rho}_2 \).

Now, suppose that Firm 1 and Firm 2 set their base stock levels independently to maximize their expected profit rates. Then, the maximum total expected profit of Firm 1 and Firm 2, when they operate independently, can be written as

\[
\hat{\Pi}^* = \max_{\hat{S}_1, \hat{\rho}_1} \hat{\Pi}_1(\hat{S}_1, \hat{\rho}_1) + \max_{\hat{S}_2, \hat{\rho}_2} \hat{\Pi}_2(\hat{S}_2, \hat{\rho}_2) = \max_{\hat{S}_1, \hat{S}_2} \left[ \lambda_A \hat{f}_A(\hat{S}_1, \hat{\rho}_1) + \lambda_B \hat{f}_B(\hat{S}_2, \hat{\rho}_2) - h_A \hat{I}_A(\hat{S}_1, \hat{\rho}_1) - h_B \hat{I}_B(\hat{S}_2, \hat{\rho}_2) \right].
\]

Based on the structural properties of the stock levels and fill rates we proved earlier, we next present our main result for the case in which Firm 1 and Firm 2 operate independently.

**Corollary 4.** \( \hat{\Pi}^* \) is nondecreasing in \( \hat{\rho}_1 \) and \( \hat{\rho}_2 \) for given \( \mu_1 \) and \( \mu_2 \).

Corollary 4 asserts that the maximum value of the total expected profit of Firm 1 and Firm 2 would improve when facing larger respective traffic intensities under optimal base stock levels.

4. Centralized cooperation

In a centralized assortment-based cooperation, firms essentially merge their product assortments to offer a wider range of products to their customers and set their base stock levels jointly to maximize the total profit of the system. Next, we present our analytical and numerical results for the centralized assortment-based cooperation.

4.1. Analytical results

In an assortment-based cooperation, we expect their effective demand rates to be different compared with the case in which they operate independently. Now let us consider Firm 1, in particular, to better understand the two core reasons for this change. Firm 1 potentially receives additional orders from customers who visit Firm 2 and demand Product A (available to Firm 2 customers after cooperation). On the flip side, Firm 1 may lose some of its own orders from customers who were initially willing to substitute their first choice with Product A (when Firm 1 was independent) but now decide on Product B (available to Firm 1 customers after cooperation). The net effect of these two oppositely directed factors depends on a number of key problem parameters such as product market shares, firm sizes, substitution probability, and the customer arrival rate. Clearly, Firm 2 faces a similar situation in terms of effective demand rates under an assortment-based cooperation.

Let \( \lambda_{A1} \) be the effective demand rate for Product A at Firm 1, including all substitutions to Product A when the two firms cooperate—i.e., \( \lambda_{A1} = \lambda q_1(\alpha_A + \alpha_0 \delta_A) \)—and \( \lambda_{A2} \) be the effective demand rate for Product A at Firm 2, including all substitutions to Product A; i.e., \( \lambda_{A2} = \lambda q_2(\alpha_A + \alpha_0 \delta_A) \). On the other hand, \( \lambda_{B2} \) denotes the effective demand rate for Product B at Firm 2, including all substitutions to Product B when the two firms cooperate—i.e., \( \lambda_{B2} = \lambda q_2(\alpha_B + \alpha_0 \delta_B) \)—and \( \lambda_{B1} \) denotes the effective demand rate for Product B at Firm 1, including all substitutions to Product B. The flow of customer orders is illustrated in Fig. 2. Accordingly, the traffic intensity at Firm 1 can be defined as \( \tilde{\xi}_1 = (\lambda_{A1} + \lambda_{A2})/\mu_1 \), and the traffic intensity at Firm 2 is defined as \( \tilde{\xi}_2 = (\lambda_{B1} + \lambda_{B2})/\mu_2 \). Then, the expected total profit rate of a centralized assortment-based cooperation is given by

\[
\Pi(S_1, S_2, \xi_1, \xi_2) = p_A(\lambda_{A1} + \lambda_{A2}) f_A(S_1, \xi_1) + p_B(\lambda_{B1} + \lambda_{B2}) f_B(S_2, \xi_2) - h_A I_A(S_1, \xi_1) - h_B I_B(S_2, \xi_2),
\]

where \( f_A(S_1, \xi_1) \) and \( f_B(S_2, \xi_2) \) are the stationary probabilities that the inventories of Firms 1 and 2 are not zero and \( I_A(S_1, \xi_1) \) and \( I_B(S_2, \xi_2) \) are the average inventory levels of the two firms.

Comparing Equations (6) and (2), and similarly Equations (7) and (1), we can express the total expected profit rate function in Equation (5) as \( \Pi(S_1, S_2, \xi_1, \xi_2) = \Pi_1(S_1, \xi_1) + \Pi_2(S_2, \xi_2) \). We summarize this observation in the following corollary.

**Corollary 5.** The optimal total expected profit of Firm 1 and Firm 2, generated under a centralized assortment-based cooperation, is equivalent to the sum of the expected profits of the two firms as if they were operating independently and facing higher traffic intensities (compared with what they would have faced if they actually operated independently) of \( \xi_1 \) and \( \xi_2 \), respectively.
As explained in Section 3, the average stock levels and fill rates of Firm 1 and Firm 2 in Equation (5) can be computed by solving the balance equations corresponding to the CTMC models associated with the production processes of the two firms as follows:

\[
\begin{align*}
I_A(S_1, \xi_1) &= \frac{S_1 + \xi_1 S_1^{\xi_1+1}}{1 - \xi_1 S_1^{\xi_1+1}} - \frac{\xi_1}{1 - \xi_1} \\
I_B(S_2, \xi_2) &= \frac{S_2 + \xi_2 S_2^{\xi_2+1}}{1 - \xi_2 S_2^{\xi_2+1}} - \frac{\xi_2}{1 - \xi_2}.
\end{align*}
\]

Consequently, the maximum expected profit rate of the firms under a centralized assortment-based cooperation is given by \(\Pi^* = \max_{S_1, S_2} \Pi(S_1, S_2, \xi_1, \xi_2)\).

We should note that the structural properties of the fill rates and expected stock levels that we derived for independent firms in Section 3 naturally extend to fill rates and expected stock levels under a centralized assortment-based cooperation. Furthermore, when Firm 1 and Firm 2 form a centralized assortment-based cooperation, we define the firm-based service level as the probability that a customer visiting a particular firm in the cooperation ends up purchasing either Product A or Product B (any product in the assortment of the cooperation). Let \(\varphi_1\) and \(\varphi_2\) denote the firm-based service levels of Firm 1 and Firm 2 under a centralized assortment-based cooperation. These service levels can be expressed as follows:

\[
\varphi_1(S_1, S_2, \xi_1, \xi_2) = \varphi_2(S_1, S_2, \xi_1, \xi_2) = (\alpha_A + \alpha_0 \delta_A) f_A(S_1, \xi_1) + (\alpha_B + \alpha_0 \delta_B) f_B(S_2, \xi_2).
\]

Let \(\gamma_A\) and \(\gamma_B\) denote the customer-based service levels for Product A and Product B, defined as the probability that a customer is able to purchase her first choice product, when the two firms form a centralized assortment-based cooperation. Subsequently, the Product A and Product B service levels can be written as follows:

\[
\begin{align*}
\gamma_A(S_1, \xi_1) &= \frac{q_1 - q_2 f_A(S_1, \xi_1)}{q_1 + q_2} = f_A(S_1, \xi_1), \\
\gamma_B(S_2, \xi_2) &= \frac{q_1 f_B(S_2, \xi_2)}{q_1 + q_2} = f_B(S_2, \xi_2).
\end{align*}
\]

Next, we present our analytical findings for a centralized assortment-based cooperation between two symmetrical firms. Accordingly, suppose Firm 1 and Firm 2 have identical parameters; i.e., \(q_1 = q_2 = q\), \(\alpha_A = \alpha_B = \alpha\), \(\mu_1 = \mu_2 = \mu\), \(p_A = p_B = p\), and \(h_A = h_B = h\). Akçay and Tan (2008) establish that assortment-based cooperation always benefits two symmetrical firms with unlimited capacities in terms of increasing the total expected profit compared with independent operations. Next, we extend this result to symmetrical firms with capacitated production facilities.

Note that when Firm 1 and Firm 2 have infinite capacities, we have \(\hat{\rho}_1 = \hat{\rho}_2 = \xi_1 = \xi_2 = 0\). Also, \(\hat{f}_A(\hat{S}_1, \xi_1) = \hat{f}_B(\hat{S}_2, \xi_2) = f_A(S_1, \xi_1) = f_B(S_2, \xi_2) = 1\), and \(I_A(S_1, \hat{\rho}_1) = \hat{S}_1\), \(I_B(S_2, \hat{\rho}_2) = \hat{S}_2\). Consequently, the two firms would set their base stock levels at zero. The ensuing model is equivalent to the model investigated in Akçay and Tan (2008) and we refer the reader to this particular study for a detailed analysis on the benefits of assortment-based cooperation for uncapacitated manufacturers.

**Proposition 6.** A centralized assortment-based cooperation is always beneficial for two symmetrical firms; i.e., \(\Pi^* \geq \hat{\Pi}^*\).

Moreover, assortment-based cooperation creates a win-win situation for firms and customers, since they both achieve higher service levels than what they would have if the firms were independent. Essentially, a centralized assortment-based cooperation not only acts as a profit improvement mechanism but also boosts up the firm-based and customer-based service levels.

**Proposition 7.** Under a centralized assortment-based cooperation of two symmetrical firms, the firm-based and customer-based service levels always improve.

Understanding the effect of each problem parameter on the benefits of assortment-based cooperation is of critical importance when firms search for answers for strategic decisions regarding whether or not to cooperate with another firm, as well as choosing the most suitable partner firm (in terms of firm characteristics) with which to cooperate. Since such a study is analytically intractable, we resort to numerical experiments to develop insights on the subject matter.
4.2. Numerical results

Here, we provide the results of our extensive numerical study that investigates the benefits of centralized assortment-based cooperation for symmetrical as well as asymmetrical firms.

4.2.1. Symmetrical firms

We use a wide range of test problems in order to assess the potential benefits of assortment-based cooperation, which we measure in terms of the percentage change in the expected total profit rate of the two firms as a result of cooperation; i.e., \( \frac{\Pi^* - \Pi^*_1}{\Pi^*} \). For this purpose, we consider variations around a base scenario with two symmetrical firms of size \( q = \frac{1}{3} \), each producing a single product with a market share of \( \alpha = \frac{1}{3} \). The manufacturing facility of each firm can process orders with rate \( \mu = 2 \). The per-unit profit margin is \( p = 1 \) (by assuming \( s = 1 \) and \( c = 0 \)), whereas the holding cost is \( h = 0.05 \). Customers arrive to the market with rate \( \lambda = 10 \) and are willing to substitute (from their first choice product to their second choice product in case their first choice is not produced by the firm that they visit) with probability \( \theta = 0.5 \). Table 1 presents the expected total profit rates, traffic intensities, base stock levels, average inventory levels, and firm-based and customer-based service levels when the two firms in the base scenario operate independently and when they form an assortment-based cooperation.

In our analysis, we systematically vary key problem parameters, one at a time, in the base scenario in order to understand their impact on the percentage increase in total expected profit rate realized as a result of the assortment-based cooperation between two firms and also their effects on the independent traffic intensity.

Figure 3 depicts the effects of product market share, firm size, production capacity, and substitution probability on

<table>
<thead>
<tr>
<th></th>
<th>Independent firms</th>
<th>Assortment-based cooperation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expected total profit rate</td>
<td>2.70</td>
<td>3.74</td>
</tr>
<tr>
<td>Traffic intensity</td>
<td>0.83</td>
<td>1.39</td>
</tr>
<tr>
<td>Base stock level</td>
<td>6</td>
<td>18</td>
</tr>
<tr>
<td>Average inventory</td>
<td>3.71</td>
<td>2.53</td>
</tr>
<tr>
<td>Firm-based service level (%)</td>
<td>45.9</td>
<td>59.5</td>
</tr>
<tr>
<td>Customer-based service level (%)</td>
<td>45.9</td>
<td>71.9</td>
</tr>
</tbody>
</table>

Fig. 3. Impact of (a) product market share, (b) firm size, (c) firm capacity, and (d) substitution probability on the change in expected total profit rate.
Assortment-based cooperation: make-to-stock firms

Fig. 4. Impact of traffic intensity on the change in expected total profit rate.

the percentage increase in the profit (solid black lines) and also on the independent traffic intensity at each firm (gray lines) when two symmetrical firms engage in an assortment-based cooperation. In Section 3, we show that $\dot{\rho}$ is an increasing function of the market share ($\alpha$), firm size ($q$), and the substitution probability ($\theta$) and a decreasing function of the production capacity ($\mu$). Accordingly, in Figs. 3(a) and 3(c), we observe that smaller firms with niche products (smaller market shares) and excess capacities benefit more from participating in an assortment-based cooperation. In other words, two symmetrical firms that are not dominant players in the market have the potential to take better advantage of the possibilities offered by an assortment-based cooperation (see also Fig. 3(b)).

Figure 3(d) illustrates the effects of the substitution probability. When the substitution probability is high, firms can satisfy a majority of the demand since customers are willing to substitute to the firm’s own product. In this case, the benefit of cooperation is lower. On the other hand, if the product of each firm is what differentiated and has distinguishing features from other products in the market, customers are less likely to give up on their first choice products (small $\theta$). In this case the benefit from cooperation is more accentuated.

Figure 4 depicts the distribution of the benefit from cooperation as a function of $\dot{\rho}$, aggregated over all different problem scenarios presented in Fig. 3. As we have already shown in our analysis of symmetrical firms (see proof of Proposition 6), assortment-based cooperation leads to an increase in each firm’s traffic intensity; i.e., $\xi \geq \dot{\rho}$, which in turn triggers an increase in their total profits (due to Corollary 4). On the other hand, one would intuitively expect a firm whose manufacturing facility initially operates at lower utilization levels, to enjoy greater returns from cooperation (induced by the additional demand streams allowed by the assortment-based cooperation) compared with a firm whose manufacturing facility is already highly utilized. This conjecture is clearly supported by Fig. 4.

Figure 5 shows how the profit margin ($p$) and the holding cost ($h$) of the two products affect the change in total expected profit rate. We observe that, as the profit margins diminish and holding costs increase, assortment-based cooperation becomes more beneficial for symmetrical firms. In a competitive market where firms have to survive on meager profit margins, additional demand streamed through cooperation turns out to be extremely valuable in generating significant incremental profits. Similarly, under adverse conditions in which keeping and maintaining stock becomes relatively more costly, assortment-based cooperation facilitates faster (and more effective) liquidation of inventories and proves to be more beneficial for cooperation participants.

4.2.2. Asymmetrical firms

In Section 4, we prove that assortment-based cooperation always benefits two symmetrical firms. However, when the firms do not have symmetrical parameters, this need not be the case. For the following numerical experiments, we focus on the base scenario introduced in Section 4.2.1 and let the two firms assume different values for a single parameter while keeping all remaining parameters symmetrical (at the base values). By doing so, we try to isolate the impact of each key problem parameter on the change in the expected

Fig. 5. Impact of (a) product profitability and (b) holding cost on the change in expected total profit rate.
Fig. 6. Impact of traffic intensity on the change in expected total profit rate for non-symmetrical firms.

total profit rate after the two nonidentical firms form an assortment-based cooperation.

The traffic intensity at a firm (when firms operate independently) is a function of the market shares of the two products, size of the firm, and its respective production capacity. The contour plot in Fig. 6 depicts the percentage change in the expected total profit when two firms with different traffic intensities (due to different product market shares, firm sizes, and production capacities) engage in an assortment-based cooperation. Figure 6(a) shows that cooperation is most beneficial when the two products have relatively small market shares. Consistent with our observation for Fig. 3(a), we observe that the potential benefit from cooperation diminishes as the products have larger market shares. Note that the traffic intensity at a firm does not change monotonically with the market shares of the two products. If the product of a firm has a large market share, the traffic intensity at the firm is high as expected. However, even when the product of a firm has just a marginal share in the market, the firm faces a high traffic intensity if the product of the other firm dominates the market, mainly due to substitution effects. It is important to highlight the fact that in such problem scenarios, where the production facility of a firm is already highly utilized, assortment-based cooperation can even decrease the total expected profit rate of the firms compared with the case when they operate independently. On the other hand, the traffic intensity at a firm monotonically increases with its own size. Therefore, it is essential that at least one of the two firms is relatively small (low utilization at the production facility of the firm) to achieve a more beneficial cooperation; see Fig. 6(b). As we already pointed out in our observation for Fig. 3(b), the maximum benefit is realized when two small firms cooperate. Similarly, the traffic intensity at a firm monotonically decreases with the capacity of its production facility. When both firms have large capacities and effectively low

Fig. 7. Impact of (a) product profitability and (b) holding cost on the change in expected total profit rate for non-symmetrical firms.
traffic intensities, they would enjoy greater benefits, as illustrated in Fig. 6(c) (and reported as well for Fig. 3(c)), since they both have sufficient capacities to fulfill additional customer orders induced by assortment-based cooperation. Even when only one of the firms has a relatively low utilization level at its production facility, firms would be able to realize significant benefits from cooperation. In contrast, when traffic intensities of both firms are high, they would not benefit from the additional traffic intensity induced by cooperation since they do not possess the capacity to convert this extra traffic into additional sales. As a result, the benefit from cooperation is rather limited.

Figure 7 shows how the profit margins and the holding costs of the products affect the change in total expected profit rate of two asymmetrical firms. We observe that if both products have high profit margins, the relative impact of the additional profits generated by the assortment-based cooperation will be slim on top of the already-high profits. When the holding costs are high, firms set relatively low base stock levels and subsequently lose part of their demand due to unavailability of products in their inventories. Under these circumstances, they each relish the sales opportunity of the other firm's product, facilitated through the assortment-based cooperation, and the benefits are rather significant relative to the meager profits before cooperation.

5. Decentralized cooperation

When the two firms participate in a decentralized assortment-based cooperation, they virtually merge their product assortments and agree to transfer products among themselves at discount rather than market prices, to satisfy customer demands at agreed-upon fill rate levels. We refer to this mechanism that basically governs the effective flow of products as well as finances among the two firms as a discount-based contract. Moreover, Firm 1 and Firm 2 make their own decisions in setting replenishment policies to maximize their individual expected profit rates under assortment-based cooperation.

5.1. Analytical results

If Firm 1 and Firm 2 decide to cooperate, Firm 1 agrees to supply Product A to Firm 2 at a discount price of \( \beta_1 \), where \( 0 \leq \beta_1 \leq 1 \) with a fill rate of \( \phi_1 \). In return, Firm 2 provides Firm 1 with Product B at a discount price of \( \beta_2 \), where \( 0 \leq \beta_2 \leq 1 \) with a fill rate of \( \phi_2 \). Therefore, the contract Firm i offers to the other party is a tuple \( (\beta_1, \phi_1) \) and the contract of the system is completely described as \( (\beta_1, \phi_1, (\beta_2, \phi_2)) \).

As a result of assortment-based cooperation, Firm 1 and Firm 2 make a profit from the sales of a product that they are not producing; i.e., Firm 1 makes a profit of \( (1 - \beta_1)S_A - c_A \) on each Product B it sells and similarly Firm 2 makes a profit of \( (1 - \beta_2)S_B - c_B \) on each Product A it sells. At the same time, Firm 1 makes a profit of \( \beta_1S_A - c_A \) on each Product A it sells to a customer through Firm 2 and Firm 2 makes a profit of \( \beta_2S_B - c_B \) on each Product B it sells to a customer through Firm 1.

In practice, discount parameters \( \beta_1 \) and \( \beta_2 \) and the service rates \( \phi_1 \) and \( \phi_2 \) are predetermined through negotiations among cooperating firms including various factors in addition to the ones captured in this model. In this study, we consider the case where the contract parameters are set endogenously by the firms to maximize their own benefit. We will show that it is possible to reach an equilibrium where the decentralized solution yields the centralized solution for particular values of the contract parameters.

5.1.1. Performance evaluation

As a direct consequence of the discount-based contract that regulates the decentralized assortment-based cooperation, each firm essentially faces two classes of customers demanding its own product to be supplied by its production facility. The higher-class demand corresponds to the firm’s own customers that pay the full price, and the lower-class demand corresponds to the group of customers directed from the other firm and pay a discounted price. When a firm faces this particular demand structure with two demand classes, it is optimal to establish a rationing policy for maximizing the expected profit (see Ha (1997) and De Véricourt et al. (2002)). The rationing level is a critical inventory level at or below which the rejection of the lower class demand is more beneficial for the firm. We denote the rationing threshold level at Firm \( i \) as \( R_i, i = 1, 2 \). The production facility of Firm \( i \) continues to produce until its inventory level reaches the base stock level \( S_i, i = 1, 2 \). Note that the base stock level and the rationing threshold level of each firm determine the fill rates for its own customers as well as the customers directed by the other firm and the average inventory levels.

We define \( \lambda_{A1}, \lambda_{A2}, \lambda_{B1}, \lambda_{B2} \) and \( \rho_1, \rho_2, \xi_1, \xi_2 \) exactly as in Section 4. Subsequently, the traffic intensity at Firm 1 due to aggregate demand can be denoted as \( \xi_1 = (\lambda_{A1} + \lambda_{A2})/\mu_1 \) and the traffic intensity due to own demand as \( \rho_1 = \lambda_{A1}/\mu_1 \). Similarly, the traffic intensity at Firm 2 due to aggregate demand is defined as \( \xi_2 = (\lambda_{B1} + \lambda_{B2})/\mu_2 \) and the traffic intensity due to its own demand as \( \rho_2 = \lambda_{B2}/\mu_2 \). Using these definitions, the expected profit rate of Firm 1 under a decentralized assortment-based cooperation with Firm 2 (for given \( S_2, R_2, \rho_2, \xi_2 \)) can be written as

\[
\tilde{\Pi}_1(S_1, R_2, S_2, \rho_1, \xi_1, \rho_2, \xi_2) = \lambda_{A1} f_A(S_1, R_1, \rho_1, \xi_1)(S_A - c_A) + \lambda_{A2} f_A(S_1, R_1, \rho_1, \xi_1)(S_A - c_A) + \lambda_{B1} f_B(S_2, R_2, \rho_2, \xi_2)(S_B - \beta_2 S_B) - h_A I_A(S_1, R_1, \rho_1, \xi_1),
\]

where \( f_A(S_1, R_1, \rho_1, \xi_1) \) is the fill rate of Product A at Firm 1, \( f_A(S_1, R_1, \rho_1, \xi_1) \) is the fill rate of Product A at Firm 2, \( f_B(S_2, R_2, \rho_2, \xi_2) \) is the fill rate of Product B at Firm 1, and

\[
\]
\(I_A(S_1, R_1, \rho_1, \xi_1)\) is the expected inventory level of Product A at Firm 1. The first term in Equation (11) is the profit generated at Firm 1 through the sales of its own product. On the other hand, the second (Product A sales at Firm 2) and third (Product B sales at Firm 1) terms represent additional profit streams of Firm 1 facilitated by the assortment-based cooperation with Firm 2. Note that the expected profit rate of Firm 1 under assortment-based cooperation is a function of its own base stock level, rationing threshold, and traffic intensity as well as their counterparts in Firm 2.

Similar to our analysis in Section 3, we determine the stationary probability distribution of the inventory level of each product by solving the balance equations of the CTMC models associated with the production processes of the two firms. Next, we derive the average inventory level of Product A and the fill rate Firm 1 by using these stationary distributions.

Let \(X_A(t)\) be the inventory level of Product A at Firm 1 at time \(t\). Then, the average Product A inventory at Firm 1, denoted by \(I_A(S_1, R_1, \rho_1, \xi_1)\), can be written as

\[
I_A(S_1, R_1, \rho_1, \xi_1) = E[X_A(t)] = S_1 - \frac{(\xi_1(1 - \rho_1)/(1 - \xi_1)) \left( 1 - \xi_1^{S_1 - R_1} - (1 - \xi_1)(S_1 - R_1)\xi_1^{S_1 - R_1 - 1} \right)}{(1 - \rho_1)(1 - \xi_1^{S_1 - R_1}) + (1 - \xi_1)\xi_1^{S_1 - R_1}(1 - \rho_1 R_1 + 1)} - \frac{(\rho_1(1 - \xi_1)/(1 - \rho_1))(\xi_1/\rho_1)^{S_1 - R_1} \left( \rho_1 S_1 - R_1 - \rho_1 S_1 + 1 + (1 - \rho_1)(S_1 - R_1)\rho_1^{S_1 - R_1 - 1} - (S_1 + 1)\rho_1^S \right)}{(1 - \rho_1)(1 - \xi_1^{S_1 - R_1}) + (1 - \xi_1)\xi_1^{S_1 - R_1}(1 - \rho_1 R_1 + 1)}.
\] (12)

We also compute the steady-state probability that Firm 1 fills its own Product A demand, denoted by \(f_A(S_1, R_1, \rho_1, \xi_1)\), as

\[
f_A(S_1, R_1, \rho_1, \xi_1) = P(X_A(t) > 0) = 1 - \frac{(1 - \xi_1)(1 - \rho_1)\xi_1^{S_1 - R_1} - \rho_1 R_1}{(1 - \rho_1)(1 - \xi_1^{S_1 - R_1}) + (1 - \xi_1)\xi_1^{S_1 - R_1}(1 - \rho_1 R_1 + 1)}.
\] (13)

Similarly, the probability that Firm 1 fills Product A demand received through Firm 2, denoted by \(f'_A(S_1, R_1)\), can be expressed as

\[
f'_A(S_1, R_1, \rho_1, \xi_1) = P(X_A(t) > R_1) = 1 - \frac{(1 - \xi_1)\xi_1^{S_1 - R_1} - \rho_1 R_1}{(1 - \rho_1)(1 - \xi_1^{S_1 - R_1}) + (1 - \xi_1)\xi_1^{S_1 - R_1}(1 - \rho_1 R_1 + 1)}.
\] (14)

Also, the probability that Firm 1 fills Product B demand using supplies from Firm 2, denoted by \(f'_B(S_2, R_2, \rho_2, \xi_2)\), is determined by

\[
f'_B(S_2, R_2, \rho_2, \xi_2) = P(X_B(t) > R_2) = 1 - \frac{(\rho_2(1 - \xi_2)\xi_2^{S_2 - R_2} - (1 - \rho_2 R_2 + 1)}{(1 - \rho_2)(1 - \xi_2^{S_2 - R_2}) + (1 - \xi_2)\xi_2^{S_2 - R_2}(1 - \rho_2 R_2 + 1)}.
\] (15)

Note that \(f'_B(S_2, R_2, \rho_2, \xi_2)\), which is a component of the expected profit of Firm 1 given in Equation (11), is a function of the base stock level, rationing threshold, as well as the aggregate and own traffic intensities of Firm 2.

Parallel to our service level definitions for independent firms, we define the firm-based service levels of Firm 1 and Firm 2 under an assortment-based cooperation as the probability that a customer visiting a particular firm in the cooperation ends up purchasing either Product A or Product B (any product in the assortment of the firm in the cooperation). Let \(\varphi_1\) and \(\varphi_2\) denote the firm-based service levels of Firm 1 and Firm 2 under an assortment-based cooperation. These service levels can be expressed as follows:

\[
\varphi_1(S_1, R_1, S_2, R_2, \rho_1, \xi_1, \rho_2, \xi_2) = (\alpha_A + \alpha_0 \delta_A) f_A(S_1, R_1, \rho_1, \xi_1) + (\alpha_S + \alpha_0 \delta_B) f'_B(S_2, R_2, \rho_2, \xi_2),
\] (16)

\[
\varphi_2(S_1, S_2, R_1, \rho_1, \xi_1, \rho_2, \xi_2) = (\alpha_A + \alpha_0 \delta_A) f_A(S_1, R_1, \rho_1, \xi_1) + (\alpha_S + \alpha_0 \delta_B) f'_B(S_2, R_2, \rho_2, \xi_2).
\] (17)

On the other hand, we define the customer-based service level as the probability that a customer is able to purchase her first choice product. Let \(\gamma_A\) and \(\gamma_B\) be the customer-based service levels for Product A and Product B when the two firms form a decentralized assortment-based cooperation. Then, the Product A and Product B service levels can be written as follows:

\[
\gamma_A(S_1, R_1, \rho_1, \xi_1) = \frac{q_1}{q_1 + q_2} f_A(S_1, R_1, \rho_1, \xi_1) + \frac{q_2}{q_1 + q_2} f'_B(S_2, R_2, \rho_2, \xi_2),
\] (18)

\[
\gamma_B(S_2, R_2, \rho_2, \xi_2) = \frac{q_1}{q_1 + q_2} f_A(S_1, R_1, \rho_1, \xi_1) + \frac{q_2}{q_1 + q_2} f'_B(S_2, R_2, \rho_2, \xi_2).
\] (19)
When firms engage in a decentralized assortment-based cooperation, their effective demand rates and subsequently their traffic intensities change compared with the case in which they operate independently. Therefore, each firm updates its base stock level to accommodate the modified traffic intensities generated by its own customers as well as the customers of the other firm. Furthermore, firms also establish rationing thresholds to deal with the two-class demand structure imposed by the discount-based contract under cooperation. Next, we present our analytical results about the impact of base stock level and rationing threshold on the fill rates and average inventory level.

One can intuitively expect Firm 1 to potentially increase the fill rates of customer orders for Product A, namely, \( f_A(S_1, R_1, \rho_1, \xi_1) \) and \( f'_A(S_1, R_1, \rho_1, \xi_1) \), regardless of whether the demand is received by Firm 1 or Firm 2 (and directed to Firm 1), by increasing its base stock level \( S_1 \). On the other hand, the impact of the rationing threshold \( R_1 \) on the fill rates of customer orders for Product A depends on whether the customer actually belongs to Firm 1 or Firm 2. By increasing \( R_1 \), Firm 1 would improve the Product A fill rate for its own customers, \( f_A(S_1, R_1, \rho_1, \xi_1) \), and in contrast reduce the Product A fill rate at Firm 2, \( f'_A(S_1, R_1, \rho_1, \xi_1) \). However, Firm 1’s ability to control the fill rates for Product A by manipulating the base stock level and the rationing threshold is limited. \( f_A(S_1, R_1, \rho_1, \xi_1) \) is bounded from above by \( \min(1, 1/\rho_1) \) (as \( S_1 \to \infty \) and \( R_1 \to \infty \)), whereas \( f'_A(S_1, R_1, \rho_1, \xi_1) \) is bounded from above by \( \min(1, 1/\xi_1) \) (as \( S_1 \to \infty \) and \( R_1 \to 0 \).

**Proposition 8.** \( f_A(S_1, R_1, \rho_1, \xi_1) \) is nondecreasing in \( S_1 \) and \( R_1 \), and \( f'_A(S_1, R_1, \rho_1, \xi_1) \) is nondecreasing in \( S_1 \) but nonincreasing in \( R_1 \).

Since Proposition 8 directly extends to Firm 2 and Product B, we next show that the firm-based service level of Firm 1, \( \varphi_1(S_1, R_1, S_2, R_2, \rho_1, \xi_1, \rho_2, \xi_2) \), is nondecreasing in the base stock level of both Firm 1 and Firm 2, as well as the rationing threshold of Firm 1, but nonincreasing in the rationing threshold of Firm 2. In other words, Firm 1 can possibly increase its own service level by setting a higher base stock level \( S_1 \) and a higher rationing threshold \( R_1 \) for given \( S_2 \) and \( R_2 \). However, we should note that the impact of such an action (Firm 1 increasing \( S_1 \) and \( R_1 \)) on the firm-based service level of Firm 2, \( \varphi_2(S_1, R_1, S_2, R_2, \rho_1, \xi_1, \rho_2, \xi_2) \), is not as obvious because a larger \( S_1 \) pushes the firm-based service level of Firm 2 up while a larger \( R_1 \) brings the firm-based service level of Firm 2 down. The net effect of these two oppositely directed components clearly depends on other problem parameters. On other hand, product-based service level of Product A, \( \gamma_A(S_1, R_1, S_2, R_2, \rho_1, \xi_1, \rho_2, \xi_2) \), improves with \( S_1 \) but deteriorates with \( R_1 \). In essence, Firm 1 can increase its own product’s product-based service level by setting a larger \( S_1 \) and a lower \( R_1 \). Clearly, Firm 2 has no direct control over the product-based service level of Product A.

**Proposition 9.** \( \varphi_1(S_1, R_1, S_2, R_2, \rho_1, \xi_1, \rho_2, \xi_2) \) is nondecreasing in \( S_1, S_2 \) and \( R_1 \) but nonincreasing in \( R_2 \), and \( \gamma_A(S_1, R_1, S_2, R_2, \rho_1, \xi_1, \rho_2, \xi_2) \) is nondecreasing in \( S_1 \) but nonincreasing in \( R_1 \) (and also independent of \( S_2 \) and \( R_2 \)).

Proposition 9 highlights the trade-off between firm-based and customer-based service levels. At a given base stock level \( S_1 \), Firm 1 would have to partially sacrifice its firm-based service level in order to improve the product-based service level of its own product, Product A. The next proposition states how the average Product A inventory level at Firm 1 changes with the base stock level and rationing threshold.

**Proposition 10.** \( I_A(S_1, R_1, \rho_1, \xi_1) \) is nondecreasing in \( S_1 \) and \( R_1 \).

As expected, increasing the base stock level \( S_1 \) leads to larger \( I_A(S_1, R_1, \rho_1, \xi_1) \) since Firm 1 essentially stocks more of Product A. For given \( S_1 \), Product A inventory is more leniently used for customers of Firm 2, as Firm 1 decreases its rationing threshold \( R_1 \). Since Product A is sold at a faster rate in such a situation, the expected inventory level of Product A intuitively decreases.

### 5.1.2. Discount-based contract

When the firms operate in a decentralized way, they first offer each other a discount-based contract that consists of the discount factor and the fill rate simultaneously. We assume that these parameters are binding. However, they can choose the base stock level and the rationing threshold independently in order to maximize their expected profit. For a particular discount-based contract—i.e., for set values of \((\beta_1, \phi_1)\) and \((\beta_2, \phi_2)\)—Firm 1’s expected profit \( \tilde{\Pi}_1(S_1, R_1, S_2, R_2, \rho_1, \xi_1, \rho_2, \xi_2) \), defined in Equation (11), depends on a number of Firm 2 parameters, specifically \( S_2 \), \( R_2 \), \( \rho_2 \), and \( \xi_2 \). However, it would suffice for Firm 1 to know \( \phi_2 = f'_B(S_2, R_2, \rho_2, \xi_2) \) (Product B is fill rate at Firm 1 guaranteed by Firm 2) to compute its own expected profit. Therefore, we essentially have

\[
\tilde{\Pi}_1(S_1, R_1, S_2, R_2, \rho_1, \xi_1, \rho_2, \xi_2) = \tilde{\Pi}_1(S_1, R_1, \rho_1, \xi_1, f'_B(S_2, R_2, \rho_2, \xi_2)).
\]

Accordingly, Firm 1 determines the optimal values of \( S_1 \) and \( R_1 \) by solving

\[
\tilde{\Pi}_1^* = \max_{S_1, R_1} \tilde{\Pi}_1(S_1, R_1, \rho_1, \xi_1, f'_B(S_2, R_2, \rho_2, \xi_2))
\]

subject to \( f_A(S_1, R_1, \rho_1, \xi_1) = \phi_1 \),

and Firm 2 determines the optimal values of \( S_2 \) and \( R_2 \) by solving

\[
\tilde{\Pi}_2^* = \max_{S_2, R_2} \tilde{\Pi}_2(S_2, R_2, \rho_2, \xi_2, f'_A(S_1, R_1, \rho_1, \xi_1))
\]

subject to \( f_B(S_2, R_2, \rho_2, \xi_2) = \phi_2 \),

where \( \tilde{\Pi}_1^* \) and \( \tilde{\Pi}_2^* \) are the optimal expected profit rates for Firm 1 and Firm 2 in a decentralized assortment-based cooperation.
Our main result indicates that for two symmetrical firms, the decentralized operation yields the results of the centralized case for particular values of the contract parameters. Furthermore, there is an equilibrium where the firms offer these contract parameters and do not deviate from their base stock and rationing levels.

**Proposition 11.** Under a decentralized assortment-based cooperation of two symmetrical firms, the maximum benefit of the centralized operation can be obtained by using the contract \( \{(\beta, \phi^*), (\beta, \phi^*)\} \). As a result, the decentralized assortment-based cooperation is always beneficial and the firm-based and customer-based service levels always improve. Furthermore, there is a Nash equilibrium where the firms use the contract \( \{(\beta, \phi^*), (\beta, \phi^*)\} \).

In the above result, the agreed-upon fill rate must be \( \phi^* = f_A(S^*, 0, \rho, \xi) \), where \( S^* \) is the optimal base stock level when the firm operates independently with the increased utilization of \( \xi \) as a result of demand increase following the assortment-based cooperation in order to reach equilibrium. In other words, the firms should guarantee the same service level they provide to their own customers to the other firm. Under this setting, for symmetrical firms, the centralized solution can be achieved at any discount rate. We note that increasing the discount rate, hence increasing the possibility of commission income from cooperation, can make the cooperation a more attractive alternative initially. Following the result that states that the assortment-based cooperation is always beneficial and also improves the service levels, the decentralized assortment-based cooperation can also achieve the same results by using specific contract parameters.

Our next result shows that the decentralized assortment-based cooperation for nonsymmetrical firms can also yield the results of the centralized operation for particular values of the contract parameters. Furthermore, these parameters also constitute a Nash equilibrium. Since assortment-based cooperation is not always beneficial for nonsymmetrical firms, the decentralized assortment-based cooperation does not always increase the profit compared with the independent case.

**Proposition 12.** Under a decentralized assortment-based cooperation of two nonsymmetrical firms, the maximum benefit of the centralized operation can be obtained by using the contract \( \{(1, \phi^*_1), (1, \phi^*_2)\} \). Furthermore, there is a Nash equilibrium where the firms use the contract \( \{(1, \phi^*_1), (1, \phi^*_2)\} \).

![Graph](image-url)

Fig. 8. (a) Total expected profit rate and (b) firm-based and (c) customer-based service levels for the base scenario as a function of the rationing threshold of Firm 1.
In this result, $\phi^*_1 = f_a(S^*_1, 0, \rho_1, \xi_1)$ and $\phi^*_2 = f_b(S^*_2, 0, \rho_2, \xi_2)$, where $S^*_1$ and $S^*_2$ are the optimal base stock levels when the firms operate independently. Similar to the symmetrical case, this result shows that the firms should guarantee the same service level they provide to their own customers to the other firm. Furthermore, they should operate without any discounts. They should agree to sell the products at the same price they sell to their customers. Under these conditions, the decentralized operation yields the same results as the centralized operation. These two results show that the discount-based contract can yield the same results as the centralized operation for specific values of the contract parameters. However, the firms may use different parameters due to different reasons; for example, to provide better service level to their own customers or to dictate different discount rates accounting for different factors, etc. These results show that setting contract parameters differently yield lower expected profits.

5.2. Numerical results

In order to illustrate the impact of the rationing policy and discount factors on the performance of a decentralized assortment-based cooperation, we use the base scenario defined in Section 4.2. However, we should first note that our numerical results and their implications for the centralized system naturally extend to the decentralized system given that the two firms agree on a discount-based contract to yield all potential benefits of a centralized cooperation. Accordingly, we assume that Firm 1 and Firm 2 initially set their base stock and rationing levels (in a decentralized manner) to achieve the total profit rate of the centralized system; i.e., $S^*_1 = S^*_2 = 18$ and $R^*_1 = R^*_2 = 0$. In Proposition 7, we formally prove that both types of service levels would always improve when two symmetrical firms cooperate compared with the case in which the firms operate independently. In our base scenario (see results in Table 1), for example, the probability that a firm realizes a sale to a visiting customer (firm-based service level) increases by 29.6%, whereas customers are also 56.7% more likely to end up purchasing their first choice products (customer-based service level) as a result of cooperation. Note that firms can potentially use the rationing threshold as a lever to manipulate firm-based and customer-based service levels (as shown in Proposition 9). Figure 8 presents what happens to the total expected profit rate and firm-based and customer-based service levels for the base scenario under an assortment-based cooperation when one of the two symmetrical firms (Firm 1) deviates from the optimal policy ($S^*_1 = S^*_2 = 18$ and $R^*_1 = R^*_2 = 0$) in terms of setting its rationing threshold $R^*_1$. We observe that by increasing...
$R_1$, Firm 1 can improve its firm-based service level. However, by doing so, it hurts the customer-based service level of its own product, namely, Product A. Moreover, its own profit rate, the total expected profit rate of the cooperation, as well as the firm-based service level of Firm 2 also decrease. At high levels of $R_1$, the total expected profit and the firm-based service level of Firm 2 can even be worse than what they would have been if the two firms operated independently. Therefore, Firm 1 does not have a clear motive to deviate from the optimal rationing level of $R_1^* = 0$.

Next, we look into how the Firm 1’s discount factor affects the cooperation’s total profit, as well as the two service levels. Since Firm 1 and Firm 2 have identical parameters, we initially set $\beta_1 = \beta_2 = 1$. As we observe in Fig. 9, firms can capture the total expected profit (as well as firm-based and customer-based service levels) of the centralized system, if they agree on a contract that does not offer significant discounts (large $\beta_1$ and $\beta_2$). If Firm 1 deviates from this initial contract and starts giving Firm 2 deeper discounts (small $\beta_1$) for exchange of products, the decentralized system becomes inefficient in the sense that benefits of assortment-based cooperation deteriorate. Furthermore, as $\beta_1$ decreases, the gap between the profit margins of the two demand classes (full-priced product sold at the firm versus the discounted product supplied to the other firm) that the manufacturing facility of Firm 1 faces becomes more significant. Therefore, Firm 1 introduces higher rationing thresholds. Since Firm 1 ends up satisfying a smaller portion of the overall demand, its base stock level also decrease with $\beta_1$ in the decentralized system. Figure 9 also illustrates that Firm 1 can potentially improve its own firm-based service level by deviating from the initial contract, but Firm 2 would be aware of the situation since its firm-based service level would start suffering. Moreover, such a deviation would also hurt the service level for the customers of Firm 1’s own product, namely, Product A.

6. Concluding remarks

Assortment-based cooperation is a viable option for SMEs with limited capacities to gain advantage in today’s fiercely competitive global economy and to improve profitability. In this study, we present an analytical model that can help firms evaluate the potential benefits from an assortment-based cooperation.

We prove that assortment-based cooperation is always beneficial when the firms have identical parameters. On the other hand, we also provide evidence that an assortment-based cooperation may yield lower total expected profits if the firms do not have identical parameter values. This is mainly due to the demand substitution and capacity limitations. For example, customers who substitute their first choice with another product that is in fact more profitable for the firm are directed to another firm that provides their first choice after these firms engage in an assortment-based cooperation. While this cooperation meets the customer expectations better, it may lower the total profit. Similarly, the capacity limitations may also cause customer loss since the assortment-based cooperation increases the effective demand rates to the firms.

Due to limited capacities and the intricate demand substitution effects, firms need to pay further attention to the impact of assortment-based cooperation on service levels. In this article, we propose two different service levels—a service level from a customer’s perspective (likelihood of purchasing first choice product) and another from a firm’s perspective (likelihood of making a sale). When firms with identical parameters cooperate, both customer-based and firm-based service levels improve. For firms with nonidentical parameters, rationing thresholds turn out to be an important lever to manipulate these two service levels. In a centralized cooperation, we prove that firms should set their rationing thresholds to zero, essentially eliminating the discounts given to the other firm. In a decentralized cooperation, rationing thresholds can be set depending on the discount rates as given in the contract between the firms. Accordingly, these discount factors not only determine the allocation of the total expected profit between the firms but also define the customer-based and firm-based service levels. We show that the discount-based contract yields the results of the centralized case for particular values of the contract parameters.

Our extensive numerical results suggest that assortment-based cooperation would be most beneficial for small-sized firms selling products with small market shares. By definition, such firms fall into the SME category. Moreover, our results verify the expectation that for an assortment-based cooperation to be fruitful, firms should have excess capacity when they operate independently. If a firm’s production facility already operates at high utilization levels, the benefit from cooperation would be marginal.

Finally, we should remark that all analytical results presented in this article directly extend to the case in which firms have multi-product assortments with dedicated capacity for each product. If the firms use a shared capacity for production and also engage in an assortment-based cooperation where the capacity is pooled, an optimal production policy to manage the pooled capacity should be developed. Investigation of a general cooperation model where the firms combine their assortments and also pool their capacity is left for future research.

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